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TEMPORALIZING LINEAR LOGIC

Dedicated to one of my teachers, Hiroakira Ono

Abstract

Completeness theorem with respect to Kripke semantics is shown for an extended intuitionistic linear logic with linear-time temporal operators.

1. Introduction

Linear-time temporal logics (LTLs), which are based on the classical logic, have widely been studied by many researchers in order to verify and specify concurrent systems. A comprehensive review for the completeness theorem w.r.t. Kripke semantics for LTLs was addressed by Lichtenstein and Pnueli [7].

Linear logics, which were introduced by Girard [2], are known as a resource-aware refinement of the classical and intuitionistic logics, and useful for obtaining more appropriate specifications of concurrent systems. In order to handle both resource-sensitive and time-dependent properties of concurrent systems, combining linear logics with temporal operators has been needed.

For this purpose, *temporal linear logics* have been proposed by Hirai [3], Tanabe [9], and Kanovich and Ito [6]. In [6], *classical and intuitionistic linear-time temporal linear logics*, which are results of combining linear logics with linear-time temporal operators, were introduced as sequent calculi, and the completeness theorems for these logics were shown using the algebraic structure of time phase semantics. Although the phase semantics for the extended linear logics were investigated in [6], other semantics such as Kripke semantics and other extended substructural logics have not been studied yet.

Substructural logics, including linear logics, relevant logics and Lambek calculus, are logics characterized by Gentzen-type sequent calculi in which applications of the structural rules are restricted. Kripke semantics for intuitionistic substructural logics have been studied by many researchers [1,4,8,10,11]. For example, Kripke semantics for substructural logics without contraction rule was originally introduced and studied by Ono and Komori [8]. A framework for proving Kripke completeness was introduced by Ishihara [4] for a wide range of the non-modal fragments of intuitionistic substructural logics. A modal extension of this framework was posed in [5].

The purpose of this paper is to obtain a Kripke semantic framework for *linear-time (temporal) intuitionistic substructural logics* including an extended intuitionistic linear logic with linear-time temporal operators. The Kripke semantics presented is based on the standard interpretation for the linear-time temporal operators X (next) and G (globally):

$$(x, i) \models \mathrm{X}\alpha \text{ iff } (x, i+1) \models \alpha,$$
$$(x, i) \models \mathrm{G}\alpha \text{ iff } (x, j) \models \alpha \text{ for all } j \in N \text{ with } i \leq j,$$

where x represents a possible world and i represents an element of the temporal domain N (the set of natural numbers). In this paper, a complete Hilbert-style axiomatization for this natural interpretation is obtained by using an extension of the method in [4,5].

The contents of this paper are then summarized as follows. In Section 2, a new logic, *linear-time intuitionistic linear logic* (LILL), which is an extension of the *intuitionistic linear logic* (ILL), is introduced based on a Hilbert-style axiomatization. In Section 3, a Kripke semantics for LILL is introduced with the addition of the interpretation for X and G. In Section 4, the completeness theorem w.r.t. the semantics is proved using an extension of the method in [4,5].

Although the explanation of Kripke semantics for other extended logics is omitted in this paper, the completeness result presented can straightforwordly be adapted and extended to other important intuitionistic substructual logics discussed in [4,5], such as linear-time relevant logics and linear-time Lambek calculus.

2. Logic

Formulas are constructed from propositional variables, **1** (multiplicative truth constant), $\top$ (additive truth constant), $\bot$ (additive falsum constant), $\rightarrow$ (implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction), ! (exponential), temporal operators X (next) and G (globally). Lower case letters $p, q, ...$ are used as metavariables for propositional variables, and lower case Greek letters $\alpha, \beta, ...$ are used as metavariables for formulas. The symbol Φ is used to denote the set of all formulas. The symbol $\equiv$ means the equality of sequences of symbols. We adopt the convention of association to the right in order to omit parentheses. For example, $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\beta\rightarrow\alpha\rightarrow\gamma \equiv (\alpha\rightarrow(\beta\rightarrow\gamma))\rightarrow(\beta\rightarrow(\alpha\rightarrow\gamma))$. The symbol N is used to represent the set of natural numbers. An expression $\mathrm{X}^i\alpha$ for any $i \in N$ is used to denote $\overbrace{\mathrm{XX}\cdots\mathrm{X}}^{i}\alpha$, i.e., $(\mathrm{X}^0\alpha \equiv \alpha)$ and $(\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha)$. Lower-case letters i, j and k are used to denote any natural numbers. An expression $\bigwedge_{j\in N} \mathrm{X}^j\alpha$ is used to denote $\alpha \wedge \mathrm{X}\alpha \wedge \mathrm{X}^2\alpha \wedge \cdots$. If a formula α is provable in a logic L, then we write $L \vdash \alpha$.

Definition 1 (LILL). The axiom schemes and inference rules for the logic LILL (*linear-time intuitionistic linear logic*) are as follows.

Non-modal part [1]:

A1: $(\mathbf{1}\rightarrow\alpha)\rightarrow\alpha$, A2: $\alpha\rightarrow\mathbf{1}\rightarrow\alpha$, A3: $\alpha\rightarrow\top$,
A4: $\bot\rightarrow\alpha$, A5: $\alpha\rightarrow\bot\rightarrow\beta$, A6: $\alpha\rightarrow\alpha$,
A7: $(\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\beta)\rightarrow\alpha\rightarrow\gamma$, A8: $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\beta\rightarrow\alpha\rightarrow\gamma$,
A9: $\alpha\rightarrow\beta\rightarrow\alpha*\beta$, A10: $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\alpha*\beta\rightarrow\gamma$,
A11: $\alpha\wedge\beta\rightarrow\alpha$, A12: $\alpha\wedge\beta\rightarrow\beta$, A13: $(\gamma\rightarrow\alpha)\wedge(\gamma\rightarrow\beta)\rightarrow\gamma\rightarrow\alpha\wedge\beta$,
A14: $\alpha\rightarrow\alpha\vee\beta$, A15: $\beta\rightarrow\alpha\vee\beta$, A16: $(\alpha\rightarrow\gamma)\wedge(\beta\rightarrow\gamma)\rightarrow\alpha\vee\beta\rightarrow\gamma$,

$$\frac{\alpha \quad \alpha\rightarrow\beta}{\beta}\,(\mathrm{mp}), \qquad \frac{\alpha \quad \beta}{\alpha\wedge\beta}\,(\mathrm{adj}).$$

Linear-exponential part [2]:

L1: $!(\alpha\rightarrow\beta)\rightarrow!\alpha\rightarrow!\beta$, L2: $!\alpha\rightarrow\alpha$, L3: $!\alpha\rightarrow!!\alpha$,
L4: $(!\alpha\rightarrow!\alpha\rightarrow\beta)\rightarrow!\alpha\rightarrow\beta$, L5: $\alpha\rightarrow!\beta\rightarrow\alpha$,

$$\frac{\alpha}{!\alpha}\,(!\mathrm{ness}).$$

[1]See e.g., [4].
[2]See [5].

Temporal-modal part: for any $i, j \in N$,

S1: $\mathrm{X}^i\mathbf{1}\rightarrow\mathbf{1}$, S2: $\mathbf{1}\rightarrow\mathrm{X}^i\mathbf{1}$, S3: $\top\rightarrow\mathrm{X}^i\top$, S4: $\mathrm{X}^i\bot\rightarrow\bot$,
S5: $\mathrm{X}^i(\alpha \wedge \beta)\rightarrow\mathrm{X}^i\alpha \wedge \mathrm{X}^i\beta$, S6: $\mathrm{X}^i\alpha \wedge \mathrm{X}^i\alpha\rightarrow\mathrm{X}^i(\alpha \wedge \beta)$,
S7: $\mathrm{X}^i(\alpha \vee \beta)\rightarrow\mathrm{X}^i\alpha \vee \mathrm{X}^i\beta$, S8: $\mathrm{X}^i\alpha \vee \mathrm{X}^i\beta\rightarrow\mathrm{X}^i(\alpha \vee \beta)$,
S9: $\mathrm{X}^i(\alpha\rightarrow\beta)\rightarrow\mathrm{X}^i\alpha\rightarrow\mathrm{X}^i\beta$, S10: $(\mathrm{X}^i\alpha\rightarrow\mathrm{X}^i\beta)\rightarrow\mathrm{X}^i(\alpha\rightarrow\beta)$,
S11: $\mathrm{X}^i(\alpha * \beta)\rightarrow\mathrm{X}^i\alpha * \mathrm{X}^i\beta$, S12: $\mathrm{X}^i\alpha * \mathrm{X}^i\beta\rightarrow\mathrm{X}^i(\alpha * \beta)$,
S13: $\mathrm{X}^i!\alpha\rightarrow!\mathrm{X}^i\alpha$, S14: $!\mathrm{X}^i\alpha\rightarrow\mathrm{X}^i!\alpha$,
S15: $\mathrm{X}^i\mathrm{G}\alpha\rightarrow\mathrm{X}^{i+j}\alpha$, S16: $(\bigwedge_{j\in N}\mathrm{X}^{i+j}\alpha)\rightarrow\mathrm{X}^i\mathrm{G}\alpha$.

In this definition, S16 is considered to be an infinite axiom scheme, since it includes a kind of infinite conjunctions. Roughly speaking, S15 and S16 means that $\mathrm{G}\alpha$ is logically equivalent to $\alpha \wedge \mathrm{X}\alpha \wedge \mathrm{X}^2\alpha \wedge \cdots \infty$. It is remarked that ILL is obtained from LILL by deleting S1–S16, and that the following rules are derivable in ILL:

$$\frac{\beta\rightarrow\gamma}{(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma)}\text{(pref)},\quad \frac{\alpha\rightarrow\beta}{(\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\gamma)}\text{(suff)},\quad \frac{\alpha\rightarrow\beta \quad \beta\rightarrow\gamma}{\alpha\rightarrow\gamma}\text{(cut)}.$$

3. Semantics

Definition 2. A *Kripke frame* is a structure $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying the following conditions.

1. N is the set of natural numbers.
2. $\langle M, \cap\rangle$ is a meet-semilattice with the greatest element ω.
3. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that [3]

   C1: $x \cdot \varepsilon = \varepsilon \cdot x = x$ for all $x \in M$,
   C2: $\omega \cdot x = \omega$ for all $x \in M$,
   C3: $\omega \leq x \cdot \omega$ for all $x \in M$,
   C4: $x \leq y$ implies $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $x, y, z \in M$,
   C5: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in M$,
   C6: $(x \cdot z) \cdot y \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,
   C7: $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ for all $x, y, z \in M$,
   C8: $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$ for all $x, y, z \in M$,

   where the order relation $x \leq y$ used is defined as $x \cap y = x$.

[3] $\langle M, \cdot, \varepsilon\rangle$ becames a commutative monoid with the identity ε.

4. $\dagger$ is a unary operation on M such that

   C9: $\dagger(x \cdot y) \leq \dagger x \cdot \dagger y$ for all $x, y \in M$,
   C10: $x \leq \dagger x$ for all $x \in M$,
   C11: $\dagger\dagger x \leq \dagger x$ for all $x \in M$,
   C12: $(x \cdot \dagger y) \cdot \dagger y \leq x \cdot \dagger y$ for all $x, y \in M$,
   C13: $x \leq x \cdot \dagger y$ for all $x, y \in M$,
   C14: $\dagger\varepsilon \leq \varepsilon$,
   C15: $x \leq y$ implies $\dagger x \leq \dagger y$ for all $x, y \in M$.

It is remarked that this frame is equivalent to that for ILL [5], and that the non-modal part of this frame is equivalent to that in [8], i.e., $\langle M, \cap, \cdot, \varepsilon, \omega\rangle$ is a *semilattice-ordered (commutative) monoid* for the non-modal version of ILL.

DEFINITION 3. A *valuation* $\models$ on a Kripke frame $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega\rangle$ is a mapping from the set of all propositional variables to $2^{M\times N}$ (where $2^{M\times N}$ denotes the power set of the direct product $M \times N$) such that $(x, i), (y, i) \in\ \models(p)$ iff $(x \cap y, i) \in\ \models(p)$, i.e., $\models(p) := X \times N \subseteq M \times N$ where X is a filter of M [4]. We will write $(x, i) \models p$ for $(x, i) \in\ \models(p)$. Each valuation $\models$ is extended to a mapping from the set Φ of all formulas to $2^{M\times N}$ by

1. $(x, i) \models \mathbf{1}$ iff $\varepsilon \leq x$,
2. $(x, i) \models \top$ for all $x \in M$ and all $i \in N$,
3. $(x, i) \models \bot$ iff $x = \omega$,
4. $(x, i) \models \alpha \rightarrow \beta$ iff $x \cdot y \leq z$ and $(y, i) \models \alpha$ imply $(z, i) \models \beta$ for all $y, z \in M$,
5. $(x, i) \models \alpha \wedge \beta$ iff $(x, i) \models \alpha$ and $(x, i) \models \beta$,
6. $(x, i) \models \alpha \vee \beta$ iff $(y, i) \models \alpha$ or $(y, i) \models \beta$, and $(z, i) \models \alpha$ or $(z, i) \models \beta$ for some $y, z \in M$ with $y \cap z \leq x$,
7. $(x, i) \models \alpha * \beta$ iff $(y, i) \models \alpha$ and $(z, i) \models \beta$ for some $y, z \in M$ with $y \cdot z \leq x$,
8. $(x, i) \models\ !\alpha$ iff $(y, i) \models \alpha$ for some y with $\dagger y \leq x$,
9. $(x, i) \models \mathrm{X}\alpha$ iff $(x, i+1) \models \alpha$,
10. $(x, i) \models \mathrm{G}\alpha$ iff $(x, j) \models \alpha$ for all $j \in N$ with $i \leq_L j$, where $\leq_L$ is the linear order on N.

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[4] Of course, X is nonempty.

In the following, the symbol $\leq_L$ or $\geq_L$ is simply expressed by $\leq$ or $\geq$ without the distinction between the order on M and the order on N, i.e., we abuse the symbols.

Proposition 4. *Let $\models$ be a valuation on a Kripke frame $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega \rangle$. Then, $\models(\alpha)$ is a mapping from Φ to $2^{M\times N}$ such that $(x, i), (y, i) \in \models(\alpha)$ iff $(x \cap y, i) \in \models(\alpha)$, i.e., $\models(\alpha) := X \times N \subseteq M \times N$ where X is a filter of M.*

Using Proposition 4, we can obtain the *hereditary condition*: for any formula α, any $x, y \in M$ and any $i \in N$, $(x, i) \models \alpha$ and $x \leq y$ imply $(y, i) \models \alpha$. In the following, we will sometimes use this condition implicitly.

Definition 5. A *Kripke model* is a structure $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega, \models\rangle$ such that $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega \rangle$ is a Kripke frame and $\models$ is a valuation on $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega \rangle$. A formula α is *true* in a Kripke model $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega, \models\rangle$ if $(\varepsilon, 0) \models \alpha$, and *valid* in a Kripke frame $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega \rangle$ if it is true for any valuation $\models$ on the Kripke frame.

Theorem 6. (Soundness). *Let C be the class of all Kripke frames, $L := \{\gamma \mid \text{LILL} \vdash \gamma\}$ and $L(C) := \{\gamma \mid \gamma$ is valid in all frames of $C\}$. Then $L \subseteq L(C)$.*

Proof. By induction on the proof P of γ in LILL. Let $\models$ be a valuation on $\langle M, N, \dagger, \cap, \cdot, \varepsilon, \omega \rangle \in C$. The cases for the ILL-part are almost the same as those for ILL in [4,5]. We show only the following cases.

(Case S15): γ is of the form $\mathrm{X}^i\mathrm{G}\alpha \rightarrow \mathrm{X}^{i+j}\alpha$. We will show $(\varepsilon, 0) \models \mathrm{X}^i\mathrm{G}\alpha \rightarrow \mathrm{X}^{i+j}\alpha$, i.e., $\forall y, z \in M$ $[\varepsilon \cdot y \leq z$ and $(y, 0) \models \mathrm{X}^i\mathrm{G}\alpha$ imply $(z, 0) \models \mathrm{X}^{i+j}\alpha]$. Suppose (1) $\varepsilon \cdot y \leq z$, i.e., $y \leq z$, and (2) $(y, 0) \models \mathrm{X}^i\mathrm{G}\alpha$, i.e., $\forall k \geq i$ $[(y, k) \models \alpha]$. In (2), taking $i + j$ for k, we obtain (3) $(y, i + j) \models \alpha$. By (1), (3) and the hereditary condition, we obtain $(z, i + j) \models \alpha$, and hence $(z, 0) \models \mathrm{X}^{i+j}\alpha$.

(Case S16): γ is of the form $\bigwedge_{j\in N} \mathrm{X}^{i+j}\alpha \rightarrow \mathrm{X}^i\mathrm{G}\alpha$. We show $(\varepsilon, 0) \models \bigwedge_{j\in N} \mathrm{X}^{i+j}\alpha \rightarrow \mathrm{X}^i\mathrm{G}\alpha$, i.e., $\forall y, z \in M$ $[\varepsilon \cdot y \leq z$ and $(y, 0) \models \bigwedge_{j\in N} \mathrm{X}^{i+j}\alpha$ imply $(z, 0) \models \mathrm{X}^i\mathrm{G}\alpha]$. Suppose (1) $y \leq z$ and (2) $(y, 0) \models \bigwedge_{j\in N} \mathrm{X}^{i+j}\alpha$, i.e., $\forall j \in N$ $[(y, 0) \models \mathrm{X}^{i+j}\alpha]$, i.e., $\forall j \in N$ $[(y, i + j) \models \alpha]$, i.e., $\forall k \geq i$ $[(y, k) \models \alpha]$. By

(1), (2) and the hereditary condition, we obtain $\forall k \geq i\ [(z,k) \models \alpha]$, i.e., $(z,i) \models \mathrm{G}\alpha$, i.e., $(z,0) \models \mathrm{X}^i\mathrm{G}\alpha$. Q.E.D.

4. Completeness

DEFINITION 7. Let $L := \{\alpha \mid \mathrm{LILL} \vdash\ \alpha\}$. An L-pretheory x [5] is a subset of Φ such that

1. $\top \in x$,
2. if $\alpha \in x$ and $\alpha \rightarrow \beta \in L$, then $\beta \in x$,
3. if $\alpha_i \in x$ for all $i \in N$, then $\bigwedge_{i \in N} \alpha_i \in x$ (esp., if $\alpha, \beta \in x$, then $\alpha \wedge \beta \in x$).

LEMMA 8. *Let M_L be the set of all L-pretheories, and $\dagger x := \{\beta \mid \exists\ \alpha \in x\ (!\alpha \rightarrow \beta \in L)\}$. For all $\alpha \in \Phi$ and all $x \in M_L$, if $\alpha \in \dagger x$, then $!\alpha \in \dagger x$.*

PROOF. See [5]. Q.E.D.

LEMMA 9. *Let M_L and $\dagger x$ be the same as those in Lemma 8, and $x \cdot y := \{\beta \mid \exists \alpha \in y\ (\alpha \rightarrow \beta \in x)\}$. Then*

1. if $x, y \in M_L$, then $x \cap y, x \cdot y, \dagger x \in M_L$,
2. $L \cdot \{\alpha\} \in M_L$,
3. if $x, y, z \in M_L$, then $x \cdot L = L \cdot x = x$, $\Phi \cdot x = \Phi$, $\Phi \subseteq x \cdot \Phi$, $[x \subseteq y$ implies $z \cdot x \subseteq z \cdot y$ and $x \cdot z \subseteq y \cdot z]$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $(x \cdot z) \cdot y \subseteq (x \cdot y) \cdot z$, $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ and $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$.
4. if $x, y \in M_L$, then $\dagger(x \cdot y) \subseteq \dagger x \cdot \dagger y$, $x \subseteq \dagger x$, $\dagger\dagger x \subseteq \dagger x$, $x \subseteq x \cdot \dagger y$, $\dagger L \subseteq L$, and $[x \subseteq y$ implies $\dagger x \subseteq \dagger y]$,
5. if $x, y \in M_L$, then $(x \cdot \dagger y) \cdot \dagger y \subseteq x \cdot \dagger y$.

PROOF. [4, 5]. The case 5 is proved using Lemma 8. Q.E.D.

By Lemma 9, we can conclude the following.

PROPOSITION 10. $F_L := \langle M_L, N, \dagger, \cap, \cdot, L, \Phi \rangle$ *is a Kripke frame for* LILL.

[5] The notion of L-pretheory is from [8].

LEMMA 11. *Let $x \in M_L$. Then*

1. $\alpha \in x$ iff $L \cdot \{\alpha\} \subseteq x$,
2. $(L \cdot \{\alpha\}) \cap (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha \vee \beta\}$,
3. $(L \cdot \{\alpha\}) \cdot (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha * \beta\}$,
4. $\dagger(L \cdot \{\alpha\}) \subseteq L \cdot \{!\alpha\}$.

PROOF. See [4,5]. Q.E.D.

LEMMA 12. *Let $\models_L$ be a mapping from the set Ψ of all propositional variables to $2^{M_L \times N}$ defined by $\models_L (p) := \{(x, i) \in M_L \times N \mid \mathrm{X}^i p \in x\}$. Then, $\models_L$ can be extended to a mapping from Φ to $2^{M_L \times N}$, i.e., we have the following: for any $\alpha \in \Phi$, any $i \in N$ and any $x \in M_L$, $\mathrm{X}^i\alpha \in x$ iff $(x, i) \models_L \alpha$.*

PROOF. By induction on the complexity of α. The base step is obvious by the definition of $\models_L (p)$. We show some cases in the induction step.

(Case $\alpha \equiv \beta \vee \gamma$): Suppose $\mathrm{X}^i(\beta \vee \gamma) \in x$. Then we have $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in x$ by $\mathrm{X}^i(\beta \vee \gamma) \rightarrow \mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in L$ (S7) and $x \in M_L$. By Lemma 11 (1), we obtain $L \cdot \{\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma\} \subseteq x$, and hence (*) $(L \cdot \{\mathrm{X}^i\beta\}) \cap (L \cdot \{\mathrm{X}^i\gamma\}) \subseteq x$ by Lemma 11 (2). Of course we have $L \cdot \{\mathrm{X}^i\beta\}, L \cdot \{\mathrm{X}^i\gamma\} \in M_L$ by Lemma 9 (2). Moreover we have $\mathrm{X}^i\beta \in L \cdot \{\mathrm{X}^i\beta\}$ and $\mathrm{X}^i\gamma \in L \cdot \{\mathrm{X}^i\gamma\}$. Then, by the induction hypothesis, we obtain (**) $(L \cdot \{\mathrm{X}^i\beta\}, i) \models_L \beta$ and $(L \cdot \{\mathrm{X}^i\gamma\}, i) \models_L \gamma$. By (*) and (**), we have $(x, i) \models_L \beta \vee \gamma$. Conversely, suppose $(x, i) \models_L \beta \vee \gamma$, i.e., there are $y, z \in M_L$ such that (*) $y \cap z \subseteq x$, (**) $(y, i) \models_L \beta$ or $(y, i) \models_L \gamma$, and (***) $(z, i) \models_L \beta$ or $(z, i) \models_L \gamma$. Applying the induction hypothesis to (**) and (***), we have $\mathrm{X}^i\beta \in y$ or $\mathrm{X}^i\gamma \in y$, and $\mathrm{X}^i\beta \in z$ or $\mathrm{X}^i\gamma \in z$. Since $\mathrm{X}^i\beta \rightarrow \mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in L$ (A14), $\mathrm{X}^i\gamma \rightarrow \mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in L$ (A15) and $y, z \in M_L$, we have $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in y$ and $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in z$, and hence $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in y \cap z$. Thus we have $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in x$ by (*). Therefore we obtain $\mathrm{X}^i(\beta \vee \gamma) \in x$ by $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \in x$ and $\mathrm{X}^i\beta \vee \mathrm{X}^i\gamma \rightarrow \mathrm{X}^i(\beta \vee \gamma) \in L$ (S8).

(Case $\alpha \equiv \beta \rightarrow \gamma$): Suppose $\mathrm{X}^i(\beta \rightarrow \gamma) \in x$, $x \cdot y \subseteq z$ and $(y, i) \models_L \beta$ for any $y, z \in M_L$. Then we have (*) $\mathrm{X}^i\beta \rightarrow \mathrm{X}^i\gamma \in x$ by $\mathrm{X}^i(\beta \rightarrow \gamma) \rightarrow \mathrm{X}^i\beta \rightarrow \mathrm{X}^i\gamma \in L$ (S9) and $x \in M_L$. Applying the induction hypotesis to $(y, i) \models_L \beta$, we have (**) $\mathrm{X}^i\beta \in y$. By (*),(**) and $x \cdot y \subseteq z$, we have $\mathrm{X}^i\gamma \in x \cdot y \subseteq z$, and hence $\mathrm{X}^i\gamma \in z$. Therefore $(z, i) \models_L \gamma$ by the induction hypothesis. Conversely, suppose $(x, i) \models_L \beta \rightarrow \gamma$. We have $L \cdot \{\mathrm{X}^i\beta\} \in M_L$ by Lemma 9 (2). We then have $(L \cdot \{\mathrm{X}^i\beta\}, i) \models_L \beta$ by applying the induction hypothesis to $\mathrm{X}^i\beta \in L \cdot \{\mathrm{X}^i\beta\} \in M_L$. Then we obtain $(x \cdot (L \cdot \{\mathrm{X}^i\beta\}), i) \models_L \gamma$ by the hypothesis, and hence $X^i\gamma \in x \cdot (L \cdot \{X^i\beta\})$ by the induction hypothesis. Thus there is $\delta \in L \cdot \{X^i\beta\} \in M_L$ such that $\delta \to X^i\gamma \in x$. Therefore we have $X^i\beta \to \delta \in L$, and hence $(\delta \to X^i\gamma) \to (X^i\beta \to X^i\gamma) \in L$ by (suff). Then we have $X^i\beta \to X^i\gamma \in x$ by $\delta \to X^i\gamma \in x$, $(\delta \to X^i\gamma) \to (X^i\beta \to X^i\gamma) \in L$ and $x \in M_L$. Therefore we obtain $X^i(\beta \to \gamma) \in x$ by $X^i\beta \to X^i\gamma \in x$, $(X^i\beta \to X^i\gamma) \to X^i(\beta \to \gamma) \in L$ (S10) and $x \in M_L$.

(Case $\alpha \equiv\ !\beta$): Suppose $X^i!\beta \in x$. Then we have $!X^i\beta \in x$ by $X^i!\beta \to !X^i\beta \in L$ (S13) and $x \in M_L$. Then we have $L \cdot \{!X^i\beta\} \subseteq x$ by Lemma 11 (1), $!X^i\beta \in x$ and $x \in M_L$. By Lemma 11 (4), we have $\dagger(L \cdot \{X^i\beta\}) \subseteq L \cdot \{!X^i\beta\} \subseteq x$. Thus we obtain (*) $\dagger(L \cdot \{X^i\beta\}) \subseteq x$ and $L \cdot \{X^i\beta\} \in M_L$ (by Lemma 9 (2)). On the other hand, we have $X^i\beta \in L \cdot \{X^i\beta\}$, and hence (**) $(L \cdot \{X^i\beta\}, i) \models_L \beta$ by the induction hypothesis. By (*) and (**), we obtain $(x, i) \models_L !\beta$. Conversely, suppose $(x, i) \models_L !\beta$. Then there is $y \in M_L$ such that (1) $\dagger y \subseteq x$ and (2) $(y, i) \models_L \beta$. Applying the induction hypothesis to (2), we obtain (3) $X^i\beta \in y$. By $!X^i\beta \to !X^i\beta \in L$ and (3), we obtain (4) $!X^i\beta \in \dagger y$. By (4) and (1), we have (5) $!X^i\beta \in x$. By (5), $!X^i\beta \to X^i!\beta \in L$ (S14) and $x \in M_L$, we obtain $X^i!\beta \in x$.

(Case $\alpha \equiv X\beta$): $X^i(X\beta) \in x$ iff $X^{i+1}\beta \in x$ iff $(x, i+1) \models_L \beta$ (by the induction hypothesis) iff $(x, i) \models_L X\beta$.

(Case $\alpha \equiv G\beta$): Suppose $X^i(G\beta) \in x$. Then, we have $\forall k \in N\ [X^{i+k}\beta \in x]$, i.e., $\forall j \geq i\ [X^j\beta \in x]$, by $X^iG\beta \to X^{i+k}\beta \in L$ (S15) and $x \in M_L$. Thus, by the hypothesis of induction, we obtain $\forall j \geq i\ [(x, i) \models_L \beta]$, and hence obtain $(x, i) \models_L G\beta$. Conversely, suppose $(x, i) \models_L G\beta$, i.e., $\forall j \geq i\ [(x, j) \models_L \beta]$. Then we obtain $\forall j \geq i\ [X^j\beta \in x]$, i.e., $\forall k \in N\ [X^{i+k}\beta \in x]$, by the hypothesis of induction. Then we obtain $\bigwedge_{j \in N} X^{i+j}\beta \in x$ by $x \in M_L$. Thus we obtain $X^i(G\beta) \in x$ by $\bigwedge_{j \in N} X^{i+j}\beta \to X^iG\beta \in L$ (S16) and $x \in M_L$. Q.E.D.

Using Lemma 12, we can obtain the following.

LEMMA 13. $\mathbf{M}_L := \langle M_L, N, \dagger, \cap, \cdot, L, \Phi, \models_L \rangle$ *is a Kripke model such that* $\alpha \in L$ *iff* α *is true in* $\mathbf{M}_L$.

Using Lemma 13 and Theorem 6, we can conclude the following.

THEOREM 14 (COMPLETENESS). *Let* C *be the class of all Kripke frames,* $L(C) := \{\alpha \mid \alpha$ *is valid in all frames of* $C\}$ *and* $L := \{\alpha \mid \text{LILL} \vdash \alpha\}$. *Then* $L = L(C)$.

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