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PSEUDO-BCH SEMILATTICES

Abstract
In this paper we study pseudo-BCH algebras which are semilattices or lattices with respect to the natural relation $\leq$; we call them pseudo-BCH join-semilattices, pseudo-BCH meet-semilattices and pseudo-BCH lattices, respectively. We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities defining pseudo-BCH meet-semilattices and pseudo-BCH lattices.

Keywords: (pseudo-)BCK/BCI/BCH algebra, pseudo-BCH join (meet)-semilattice, weakly regular, arithmetical at 1.

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1. Introduction
In 1966, Imai and Iséki ([8], [11]) introduced BCK and BCI algebras as algebras connected to certain kinds of logics. In 1983, Hu and Li ([7]) defined BCH algebras. It is known that BCK and BCI algebras are contained in the class of BCH algebras. In [9], [10], Iorgulescu introduced many interesting generalizations of BCI or of BCK algebras.

In 2001, Georgescu and Iorgulescu ([9]) defined pseudo-BCK algebras as an extension of BCK algebras. In 2008, Dudek and Jun ([2]) introduced pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCK algebras. These algebras have also connections with other algebras of logic such as pseudo-MV algebras and pseudo-BL algebras defined by Georgescu and Iorgulescu in [4] and [5], respectively. Recently, Walendziak ([14]) introduced pseudo-BCH algebras as an extension of BCH algebras.
In [13], Kühr investigated pseudo-BCK algebras whose underlying posets are semilattices. In this paper we study pseudo-BCH join-semilattices, that is, pseudo-BCH algebras which are join-semilattices with respect to the natural relation \( \leq \). We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities defining pseudo-BCH meet-semilattices and pseudo-BCH lattices.

2. Preliminaries

We recall that an algebra \( (X; \rightarrow, 1) \) of type \((2, 0)\) is called a BCH algebra if it satisfies the following axioms:

\begin{align*}
(BCH-1) & \quad x \rightarrow x = 1; \\
(BCH-2) & \quad x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z); \\
(BCH-3) & \quad x \rightarrow y = y \rightarrow x = 1 = \Rightarrow x = y.
\end{align*}

A BCI algebra is a BCH algebra \((X; \rightarrow, 1)\) satisfying the identity

\[ (BCI) \quad (y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = 1. \]

A BCK algebra is a BCI algebra \((X; \rightarrow, 1)\) such that \(x \rightarrow 1 = 1\) for all \(x \in X\).

A pseudo-BCI algebra \((X; \leq, \rightarrow, \Rightarrow, 1)\) is a structure on the set \(X\), \(\rightarrow\) and \(\Rightarrow\) are binary operations on \(X\) and 1 is an element of \(X\), verifying the axioms:

\begin{align*}
(pBCI-1) & \quad y \rightarrow z \leq (z \rightarrow x) \Rightarrow (y \rightarrow x), \quad y \Rightarrow z \leq (z \Rightarrow x) \rightarrow (y \Rightarrow x); \\
(pBCI-2) & \quad x \leq (x \Rightarrow y) \rightarrow y, \quad x \leq (x \rightarrow y) \Rightarrow y; \\
(pBCI-3) & \quad x \leq x; \\
(pBCI-4) & \quad x \leq y, y \leq x \Rightarrow x = y; \\
(pBCI-5) & \quad x \leq y \iff x \rightarrow y = 1 \iff x \Rightarrow y = 1.
\end{align*}

A pseudo-BCI-algebra \((X; \leq, \rightarrow, \Rightarrow, 1)\) is called a pseudo-BCK algebra if it satisfies the identities

\[ (pBCK) \quad x \rightarrow 1 = x \Rightarrow 1 = 1. \]

**Definition 2.1.** \((173)\) A (dual) pseudo-BCH algebra is an algebra \(X = (X; \rightarrow, \leftarrow, 1)\) of type \((2, 2, 0)\) satisfying the axioms:

\begin{align*}
(pBCH-1) & \quad x \rightarrow x = x \leftarrow x = 1; \\
(pBCH-2) & \quad x \rightarrow (y \leftarrow z) = y \leftarrow (x \rightarrow z);
\end{align*}
(pBCH-3) \( x \to y = y \Rightarrow x = 1 \Rightarrow x = y \);
(pBCH-4) \( x \to y = 1 \iff x \adj y = 1 \).

Remark 2.2. Observe that if \((X; \to, 1)\) is a BCH algebra, then letting \(x \to y := x \adj y\), produces a pseudo-BCH algebra \((X; \to, \adj, 1)\). Therefore, every BCH algebra is a pseudo-BCH algebra in a natural way. It is easy to see that if \((X; \adj, \to, 1)\) is a pseudo-BCH algebra, then \((X; \to, \adj, 1)\) is also a pseudo-BCH algebra. From Proposition 3.2 of [15] we conclude that if \((X; \leq, \to, \adj, 1)\) is a pseudo-BCI algebra, then \((X; \to, \adj, 1)\) is a pseudo-BCH algebra.

In any pseudo-BCH algebra we can define a natural relation \(\leq\) by putting
\[
x \leq y \iff x \to y = 1 \iff x \adj y = 1.
\]
It is easy to see that \(\leq\) is reflexive and anti-symmetric but it is not transitive in general (see Example 2.3 below). We note that in pseudo-BCK/BCI algebras the relation \(\leq\) is a partial order.

Example 2.3. Let \(X = \{a, b, c, d, e, f, 1\}\). We define the binary operations \(\to\) and \(\adj\) on \(X\) as follows

\[
\begin{array}{cccccccc}
\to & | & a & b & c & d & e & f & 1 \\
\hline
a & | & 1 & b & b & d & e & f & 1 \\
b & | & a & 1 & c & d & e & f & 1 \\
c & | & 1 & 1 & 1 & d & e & f & 1 \\
d & | & a & b & c & 1 & 1 & f & 1 \\
e & | & a & b & c & 1 & 1 & 1 & 1 \\
f & | & a & b & c & d & e & 1 & 1 \\
1 & | & a & b & c & d & e & f & 1 \\
\end{array}
\begin{array}{cccccccc}
\adj & | & a & b & c & d & e & f & 1 \\
\hline
a & | & 1 & b & c & d & e & f & 1 \\
b & | & a & 1 & a & d & e & f & 1 \\
c & | & 1 & 1 & 1 & d & e & f & 1 \\
d & | & a & b & c & 1 & 1 & f & 1 \\
e & | & a & b & c & e & 1 & 1 & 1 \\
f & | & a & b & c & d & e & 1 & 1 \\
1 & | & a & b & c & d & e & f & 1 \\
\end{array}
\]

Then \(X = (X; \to, \adj, 1)\) is a pseudo-BCH algebra (see Example 2.6 of [15]). We have \(d \leq e\) and \(e \leq f\) but \(d \nleq f\), and therefore \(\leq\) is not transitive.

Proposition 2.4. ([14]) Every pseudo-BCH algebra \(X\) satisfies, for all \(x, y \in X\), the following conditions:

(i) \(1 \to x = 1 \adj x = x\),
(ii) \(x \leq (x \adj y) \to y\), and \(x \leq (x \to y) \adj y\).
Proposition 2.5. \((\text{[L4]}\)) Let \(X\) be a pseudo-BCH algebra. Then \(X\) is a pseudo-BCI algebra if and only if it verifies the following implication: for all \(x, y, z \in X\),
\[
x \leq y \implies (z \to x \leq z \to y, z \leq x \leq z \leq y).
\] (2.1)

3. Pseudo-BCI semilattices

Generalizing the notion of a pseudo-BCK semilattice (see \([13]\)) we define pseudo-BCH join-semilattices.

Definition 3.1. We say that an algebra \((X; \vee, \to, \dashedrightarrow, 1)\) is a pseudo-BCH join-semilattice if \((X; \vee)\) is a join-semilattice, \((X; \to, \dashedrightarrow, 1)\) is a pseudo-BCH-algebra and \(x \vee y = y \iff x \to y = 1\) for all \(x, y \in X\).

Example 3.2. Let \(X = \{a, b, c, 1\}\). We define the binary operations \(\to\) and \(\dashedrightarrow\) on \(X\) as follows:

\[
\begin{array}{c|cccc}
\to & a & b & c & 1 \\
\hline
a & 1 & b & b & 1 \\
b & 1 & 1 & b & 1 \\
c & 1 & 1 & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\quad
\begin{array}{c|cccc}
\dashedrightarrow & a & b & c & 1 \\
\hline
a & 1 & b & c & 1 \\
b & 1 & 1 & a & 1 \\
c & 1 & 1 & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\]

It is easy to check that \((X; \to, \dashedrightarrow, 1)\) is a pseudo-BCH algebra. Since \(X\) is a join-semilattice with respect to \(\vee\) (under \(\leq\)), we conclude that \(X = (X; \vee, \to, \dashedrightarrow, 1)\) is a pseudo-BCH join-semilattice; it is even a chain with \(c < b < a < 1\).

Example 3.3. Let \(X = (\{a, b, c, d, e, f, 1\}; \to, \dashedrightarrow, 1)\) be the pseudo-BCH algebra from Example 2.3. Since the relation \(\leq\) is not transitive, \(X\) is not a join-semilattice with respect to \(\leq\). Therefore it is not a pseudo-BCH join-semilattice.

Proposition 3.4. Let \((X; \vee, \to, \dashedrightarrow, 1)\) be a pseudo-BCH join-semilattice. The following properties hold (for all \(x, y, z \in X\)):

(a1) \(x \vee y = y \vee x\),
(a2) \((x \vee y) \vee z = x \vee (y \vee z)\),
(a3) \(x \to (y \dashedrightarrow z) = y \dashedrightarrow (x \to z)\),
Pseudo-BCH Semilattices

(a4) \(1 \rightarrow x = 1 \leadsto x = x\),
(a5) \(x \rightarrow (x \lor y) = x \leadsto (x \lor y) = 1\),
(a6) \(((x \leadsto y) \rightarrow y) \lor x = (x \leadsto y) \rightarrow y\),
(a7) \(((x \rightarrow y) \leadsto y) \lor x = (x \rightarrow y) \leadsto y\).

**Proof:** (a1)–(a3) and (a5) are obvious. By Proposition 2.4 (i) we get (a4). Identities (a6) and (a7) follow from Proposition 2.4 (ii).

**Proposition 3.5.** Let \((X; \lor, \rightarrow, \leadsto, 1)\) be an algebra of type \(2, 2, 2, 0\) satisfying (a1)–(a7). Define \(\leq\) on \(X\) by

\[x \leq y \iff x \lor y = y.\]

Then, for all \(x, y, z \in X\), we have:

1. \(x \leq y\) and \(y \leq x\) imply \(x = y\),
2. \(x \leq y\) and \(y \leq z\) imply \(x \leq z\),
3. \(x \leq y \iff x \rightarrow y = 1\),
4. \(x \leq y \iff x \leadsto y = 1\),
5. \(x \lor 1 = 1 \lor x = 1\) (that is, \(x \leq 1\)),
6. \(x \rightarrow 1 = x \leadsto 1 = 1\),
7. \(x \rightarrow x = x \leadsto x = 1\) (that is, \(x \leq x\)).

**Proof:** Statements (1) and (2) follow from (a1) and (a2), respectively.

To prove (3), let \(x, y \in X\) and \(x \lor y = y\). Applying (a5), we get \(x \rightarrow y = 1\).

Conversely, suppose that \(x \rightarrow y = 1\). Hence \((x \rightarrow y) \leadsto y = 1 \leadsto y = y\) by (a4). From (a7) we see that \(x \lor y = y\), that is, \(x \leq y\).

(4) The proof of (4) is similar to that of (3).

(5) Applying (a5) and (a4), we obtain \(1 = 1 \rightarrow (1 \lor x) = 1 \lor x\). This clearly forces (5).

(6) By (5), \(x \leq 1\). Using (3) and (4), we get (6).

(7) We have

\[
\begin{align*}
1 & = ((1 \leadsto x) \rightarrow x) \lor 1 \quad \text{[by (5)]} \\
& = (1 \leadsto x) \rightarrow x \quad \text{[by (a6)]} \\
& = x \rightarrow x. \quad \text{[by (a4)]}
\end{align*}
\]

Similarly, \(x \leadsto x = 1\).

Combining Propositions 3.4 and 3.5 we get
Theorem 3.6. An algebra \((X; \lor, \rightarrow, \rightsquigarrow, 1)\) of type \((2, 2, 2, 0)\) is a pseudo-BCH join-semilattice if and only if it satisfies the identities (a1)–(a7).

From Proposition 3.5 (6) we have

Corollary 3.7. Every pseudo-BCH join-semilattice verifies \((pBCK)\).

Let us denote by \(J\) the class of all pseudo-BCH join-semilattices.

Remark 3.8. The class \(J\) is a variety. Therefore \(J\) is closed under the formation of homomorphic images, subalgebras, and direct products.

The disjoint union of BCK algebras was introduced by Iséki and Tanaka in [12] and next generalized to BCH algebras ([3]) and pseudo-BCH algebras ([15]). Below we extend this concept to the case of pseudo-BCH join-semilattices.

Let \(T\) be any set and, for each \(t \in T\), let \(X_t = (X_t; \lor_t, \rightarrow_t, \rightsquigarrow_t, 1)\) be a pseudo-BCH join-semilattice. Suppose that \(X_s \cap X_t = \{1\}\) for \(s, t \in T, s \neq t\). Set \(X = \bigcup_{t \in T} X_t\) and define the binary operations \(\lor, \rightarrow\) and \(\rightsquigarrow\) on \(X\) via

\[ x \lor y = \begin{cases} x \lor_t y & \text{if } x, y \in X_t, t \in T, \\ 0 & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases} \]

\[ x \rightarrow y = \begin{cases} x \rightarrow_t y & \text{if } x, y \in X_t, t \in T, \\ x & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases} \]

and

\[ x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_t y & \text{if } x, y \in X_t, t \in T, \\ x & \text{if } x \in X_s, y \in X_t, s, t \in T, s \neq t. \end{cases} \]

It is easily seen that \(X = (X; \lor, \rightarrow, \rightsquigarrow, 1)\) is a pseudo-BCH join-semilattice; it will be called the disjoint union of \((X_t)_{t \in T}\).

Example 3.9. Let \(X_1 = X\), where \(X = \{a, b, c, 1\}; \lor, \rightarrow, \rightsquigarrow, 1\) is the pseudo-BCH join-semilattice from Example 3.3. Consider the set \(X_2 = \{d, e, f, 1\}\) with the operations \(\rightarrow_2\) and \(\lor_2\) defined by the following tables:

<table>
<thead>
<tr>
<th>(\rightarrow_2)</th>
<th>d</th>
<th>e</th>
<th>f</th>
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and

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<th>(\lor_2)</th>
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Let $\rightarrow_2 := \rightarrow_2$. Routine calculations show that $\mathfrak{X}_2 = (X_2; \vee_2, \rightarrow_2, \sim_2, 1)$ is a (pseudo)-BCH join-semilattice. Let $X' = \{a, b, c, d, e, f, 1\}$. We define the binary operations $\rightarrow'$ and $\sim'$ on $X'$ as follows

<table>
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<tr>
<th>$\sim'$</th>
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It is clear that $\mathfrak{X}' = (X'; \vee', \rightarrow', \sim', 1)$, where the operation $\vee'$ is illustrated in Figure 1, is the disjoint union of $\mathfrak{X}_1$ and $\mathfrak{X}_2$.

\begin{center}
\begin{tikzpicture}
\node (a) at (0,1) [circle,fill,inner sep=1.5pt] {};\node (d) at (1,0) [circle,fill,inner sep=1.5pt] {};
\node (e) at (1.5,1) [circle,fill,inner sep=1.5pt] {};\node (f) at (1,2) [circle,fill,inner sep=1.5pt] {};
\node (b) at (-1,0) [circle,fill,inner sep=1.5pt] {};\node (c) at (0,-1) [circle,fill,inner sep=1.5pt] {};
\node (a) at (0,1) [circle,fill,inner sep=1.5pt] {};
\node (1) at (0,3) [circle,fill,inner sep=1.5pt] {};
\draw (a) -- (d);
\draw (a) -- (e);
\draw (a) -- (f);
\draw (b) -- (c);
\draw (b) -- (d);
\draw (b) -- (e);
\draw (c) -- (d);
\draw (c) -- (e);
\draw (c) -- (f);
\end{tikzpicture}
\end{center}

Figure 1

**Proposition 3.10.** Let $\mathfrak{X} = (X; \vee, \rightarrow, \sim, 1)$ be a pseudo-BCH join-semilattice. Then the following statements are equivalent:

(i) $\mathfrak{X}$ is a pseudo-BCK join-semilattice.

(ii) $\mathfrak{X}$ satisfies (2.1) for all $x, y, z \in X$.

**Proof:** Follows immediately from Proposition 2.5 and Corollary 3.7.

**Proposition 3.11.** Let $\mathfrak{X} = (X; \vee, \rightarrow, \sim, 1)$ be a pseudo-BCH join-semilattice satisfying the following implication: for all $x, y, z \in X$,

$$x \leq y \implies (y \rightarrow x) \sim x = (y \sim x) \rightarrow x = y.$$  

(3.1)

Then $\mathfrak{X}$ is a pseudo-BCK join-semilattice.
Proof: Let $x, y, z \in X$ and $x \leq y$. By (pBCH-2), (pBCH-1) and (pBCK),
\[
(z \rightarrow x) \rightarrow (z \rightarrow y) = (z \rightarrow x) \rightarrow (z \rightarrow ((y \rightarrow x) \sim x))
\]
\[
= (y \rightarrow x) \sim ((z \rightarrow x) \rightarrow (z \rightarrow x))
\]
\[
= (y \rightarrow x) \sim 1
\]
\[
= 1.
\]
Then $z \rightarrow x \leq z \rightarrow y$. Similarly, $z \sim x \leq z \sim y$. From Proposition 3.10, we see that $X$ is a pseudo-BCK join-semilattice.

Remark 3.12. The converse of Proposition 3.11 is false. Indeed, let $X$ be the pseudo-BCH join-semilattice from Example 3.2. It is easy to check that $X$ satisfies implication (2.1), and therefore it is a pseudo-BCK join-semilattice. However, (3.1) does not hold in $X$, because we have $c < a$ and $(a \sim c) \rightarrow c = 1$.

Definition 3.13. An algebra $(X; \wedge, \rightarrow, \sim, 1)$ is called a pseudo-BCH meet-semilattice if $(X; \wedge)$ is a meet-semilattice, $(X; \rightarrow, \sim, 1)$ is a pseudo-BCH algebra, and $x \wedge y = x \iff x \rightarrow y = 1$ for all $x, y \in X$.

Denote by $\mathcal{M}$ the class of all pseudo-BCH meet-semilattices.

Proposition 3.14. An algebra $X = (X; \wedge, \rightarrow, \sim, 1)$ of type $(2, 2, 2, 0)$ is a pseudo-BCH meet-semilattice if and only if it satisfies the following identities:

(b1) $x \wedge x = x$,
(b2) $x \wedge y = y \wedge x$,
(b3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(b4) $x \rightarrow (y \sim z) = y \sim (x \rightarrow z)$,
(b5) $1 \rightarrow x = 1 \sim x = x$,
(b6) $(x \wedge y) \rightarrow y = 1 = (x \wedge y) \sim y$,
(b7) $x \wedge ((x \rightarrow y) \rightarrow y) = x \wedge ((x \rightarrow y) \sim y)$.

Proof: Obviously, every pseudo-BCH meet-semilattice satisfies the axioms (b1)–(b7).

Conversely, let (b1)–(b7) hold in $X$. Clearly, $(X; \wedge)$ is a meet-semilattice. Define $\leq$ on $X$ by
\[
x \leq y \iff x = x \wedge y.
\]

Observe that
\[
x \leq y \iff x \rightarrow y = 1 \iff x \sim y = 1 \quad (3.2)
\]
for all $x, y \in X$. Let $x \leq y$, that is, $x \land y = x$. By (b6), $x \rightarrow y = 1$ and $x \preceq y = 1$. Suppose now that $x \rightarrow y = 1$. Applying (b7) and (b5), we get

$$x = x \land ((x \rightarrow y) \preceq y) = x \land (1 \preceq y) = x \land y.$$ 

Hence $x \leq y$. Similarly, if $x \preceq y = 1$, then $x \leq y$. Thus (3.2) holds. Therefore, we deduce that $(X; \land, \rightarrow, 1)$ is a pseudo-BCH algebra, and finally that $(X; \land, \rightarrow, \preceq, 1)$ is a pseudo-BCH meet-semilattice.

**Corollary 3.15.** The class $\mathcal{M}$ is a variety.

**Definition 3.16.** An algebra $(X; \lor, \land, \rightarrow, \preceq, 1)$ is called a pseudo-BCH lattice if $(X; \lor, \land)$ is a lattice, $(X; \rightarrow, \preceq, 1)$ is a pseudo-BCH algebra, and $x \rightarrow y = 1 \iff x \lor y = y \iff x \land y = x$ for all $x, y \in X$.

Denote by $\mathcal{L}$ the class of all pseudo-BCH lattices.

**Example 3.17.** Let $X = \{a, b, c, d, 1\}$. Define binary operations $\rightarrow$ and $\preceq$ on $X$ by the following tables:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>1</td>
<td>$a$</td>
<td>$d$</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$d$</td>
<td>1</td>
<td>1</td>
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<td>1</td>
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<tr>
<td>1</td>
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<td>1</td>
<td>1</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>$b$</td>
<td>$b$</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>1</td>
<td>$a$</td>
<td>$a$</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$a$</td>
<td>1</td>
</tr>
<tr>
<td>$d$</td>
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<td>1</td>
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<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

By routine calculation, $\mathcal{X} = (X; \rightarrow, \preceq, 1)$ is a pseudo-BCH algebra. We shall represent the set $X$ and the binary relation $\leq$ by the following Hasse diagram:

![Hasse diagram](image.png)
Therefore, \((X; \lor, \land, \rightarrow, \neg, 1)\) is a pseudo-BCH lattice.

**Remark 3.18.** The class \(L\) is a variety that is axiomatized by the defining identities of lattices and by the identities \((a3)-(a7)\) or by \((b4)-(b7)\), respectively.

Now we recall several universal algebraic notions (see e.g. \([1]\)). We will denote by \(\text{Con}\mathfrak{A}\) the congruence lattice of an algebra \(\mathfrak{A}\). For \(\theta, \phi \in \text{Con}\mathfrak{A}\) and \(x \in A\), let \(x/\theta\) denote the equivalence class of \(x\) modulo \(\theta\). An algebra \(\mathfrak{A}\) with a constant 1 is called:

- **weakly regular** (at 1) if \(1/\theta = 1/\phi\) implies \(\theta = \phi\), for all \(\theta, \phi \in \text{Con}\mathfrak{A}\);
- **permutable at** 1 if \(1/(\theta \circ \phi) = 1/(\phi \circ \theta)\) for all \(\theta, \phi \in \text{Con}\mathfrak{A}\);
- **distributive at** 1 if \(1/\theta \cap (\phi \lor \psi) = 1/(\theta \cap \phi) \lor (\theta \cap \psi)\) for all \(\theta, \phi, \psi \in \text{Con}\mathfrak{A}\);
- **arithmetical at** 1 if it is both permutable at 1 and distributive at 1.

Let \(V\) be a variety of algebras with a constant 1. We say that \(V\) is **weakly regular** (resp., **permutable at** 1, **distributive at** 1, and **arithmetical at** 1) if every algebra \(A \in V\) is weakly regular (resp., permutable at 1, distributive at 1, and arithmetical at 1). It is known that a variety \(V\) is weakly regular if and only if there exist binary terms \(t_1, \ldots, t_n\) for some \(n \in \mathbb{N}\) such that

\[
    t_1(x, y) = \cdots = t_n(x, y) = 1 \iff x = y. \quad (3.3)
\]

A variety is **arithmetical at** 1 if and only if there exists a binary term \(t\) satisfying \(t(x, x) = t(1, x) = 1\) and \(t(x, 1) = x\). A variety \(V\) is **congruence distributive** if \(\text{Con}\mathfrak{A}\) is a distributive lattice for every \(\mathfrak{A} \in V\).

**Theorem 3.19.** The variety \(J\), \(M\) and \(L\) are weakly regular. Moreover, \(J\) and \(L\) are arithmetical at 1 and congruence distributive.

**Proof:** \(J\), \(M\) and \(L\) are weakly regular since the terms \(t_1(x, y) = x \rightarrow y\) and \(t_2(x, y) = y \leftarrow x\) satisfy (3.3) for \(n = 2\).

Let \(\mathcal{X}\) be a pseudo-BCH join-semilattice and \(t(x, y) = y \rightarrow x\). Clearly, \(t(x, x) = 1\) and \(t(x, 1) = x\). By Corollary 3.7, \(\mathcal{X}\) satisfies \((\text{pBCK})\), and hence \(t(1, x) = 1\). Then \(\mathcal{X}\) is arithmetical at 1, and consequently distributive at 1.

Let \(\theta, \phi, \psi \in \text{Con}\mathcal{X}\). By distributivity at 1, \(1/\theta \cap (\phi \lor \psi) = 1/(\theta \cap \phi) \lor (\theta \cap \psi)\). From weak regularity we obtain \(\theta \cap (\phi \lor \psi) = (\theta \cap \phi) \lor (\theta \cap \psi)\). Therefore \(\text{Con}\mathcal{X}\) is a distributive lattice.
Thus pseudo-BCH join-semilattices (and hence pseudo-BCH lattices) are arithmetical at 1 and congruence distributive.

References


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