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ON THESES WITHOUT ITERATED MODALITIES OF MODAL LOGICS BETWEEN C1 AND S5. PART 2

Abstract

This is the second, out of two papers, in which we identify all logics between **C1** and **S5** having the same theses without iterated modalities. All these logics can be divided into certain groups. Each such group depends only on which of the following formulas are theses of all logics from this group: $(\mathtt{N})$, $(\mathtt{T})$, $(\mathtt{D})$, $\ulcorner(\mathtt{T})\vee\Box\mathtt{q}\urcorner$, and for any $n > 0$ a formula $\ulcorner(\mathtt{T}) \vee (\mathtt{alt_n})\urcorner$, where $(\mathtt{T})$ has not the atom 'q', and $(\mathtt{T})$ and $(\mathtt{alt_n})$ have no common atom. We generalize Pollack's result from [1], where he proved that all modal logics between **S1** and **S5** have the same theses which does not involve iterated modalities (i.e., the same first-degree theses).

Keywords: first-degree theses of modal logics; theses without iterated modalities; Pollack's theory of *Basic Modal Logic*; basic theories for modal logics between C1 and S5.

5. Auxiliary facts

The facts given in this section provide a basis for proofs of main theorems of the paper, given in the next section.

FACT 5.1. *Let* $\boldsymbol{\Lambda}$ *be a modal logic such that* $\mathbf{C1} \subseteq \boldsymbol{\Lambda} \subseteq \mathbf{S5}$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{S0.5}^{\circ}[\mathtt{Talt_0}]$. *Then either* $(\mathtt{T}) \in \boldsymbol{\Lambda}$ *or* $(\mathtt{D}) \in \boldsymbol{\Lambda}$.

PROOF: Suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{S0.5}^{\circ}[\mathtt{Talt_0}]$ and $\boldsymbol{\Lambda} \subseteq \mathbf{S5}$. Then there is $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ such that $\varphi \notin \mathbf{S0.5}^{\circ}[\mathtt{Talt_0}]$. Hence, by Theorem 2.9, φ is false in some model from $\mathsf{M}^{\mathsf{sa}} \cup \mathsf{M}^{\varnothing}$, but φ is true in all models from M^{sa}, since $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S5} = {}^{\mathbf{1}}\mathbf{S0.5}$. Therefore φ is false in some t-normal model $\mathfrak{M}^{\varphi} = \langle w^{\varphi}, A^{\varphi}, V^{\varphi}\rangle$ with $A^{\varphi} = \varnothing$.

In **MCNF** (see p. 115 in Part 1) there is a formula $\varphi^{\mathrm{N}} := \ulcorner \bigwedge_{i=1}^{c} \kappa_i^{\varphi} \urcorner$ such that $\ulcorner \varphi^{\mathrm{N}} \equiv \varphi \urcorner \in \mathbf{C1}$ and every conjunct of φ^{N} belongs to ${}^{\mathbf{1}}\boldsymbol{\Lambda}$ and has one of the forms (a)–(d) given in Lemma 2.8. Since $\varphi^{\mathrm{N}} \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $\mathfrak{M}^{\varphi} \not\models \varphi^{\mathrm{N}}$, so there is $\kappa_* \in \{\kappa_1^{\varphi}, \ldots, \kappa_c^{\varphi}\}$ such that $\kappa_* \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $\mathfrak{M}^{\varphi} \not\models \kappa_*$. Now we show:

Claim. *The conjunct κ_* satisfies the following conditions*:

1. $\kappa_* \notin \mathbf{For_{cl}}$.
2. κ_* *has no disjunct of the form* $\ulcorner \Box\gamma \urcorner$.
3. κ_* *has one of the following forms*:
   (i) $\ulcorner \Diamond\beta \urcorner$, *where* $\beta \in \mathbf{Taut}$,
   (ii) $\ulcorner \alpha \vee \Diamond\beta \urcorner$, *where* $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$, *but* $\alpha \notin \mathbf{Taut}$.

Proof of Claim. Ad 1. Since $\mathfrak{M}^{\varphi} \not\models \kappa_*$, so $\kappa_* \notin \mathbf{Taut}$; but $\kappa_* \in \boldsymbol{\Lambda}$ and $\mathbf{For_{cl}} \cap \boldsymbol{\Lambda} = \mathbf{Taut}$, by Corollary 2.15.

Ad 2. All formulas of the form $\ulcorner \Box\gamma \urcorner$ are true in $\mathfrak{M}^{\varphi}$, but $\mathfrak{M}^{\varphi} \not\models \kappa_*$.

Ad 3: By items 1 and 2, and Lemma 2.8, κ_* has one of two forms (b) or (c) with $k = 0$ given in this lemma. So we use Lemma 2.2(1,3). Moreover, in the case 3 we have $\alpha \notin \mathbf{Taut}$, since $\kappa_* \notin \mathbf{Taut}$. $\lhd$

Thus, by Claim, there are only two alternative forms of κ_* described in item 3.

In case 3, $\kappa_* = \ulcorner \Diamond\beta \urcorner$, for some $\beta \in \mathbf{Taut}$. So $(\mathtt{D}) \in \boldsymbol{\Lambda}$, since $\ulcorner (\mathtt{D}) \equiv \Diamond\beta \urcorner \in \mathbf{C1}$.

In case 3 we have $\kappa_* = \ulcorner \alpha \vee \Diamond\beta \urcorner$, for some $\alpha, \beta \in \mathbf{For_{cl}}$ such that $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$ and $\alpha \notin \mathbf{Taut}$. We consider three subcases.

The first case, when $\ulcorner \neg\alpha \urcorner \in \mathbf{Taut}$. Then $\ulcorner \Diamond\beta \urcorner \in \boldsymbol{\Lambda}$, since $\ulcorner \neg\alpha \supset (\kappa_* \supset \Diamond\beta) \urcorner \in \mathbf{PL}$. Moreover, $\beta \in \mathbf{Taut}$, since $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$. So $(\mathtt{D}) \in \boldsymbol{\Lambda}$, since $\ulcorner (\mathtt{D}) \equiv \Diamond\beta \urcorner \in \mathbf{C1}$.

The second case, when $\ulcorner \neg\alpha \urcorner \notin \mathbf{Taut}$ and $\beta \in \mathbf{Taut}$. Then for some uniform substitution s both $\ulcorner s(\alpha) \equiv q \urcorner$ and $s(\beta)$ belong to **Taut**. Hence $\ulcorner s(\kappa_*) \equiv (q \vee \Diamond s(\beta)) \urcorner \in \mathbf{C1}$. So $\ulcorner q \vee \Diamond s(\beta) \urcorner \in \boldsymbol{\Lambda}$, since $s(\kappa_*) \in \boldsymbol{\Lambda}$. Hence both '$q \vee \Diamond\top$' and '$\neg q \vee \Diamond\top$' belong to $\boldsymbol{\Lambda}$. So also '$\Diamond\top$' and $(\mathtt{D})$ belong to $\boldsymbol{\Lambda}$.

The third case, when $\ulcorner \neg\alpha \urcorner \notin \mathbf{Taut}$ and $\beta \notin \mathbf{Taut}$. Then, by Lemma A.2 with $k = 0$,[1] there is a uniform substitution s such that both $\ulcorner s(\alpha) \equiv p \urcorner$ and $\ulcorner s(\beta) \equiv \neg p \urcorner$ belong to **Taut**. Hence $\ulcorner s(\kappa_*) \equiv (p \vee \Diamond\neg p) \urcorner \in \mathbf{C1}$, i.e., $\ulcorner s(\kappa_*) \equiv (p \vee \neg\Box\neg\neg p) \urcorner \in \mathbf{C1}$. So '$p \vee \neg\Box p$' belongs to $\boldsymbol{\Lambda}$, since '$\Box p \equiv \Box\neg\neg p$' belongs to **C1**. Therefore $(\mathtt{T}) \in \boldsymbol{\Lambda}$. $\Box$

[1] Lemma A.2 is proved in the Appendix on p. 215.

FACT 5.2. *Let $\boldsymbol{\Lambda}$ be a modal logic such that $\mathbf{C1} \subseteq \boldsymbol{\Lambda} \subseteq \mathbf{S5}$ and ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{S0.5}^{\circ}[\mathtt{D}]$. Then either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or for some $n \geqslant 0$ we have $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$.*

PROOF: Suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{S0.5}^{\circ}[\mathtt{D}]$ and $\boldsymbol{\Lambda} \subseteq \mathbf{S5}$. Then there is $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ such that $\varphi \notin \mathbf{S0.5}^{\circ}[\mathtt{D}]$. Hence, by Theorem 2.9, φ is false in some model from M^{+}, but φ is true in all models from M^{sa}, since $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S5} = {}^{\mathbf{1}}\mathbf{S0.5}$. Therefore φ is false in some t-normal model $\mathfrak{M}^{\varphi} = \langle w^{\varphi}, A^{\varphi}, V^{\varphi}\rangle$ with $w^{\varphi} \notin A^{\varphi} \neq \varnothing$.

In **MCNF** there is a formula $\varphi^{\mathrm{N}} := \ulcorner \bigwedge_{i=1}^{c} \kappa_i^{\varphi} \urcorner$ such that $\ulcorner \varphi^{\mathrm{N}} \equiv \varphi \urcorner \in \mathbf{C1}$ and every conjunct of φ^{N} belongs to ${}^{\mathbf{1}}\boldsymbol{\Lambda}$ and has one of the forms (a)–(d) given in Lemma 2.8. Since $\varphi^{\mathrm{N}} \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $\mathfrak{M}^{\varphi} \nvDash \varphi^{\mathrm{N}}$, so there is $\kappa_* \in \{\kappa_1^{\varphi}, \ldots, \kappa_c^{\varphi}\}$ such that $\kappa_* \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $\mathfrak{M}^{\varphi} \nvDash \kappa_*$. Now we show:

CLAIM. *The conjunct κ_* satisfies the following conditions*:

1. $\kappa_* \notin \mathbf{For_{cl}}$.
2. κ_* *has no disjunct of the form* $\ulcorner \Box\gamma \urcorner$ *with* $\gamma \in \mathbf{Taut}$.
3. κ_* *has no disjunct of the form* $\ulcorner \Diamond\beta \urcorner$ *with* $\beta \in \mathbf{Taut}$.
4. κ_* *has no disjunct of the form* $\ulcorner \Diamond\beta \vee \Box\gamma \urcorner$ *with* $\ulcorner \beta \vee \gamma \urcorner \in \mathbf{Taut}$.
5. κ_* *has one of the following forms*:
   (i) $\ulcorner \alpha \vee \Diamond\beta \urcorner$, *where* $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$, *but* $\alpha, \beta \notin \mathbf{Taut}$,
   (ii) $\ulcorner \alpha \vee \Diamond\beta \vee \bigvee_{i=1}^{k} \Box\gamma_i \urcorner$, *where* $k > 1$ *and* $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$, *but* $\alpha, \beta \notin \mathbf{Taut}$ *and* $\ulcorner \beta \vee \gamma_j \urcorner \notin \mathbf{Taut}$, *for any* $j \in \{1, \ldots, k\}$.

Proof of Claim. Ad 1. As in the proof of the case 1 of Claim of Fact 5.1.

Ad 2–4. If κ_* had a disjunct of the form $\ulcorner \Box\gamma \urcorner$ (resp. $\ulcorner \Diamond\beta \urcorner$, $\ulcorner \Diamond\beta \vee \Box\gamma \urcorner$) with $\gamma \in \mathbf{Taut}$ (resp. $\beta \in \mathbf{Taut}$, $\ulcorner \beta \vee \gamma \urcorner \in \mathbf{Taut}$), then κ_* would be true in $\mathfrak{M}^{\varphi}$, since $\ulcorner \Box\gamma \urcorner$ (resp. $\ulcorner \Diamond\beta \urcorner$, $\ulcorner \Diamond\beta \vee \Box\gamma \urcorner$) would be true in $\mathfrak{M}^{\varphi}$. A contradiction.

Ad 5. By Lemma 2.8, κ_* has one of the forms (a)–(d) given in this lemma. First, by Lemma 2.2(2), if κ_* had the form (a) then either $\alpha \in \mathbf{Taut}$ or κ_* would have some disjunct of the form $\ulcorner \Box\gamma_i \urcorner$ with $\gamma_i \in \mathbf{Taut}$. However, this is excluded due to items 1 and 2. Second, by Lemma 2.2(3), if κ_* had the form (b), then either $\beta \in \mathbf{Taut}$ or it would have some disjunct of the form $\ulcorner \Diamond\beta \vee \Box\gamma_i \urcorner$ with $\ulcorner \beta \vee \gamma_i \urcorner \in \mathbf{Taut}$; this is contrary to item 3 or 4. Third, by Lemma 2.2(4), if κ_* had the form (d) then κ_* would have some disjunct of the form $\ulcorner \Box\gamma_i \urcorner$ with $\gamma_i \in \mathbf{Taut}$; what is contrary to the item 2. Thus, κ_* has the form (c) with $k = 0$ or $k > 0$. By Lemma 2.2(1) and the item 4, we obtain $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$. Moreover, $\alpha, \beta \notin \mathbf{Taut}$, by items 1 and 3. Finally, in the case 5 we have $k > 1$, by the item 4. $\lhd$

Thus, by the claim, there are only two alternative forms of κ_* described in its item 5.

In case 5 we have $\kappa_* = \ulcorner\alpha \vee \lozenge\beta\urcorner$ and $\ulcorner\neg\alpha\urcorner \notin \mathbf{Taut}$. Therefore we can prove that $(\mathtt{T}) \in \boldsymbol{\Lambda}$, as in the proof of Fact 5.1, when we considered the third subcase of the case 3 of the form of κ_*.

In case 5, when $\ulcorner\alpha \vee \lozenge\beta \vee \bigvee_{i=1}^{k} \Box\gamma_i\urcorner$, where $k > 1$ and $\ulcorner\alpha \vee \beta\urcorner \in \mathbf{Taut}$, but $\alpha, \beta \notin \mathbf{Taut}$, we consider two subcases.

The first case 5(a), when $\ulcorner\beta \vee \bigvee_{i=1}^{k} \gamma_i\urcorner \notin \mathbf{Taut}$. Then, by Lemma A.2 for $k > 0$, there is a uniform substitution s such that both $\ulcorner s(\alpha) \equiv p\urcorner \in \mathbf{Taut}$, $\ulcorner s(\beta) \equiv \neg p\urcorner \in \mathbf{Taut}$, and for any $i \in \{1, \ldots, n+1\}$ either $\ulcorner s(\gamma_i) \equiv \neg p\urcorner \in \mathbf{Taut}$ or $\ulcorner\neg s(\gamma_i)\urcorner \in \mathbf{Taut}$. Hence either $\ulcorner s(\kappa_*) \equiv (p \vee \lozenge\neg p \vee \Box\neg p)\urcorner \in \mathbf{C1}$, or $\ulcorner s(\kappa_*) \equiv (p \vee \lozenge\neg p \vee \Box\neg p \vee \Box\bot\urcorner \in \mathbf{C1}$, or $\ulcorner s(\kappa_*) \equiv (p \vee \lozenge\neg p \vee \Box\bot)\urcorner \in \mathbf{C1}$. Thus, since $s(\kappa_*) \in \boldsymbol{\Lambda}$ and $\mathbf{C1} \subseteq \boldsymbol{\Lambda}$, either '$\Box p \supset (p \vee \Box\neg p)$', or '$\Box p \supset (p \vee \Box\neg p \vee \Box\bot)$', or '$\Box p \supset (p \vee \Box\bot)$' belongs to $\boldsymbol{\Lambda}$. Therefore $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$ (see Lemma 2.6).

The second case 5(b), when $\ulcorner\beta \vee \bigvee_{i=1}^{k} \gamma_i\urcorner \in \mathbf{Taut}$. For the application of Lemma A.3(1) notice that the following implications belong to $^{\mathbf{1}}\boldsymbol{\Lambda}$:[2]

$$\begin{aligned}(\neg\alpha \wedge \Box\neg\beta) &\supset \textstyle\bigvee_{i=1}^{k} \Box\gamma_i \\ &\supset \Box\neg\beta \wedge \textstyle\bigvee_{i=1}^{k} \Box\gamma_i \\ &\supset \textstyle\bigvee_{i=1}^{k} \Box(\neg\beta \wedge \gamma_i)\end{aligned}$$

Hence $\ulcorner\alpha \vee \lozenge\beta \vee \bigvee_{i=i}^{k} \Box(\neg\beta \wedge \gamma_i)\urcorner \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$. Thus, by Lemma A.3(1), there are $n \in \{1, \ldots k-1\}$ and non-empty different subsets $\Gamma_1, \ldots, \Gamma_{n+1}$ of Γ such that $\Gamma = \bigcup_{i=1}^{n+1} \Gamma_i$ and for some uniform substitution s we have:

- $\ulcorner s(\alpha) \equiv p\urcorner$ and $\ulcorner s(\beta) \equiv \neg p\urcorner$ belong to $\mathbf{Taut}$;
- for any $\gamma \in \mathit{\Gamma}_1$: $\ulcorner s(\neg\beta \wedge \gamma) \supset q_1\urcorner$ belongs to $\mathbf{Taut}$;
- for all $i \in \{1, \ldots, n\}$ and $\gamma \in \mathit{\Gamma}_{i+1}$: $\ulcorner s(\neg\beta \wedge \gamma) \supset (\bigwedge_{j=1}^{i} q_j \supset q_{i+1})\urcorner$ belongs to $\mathbf{Taut}$.

Therefore we also have:

- $\ulcorner\lozenge s(\beta) \equiv \lozenge\neg p\urcorner \in \mathbf{C1}$.
- For any $\gamma \in \mathit{\Gamma}_1$: $\ulcorner\Box s(\neg\beta \wedge \gamma) \supset \Box q_1\urcorner \in \mathbf{C1}$.
- For all $i \in \{1, \ldots, n\}$ and $\gamma \in \mathit{\Gamma}_{i+1}$: $\ulcorner\Box s(\neg\beta \wedge \gamma) \supset \Box(\bigwedge_{j=1}^{i} q_j \supset q_{i+1})\urcorner \in \mathbf{C1}$.

Thus, both $\ulcorner p \vee \neg\Box p \vee \Box q_1 \vee \bigvee_{i=1}^{n} \Box(\bigwedge_{j=1}^{i} q_j \supset q_{i+1})\urcorner$ and $(\mathtt{Talt_m}) \in \boldsymbol{\Lambda}$. $\Box$

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[2] Lemma A.3 is proved in the Appendix on p. 216.

FACT 5.3. *Let* $\boldsymbol{\Lambda}$ *be a modal logic such that* $\mathbf{C1} \subseteq \boldsymbol{\Lambda} \subseteq \mathbf{S5}$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_1}]$. *Then either* $(\mathtt{T}) \in \boldsymbol{\Lambda}$ *or* $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$.

PROOF: Suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_1}]$ and $\boldsymbol{\Lambda} \subseteq \mathbf{S5}$. Then there is $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ such that $\varphi \notin {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_1}]$. Hence, by Corollary 2.17, φ is false in some model from $\mathsf{M}^{\mathsf{sa}} \cup (\mathsf{M}^{\leqslant 1} \cap \mathsf{M}^{+})$. But, by Theorem 2.9, φ is true in all models from M^{sa}, since $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S5} = {}^{\mathbf{1}}\mathbf{S0.5}$. Therefore φ is false in some t-normal model $\mathfrak{M}^{\varphi} = \langle w^{\varphi}, A^{\varphi}, V^{\varphi}\rangle$ with $\mathrm{Card}A^{\varphi} = 1$. Thus, we can repeat the proof of Fact 5.2. Hence there are only two alternative forms of κ_* described in the item 5 of the claim in that proof.

Now we show that either $\kappa_* = \ulcorner\alpha \vee \Diamond\beta\urcorner$ or for some $k > 0$ we have $\kappa_* = \ulcorner\alpha \vee \Diamond\beta \vee \bigvee_{i=1}^{k} \Box\gamma_i\urcorner$ and $\ulcorner\beta \vee \bigvee_{i=1}^{k} \gamma_i\urcorner \notin \mathbf{Taut}$.

Indeed, if $k > 0$ and $\ulcorner\beta \vee \bigvee_{i=1}^{k} \gamma_i\urcorner \in \mathbf{Taut}$, then $\mathfrak{M}^{\varphi} \models \ulcorner\Diamond\beta \vee \bigvee_{i=1}^{k} \Box\gamma_i\urcorner$, since $\mathrm{Card}A^{\varphi} = 1$. Hence also $\mathfrak{M}^{\varphi} \models \kappa_*$. So we obtain a contradiction, because $\mathfrak{M}^{\varphi} \not\models \kappa_*$.

Thus, as in the proof of Fact 5.2, we obtain that either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$. □

FACT 5.4. *Let* $\boldsymbol{\Lambda}$ *be a modal logic between* $\mathbf{C1}$ *and* $\mathbf{S5}$. *Then for any* $n > 0$, *if* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_n}]$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$, *then* $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$.

PROOF: Let $n > 0$. Suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_n}]$, and $\boldsymbol{\Lambda} \subseteq \mathbf{S5}$. Then there is $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$ such that $\varphi \notin {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_1}]$. Hence, by Corollary 2.17, φ is false in some model from $\mathsf{M}^{\mathsf{sa}} \cup (\mathsf{M}^{\leqslant n+1} \cap \mathsf{M}^{+})$. But, by Theorem 2.9, φ is true in all models from $\mathsf{M}^{\mathsf{sa}} \cup (\mathsf{M}^{\leqslant n} \cap \mathsf{M}^{+})$, since $\varphi \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S5} = {}^{\mathbf{1}}\mathbf{S0.5}$. Therefore φ is false in some t-normal model $\mathfrak{M}^{\varphi} = \langle w^{\varphi}, A^{\varphi}, V^{\varphi}\rangle$ with $\mathrm{Card}A^{\varphi} = n + 1$. Thus, we can repeat the proof of Fact 5.2. Hence there are only two alternative forms of κ_* described in the item 5 of the claim in that proof.

However, since $(\mathtt{T}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{Talt_0}) \notin \boldsymbol{\Lambda}$, so cases 5 and 5(a) of Claim in the proof of Fact 5.2 will not occur. So we have only case 5(b).

Let $A^{\varphi} = \{a_1, \ldots, a_{n+1}\}$, where $a_i \neq a_j$, if $1 \leqslant i < j \leqslant n + 1$. Since $\mathfrak{M}^{\varphi} \not\models \kappa_*$, so we have $V^{\varphi}(w^{\varphi}, \kappa_*) = 0$. Therefore $V^{\varphi}(a_1, \beta) = \cdots = V^{\varphi}(a_{n+1}, \beta) = 0$ and for any $\gamma \in \Gamma := \{\gamma_1, \ldots, \gamma_k\}$ there is an $i \in \{1, \ldots, n + 1\}$ such that $V^{\varphi}(a_i, \gamma) = 0$. For any $i \in \{1, \ldots, n + 1\}$ we put $\Psi_i := \{\gamma \in \Gamma : V^{\varphi}(a_i, \gamma) = 0\}$. Of course, $\Gamma = \bigcup_{i=1}^{n+1} \Psi_i$. Since κ_* is true in all models from $\mathsf{M}^{\leqslant n} \cap \mathsf{M}^{+}$, so $\Psi_i \neq \varnothing$, for any $i \in \{1, \ldots, n + 1\}$. (Indeed, otherwise κ_* would be false in some n-element model.)

For any $i \in \{1, \dots, n+1\}$ we put $\psi_i := \bigvee \Psi_i$. Because $\ulcorner \beta \vee \bigvee \Gamma \urcorner \in \mathbf{Taut}$, so also $\ulcorner \beta \vee \bigvee_{i=1}^{n+1} \psi_i \urcorner \in \mathbf{Taut}$. Since $(\mathtt{M}) \in \mathbf{C1}$, so $\ulcorner \kappa_* \supset (\alpha \vee \Diamond\beta \vee \bigvee_{i=1}^{n+1} \Box\psi_i)\urcorner$ belongs to $\mathbf{C1}$. Hence $\ulcorner \alpha \vee \Diamond\beta \vee \bigvee_{i=1}^{n+1} \Box\psi_i \urcorner \in \boldsymbol{\Lambda}$. Thus, as in the second subcase of 5 in the proof of Fact 5.2, we can show that $\ulcorner \alpha \vee \Diamond\beta \vee \bigvee_{i=1}^{n+1} \Box(\neg\beta \wedge \psi_i)\urcorner \in {}^{\mathbf{1}}\boldsymbol{\Lambda}$. Thus, by Lemma A.3(1,2), as in Fact 5.2, we obtain that $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$. □

Fact 5.5. *Let* $\boldsymbol{\Lambda}$ *be a modal logic between* $\mathbf{C1}$ *and* $\mathbf{S5}$*. Then for any* $n > 0$*, if* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$ *then either* $(\mathtt{T}) \in \boldsymbol{\Lambda}$ *or* $(\mathtt{Talt_k}) \in \boldsymbol{\Lambda}$*, for some some* $k \in \{0, \dots, n-1\}$*.*

Proof: Let $n > 0$. Suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$ and $\boldsymbol{\Lambda} \subseteq \mathbf{S5}$. This proof is done by induction on n. By Fact 5.3 the given fact holds for $n = 1$.

Inductive step. We prove that for any $n > 1$: if the given fact holds for $n-1$, then it holds for n.

For $n > 0$ we suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$. We may also suppose that ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{TAlt_{n-1}}]$, since otherwise – by inductive hypothesis – either $(\mathtt{T}) \in \boldsymbol{\Lambda}$, or $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$, for some some $k \in \{1, \dots, n-2\}$ we have $(\mathtt{Talt_k}) \in \boldsymbol{\Lambda}$. However, in such case, we have $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, by Fact 5.4. □

Fact 5.6. *Let* $\boldsymbol{\Lambda}$ *be a modal logic between* $\mathbf{C1}$ *and* $\mathbf{S5}$*. Then for any* $n > 0$*, if* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_n}]$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$*, then* $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$*.*

Proof: By Fact 5.5, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_k}) \in \boldsymbol{\Lambda}$, for some $k \in \{1, \dots, n\}$. But $(\mathtt{T}) \notin \boldsymbol{\Lambda}$, $(\mathtt{Talt_0}) \notin \boldsymbol{\Lambda}$, and $(\mathtt{Talt_k}) \notin \boldsymbol{\Lambda}$, for any $k \in \{1, \dots n-1\}$. So $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$. □

6. Main theorems

In the light of lemmas from previous section we obtain the main results of this paper.

Theorem 6.1. *For any modal logic* $\boldsymbol{\Lambda}$ *between* $\mathbf{C1}$ *and* $\mathbf{S5}$:

1. ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{E1}$*,* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_0}]$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *iff* ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^1_{\mathtt{D}} = \mathsf{B}$.
   ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{E1}$*,* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_0}]$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *iff* ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^1_{\mathtt{D}}$.
2. ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{E1}$*,* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_0}]$ *and* ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *iff* ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^0$.

$\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^0$.

3. $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $(\exists n > 0)\ \mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n_{\mathtt{D}}$.
   $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $(\exists n > 0)\ \mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n_{\mathtt{D}}$.
4. $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$, *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $(\exists n > 0)\ \mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n$.
   $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $(\exists n > 0)\ \mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n$.
5. $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^\infty_{\mathtt{D}}$.
   $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^\infty_{\mathtt{D}}$.
6. $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^\infty$.
   $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^\infty$.

Thus, either $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^\infty$, *or* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^\infty$, *or* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^\infty_{\mathtt{D}}$, *or* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^\infty_{\mathtt{D}}$, *or for some* $n \geqslant 0$ *either* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n$, $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n$, *or* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n_{\mathtt{D}}$, *or* $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n_{\mathtt{D}}$.

For items 3 and 4, for any $n > 0$, *we have the following particular cases*:

7. $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n_{\mathtt{D}}$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$, *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$.
   $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n_{\mathtt{D}}$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$, $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$, *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$.
8. $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_n}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$ $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^n$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_n}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_{n+1}}]$.
   $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^n$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_n}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$ *iff* $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_n}]$ *and* $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_{n+1}}]$.

PROOF: The proofs of all "$\Leftarrow$"-parts of items 1–6 are obvious. We shall only go through the "$\Rightarrow$"-parts.

Ad 1. Suppose that (i) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ and (ii) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$. Then, by (i) and Fact 5.1, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{D}) \in \boldsymbol{\Lambda}$. Moreover, by (ii) and Fact 5.3, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$. So $(\mathtt{T}) \in \boldsymbol{\Lambda}$, because $\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_0}] = \mathbf{S0.5^\circ}[\mathtt{T}]$. Hence if $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ then $\mathbf{^1S0.5^\circ}[\mathtt{T}] = \mathbf{^1S0.5} \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S5} = \mathbf{^1S0.5} = \mathsf{B}$, by Fact 2.19 and Theorem 4.1 (or Theorem 3.4). Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$ then $\mathbf{^1E1} = \mathbf{^1C1}[\mathtt{T}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1C1}[\mathtt{D}, \mathtt{Talt_0}] = \mathbf{^1E1} = \mathsf{rB}^1_{\mathtt{D}}$, by Theorem 4.1.

Ad 2. Let (i) $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ and (ii) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$. Then, by (ii) and Fact 5.3, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$. But $(\mathtt{T}) \notin \boldsymbol{\Lambda}$, by (i). So $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$. Hence if $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ then $\mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] = \mathsf{nB}^0$, by Fact 2.19 and Theorem 4.1. Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$ then $\mathbf{^1C1}[\mathtt{Talt_0}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] = \mathbf{^1C1}[\mathtt{Talt_0}] = \mathsf{rB}^0$, by Theorem 4.1.

Ad 3. Let (i) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$, (ii) $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$, and (iii) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$. Then, by (i) and Fact 5.1, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{D}) \in \boldsymbol{\Lambda}$. Moreover, by (iii) and Fact 5.2, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, for some $n \geqslant 0$. But, by (ii), $(\mathtt{Talt_0}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{T}) \notin \boldsymbol{\Lambda}$. So $(\mathtt{D}) \in \boldsymbol{\Lambda}$ and $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, for some $n > 0$. We put $n_* := \min\{n > 0 : (\mathtt{Talt_n}) \in \boldsymbol{\Lambda}\}$. Note that $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n_*}}]$, since otherwise, by Fact 5.5, we obtain a contradiction: $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_k}) \in \boldsymbol{\Lambda}$, for some $k \in \{0, \dots, n_* - 1\}$. Hence, by Fact 2.19, if $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$, then $\mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n_*}}] \subseteq \mathbf{^1}\boldsymbol{\Lambda}$. Thus, $\mathbf{^1}\boldsymbol{\Lambda} = \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n_*}}] = \mathsf{nB}_{\mathtt{D}}^{\mathtt{n_*}}$. Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$ then $\mathbf{^1C1}[\mathtt{D}, \mathtt{Talt_{n_*}}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n_*}}] \cap \mathbf{^1E1} = \mathbf{^1C1}[\mathtt{D}, \mathtt{Talt_{n_*}}] = \mathsf{rB}_{\mathtt{D}}^{\mathtt{n_*}}$.

Ad 4. Let (i) $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$, (ii) $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}, \mathtt{Talt_1}]$, and (iii) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$. Then, by (iii) and Fact 5.2, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, for some $n \geqslant 0$. But $(\mathtt{T}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{Talt_0}) \notin \boldsymbol{\Lambda}$, by (i) and (ii), respectively. So $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, for some $n > 0$. We put $n_* := \min\{n > 0 : (\mathtt{Talt_n}) \in \boldsymbol{\Lambda}\}$. Note that $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_{n_*}}]$, since otherwise, by Fact 5.5, we obtain a contradiction: $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{Talt_k}) \in \boldsymbol{\Lambda}$, for some $k \in \{0, \dots, n_* - 1\}$. Hence if $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ then $\mathbf{^1S0.5^\circ}[\mathtt{Talt_{n_*}}] \subseteq \mathbf{^1}\boldsymbol{\Lambda}$. Thus, $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}^{n_*}$. Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$ then $\mathbf{^1C1}[\mathtt{Talt_{n_*}}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_{n_*}}] \cap \mathbf{^1E1} = \mathbf{^1C1}[\mathtt{Talt_{n_*}}]$. Thus, $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}^{n_*}$.

Ad 5. Let (i) $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}]$ and (ii) $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$. Then, by (i) and Fact 5.1, either $(\mathtt{T}) \in \boldsymbol{\Lambda}$ or $(\mathtt{D}) \in \boldsymbol{\Lambda}$. But $(\mathtt{T}) \notin \boldsymbol{\Lambda}$, by (ii). So $(\mathtt{D}) \in \boldsymbol{\Lambda}$. Hence if $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ then $\mathbf{^1S0.5^\circ}[\mathtt{D}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}]$. So $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{nB}_{\mathtt{D}}^{\infty}$. Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1}$ then $\mathbf{C1}[\mathtt{D}] \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{D}] \cap \mathbf{^1E1} = \mathbf{C1}[\mathtt{D}]$. So $\mathbf{^1}\boldsymbol{\Lambda} = \mathsf{rB}_{\mathtt{D}}^{\infty}$.

Ad 6. If $\mathbf{^1}\boldsymbol{\Lambda} \nsubseteq \mathbf{^1E1}$ and $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}]$, then $\mathbf{S0.5^\circ} \subseteq \boldsymbol{\Lambda}$ and $\mathsf{nB}^{\infty} = \mathbf{^1S0.5^\circ} \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}] = \mathsf{nB}^0 \cap \mathsf{nB}_{\mathtt{D}}^{\infty} = \mathsf{nB}^{\infty}$, by Fact 2.19 and theorems 4.1 and 4.2(5), respectively. Moreover, if $\mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1E1} \cap \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}]$ then, by theorems 4.1(2,4) and 4.2(5), $\mathsf{rB}^{\infty} = \mathbf{^1C1} \subseteq \mathbf{^1}\boldsymbol{\Lambda} \subseteq \mathbf{^1S0.5^\circ}[\mathtt{Talt_0}] \cap \mathbf{^1S0.5^\circ}[\mathtt{D}] \cap \mathbf{^1E1} = \mathbf{C1}[\mathtt{Talt_0}] \cap \mathbf{C1}[\mathtt{D}] = \mathsf{rB}^0 \cap \mathsf{rB}_{\mathtt{D}}^{\infty} = \mathsf{rB}^{\infty}$.

The proofs of "$\Rightarrow$"-parts of items 7 and 8 are obvious. For "$\Leftarrow$"-parts we have:

Ad 7. Let (i) ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}]$, (ii) ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$, and (iii) ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_0}]$. Then $(\mathtt{T}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, by (i), (ii), and Fact 5.4. Hence $(\mathtt{D}) \in \boldsymbol{\Lambda}$, by (iii) and Fact 5.1. So if ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{E1}$ then ${}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}] = {}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\mathsf{n}}_{\mathtt{D}}$. If ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{E1}$ then ${}^{\mathbf{1}}\mathbf{C1}[\mathtt{D}, \mathtt{Talt_n}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_n}] \cap {}^{\mathbf{1}}\mathbf{E1} = \mathbf{C1}[\mathtt{D}, \mathtt{Talt_n}] = \mathsf{rB}^{\mathsf{n}}_{\mathtt{D}}$.

Ad 8. Let ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_n}]$ and ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$. Then $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, by Fact 5.6. Hence if ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{E1}$ then ${}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_n}] = {}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\mathsf{n}}$. Moreover, $\mathbf{S0.5^\circ}[\mathtt{Talt_n}] \cap \mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}] = \mathbf{S0.5^\circ}[\mathtt{Talt_{n+1}}]$, by Corollary 2.13. Hence if ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_n}]$ and ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_{n+1}}]$, then ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{D}, \mathtt{Talt_{n+1}}]$.

If ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{E1}$ then ${}^{\mathbf{1}}\mathbf{C1}[\mathtt{Talt_n}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5^\circ}[\mathtt{Talt_n}] \cap {}^{\mathbf{1}}\mathbf{E1} = {}^{\mathbf{1}}\mathbf{C1}[\mathtt{Talt_n}] = \mathsf{rB}^{\mathsf{n}}$. Moreover, $\mathbf{C1}[\mathtt{Talt_n}] \cap \mathbf{C1}[\mathtt{D}, \mathtt{Talt_{n+1}}] = \mathbf{C1}[\mathtt{Talt_{n+1}}]$, by Corollary 2.18. Hence if ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq \mathbf{C1}[\mathtt{Talt_n}]$ and ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{C1}[\mathtt{Talt_{n+1}}]$, then ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq \mathbf{C1}[\mathtt{D}, \mathtt{Talt_{n+1}}]$. □

The following theorem shows that for any modal logic $\boldsymbol{\Lambda}$ between **C1** and **S5** we are able to indicate a basic theory which corresponds to $\boldsymbol{\Lambda}$ (see figures 1–3). The proof of this theorem we obtain by theorems 3.4, 4.1, 4.2, 6.1. and facts 2.19, 5.1–5.5.

THEOREM 6.2. *For any modal logic $\boldsymbol{\Lambda}$ such that $\mathbf{C1} \subseteq \boldsymbol{\Lambda} \subseteq \mathbf{S5}$:*

1. ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^1_{\mathtt{D}}$ *iff*
   $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$ *iff* $(\mathtt{N}) \in \boldsymbol{\Lambda}$ *and* $(\mathtt{T}) \in \boldsymbol{\Lambda}$.
   ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^1_{\mathtt{D}}$ *iff*
   $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$ *and* $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$ *iff* $(\mathtt{N}) \notin \boldsymbol{\Lambda}$ *and* $(\mathtt{T}) \in \boldsymbol{\Lambda}$.
2. ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^0$ *iff* $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$.
   ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^0$ *iff* $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$.
3. *For any $n > 0$:* ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^n_{\mathtt{D}}$ *iff*
   $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$, $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$.
   For any $n > 0$: ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^n_{\mathtt{D}}$ *iff*
   $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$, $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$.
4. *For any $n > 0$:* ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^n$ *iff*
   $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$.
   For any $n > 0$: ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^n$ *iff*
   $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, *and* $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$.
5. ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^\infty_{\mathtt{D}}$ *iff* $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$, *and* $(\forall n \geqslant 0)\ (\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$.
   ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^\infty_{\mathtt{D}}$ *iff* $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \in \boldsymbol{\Lambda}$, *and* $(\forall n \geqslant 0)\ (\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$.
6. ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^\infty$ *iff* $(\mathtt{N}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, *and* $(\forall n \geqslant 0)\ (\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$.
   ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^\infty$ *iff* $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, *and* $(\forall n \geqslant 0)\ (\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$.

PROOF: For all "⇒"-parts we use Theorem 4.1. For "⇐"-parts we have:[3]

Ad 1. If $(\mathtt{N}), (\mathtt{T}) \in \boldsymbol{\Lambda}$, then $\mathbf{S0.5} \subseteq \boldsymbol{\Lambda}$. So we use Theorem 3.4. Moreover, if $(\mathtt{N}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{T}) \in \boldsymbol{\Lambda}$, then ${}^{\mathbf{1}}\mathbf{E1} = {}^{\mathbf{1}}\mathbf{C1}[\mathtt{T}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5} \cap {}^{\mathbf{1}}\mathbf{E1} = {}^{\mathbf{1}}\mathbf{E1}$, by Fact 2.19. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^1_{\mathtt{D}}$, by Theorem 4.1.

Ad 2. Suppose that $(\mathtt{Talt_0}) \in \boldsymbol{\Lambda}$ and $(\mathtt{D}) \notin \boldsymbol{\Lambda}$. If $(\mathtt{N}) \in \boldsymbol{\Lambda}$ then ${}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_0}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $(\mathtt{T}) \notin \boldsymbol{\Lambda}$. So ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_0}]$, by Fact 5.1. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^0$, by Theorem 4.1. If $(\mathtt{N}) \notin {}^{\mathbf{1}}\boldsymbol{\Lambda}$ then ${}^{\mathbf{1}}\mathbf{C1}[\mathtt{Talt_0}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda}$ and $(\mathtt{T}) \notin \boldsymbol{\Lambda}$. So ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_0}] \cap {}^{\mathbf{1}}\mathbf{E1} = \mathbf{C1}[\mathtt{Talt_0}]$, by facts 2.19 and 5.1. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^0$, by Theorem 4.1.

Ad 3. Let $n > 0$. Suppose that $(\mathtt{D}) \in \boldsymbol{\Lambda}$, $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, and $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$. Then $(\mathtt{T}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{Talt_k}) \notin \boldsymbol{\Lambda}$, for any $k \in \{0, \ldots, n-1\}$. If $(\mathtt{N}) \in \boldsymbol{\Lambda}$ then ${}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_n}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_n}]$, by Fact 5.5. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^n_{\mathtt{D}}$, by Theorem 4.1. If $(\mathtt{N}) \notin \boldsymbol{\Lambda}$ then ${}^{\mathbf{1}}\mathbf{C1}[\mathtt{D}, \mathtt{Talt_n}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_n}] \cap {}^{\mathbf{1}}\mathbf{E1} = {}^{\mathbf{1}}\mathbf{C1}[\mathtt{D}, \mathtt{Talt_n}]$, by facts 2.19 and 5.5. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^n_{\mathtt{D}}$, by Theorem 4.1.

Ad 4. Let $n > 0$. Suppose that $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$, $(\mathtt{D}) \notin \boldsymbol{\Lambda}$, $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$. Then ${}^{\mathbf{1}}\boldsymbol{\Lambda} \nsubseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $(\mathtt{T}) \notin \boldsymbol{\Lambda}$, and $(\mathtt{Talt_0}) \notin \boldsymbol{\Lambda}$. So ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_0}] \cap {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{Talt_1}]$, by facts 5.1 and 5.4, respectively. Therefore, by Theorem 6.1(4), for some $n_0 > 0$ either ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{n_0}$ or ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^{n_0}$. If $(\mathtt{N}) \in \boldsymbol{\Lambda}$ then $\mathsf{nB}^n = {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_n}] \subseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\mathtt{n_0}}$, since $(\mathtt{Talt_n}) \in \boldsymbol{\Lambda}$. Moreover, $\mathsf{nB}^{n-1} = {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{Talt_{n-1}}] \nsubseteq {}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{n_0}$, since $(\mathtt{Talt_{n-1}}) \notin \boldsymbol{\Lambda}$. So, by Theorem 4.2, $\mathsf{nB}^{n_0} \subsetneq \mathsf{nB}^{n-1}$. Thus, $\mathsf{nB}^n \subseteq \mathsf{nB}^{n_0} \subsetneq \mathsf{nB}^{n-1}$; so $n = n_0$. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^n$, by Theorem 4.1. Similarly, if $(\mathtt{N}) \notin \boldsymbol{\Lambda}$, we obtain ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^n$.

Ad 5. Suppose that $(\mathtt{D}) \in \boldsymbol{\Lambda}$ and $(\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$, for any $n \geqslant 0$. Then also $(\mathtt{T}) \notin \boldsymbol{\Lambda}$. If $(\mathtt{N}) \in \boldsymbol{\Lambda}$ then $\mathbf{S0.5}^{\circ}[\mathtt{D}] \subseteq \boldsymbol{\Lambda}$. Moreover, ${}^{\mathbf{1}}\boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}]$, by Fact 5.2. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\infty}_{\mathtt{D}}$, by Theorem 4.1. If $(\mathtt{N}) \notin \boldsymbol{\Lambda}$ then $\mathbf{C1}[\mathtt{D}] \subseteq \boldsymbol{\Lambda} \subseteq {}^{\mathbf{1}}\mathbf{S0.5}^{\circ}[\mathtt{D}] \cap {}^{\mathbf{1}}\mathbf{E1} = \mathbf{C1}[\mathtt{D}]$, by Fact 2.19. Thus, ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^{\infty}_{\mathtt{D}}$, by Theorem 4.1.

Ad 6. Suppose that $(\mathtt{D}) \notin \boldsymbol{\Lambda}$ and $(\mathtt{Talt_n}) \notin \boldsymbol{\Lambda}$, for any $n \geqslant 0$. Then, also $(\mathtt{T}) \notin \boldsymbol{\Lambda}$. Hence, by theorems 4.1 and 6.1, either ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\infty}$ or ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^{\infty}$. Thus, if $(\mathtt{N}) \in \boldsymbol{\Lambda}$ (resp. $(\mathtt{N}) \notin \boldsymbol{\Lambda}$) then ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{nB}^{\infty}$ (resp. ${}^{\mathbf{1}}\boldsymbol{\Lambda} = \mathsf{rB}^{\infty}$), by Theorem 6.1 and Fact 2.19. □

In the light of theorems 4.1 and 6.2, there is a correspondence between all "normal basic theories" and well known normal logics included in **S5**. We present graphically this correlation in Figure 3, showing a comparison of very weak t-normal logic and normal logics. (Note that $\mathbf{KB4} = \mathbf{KB5} = \mathbf{K5} \oplus (\mathtt{Talt_0})$; see p. 120 in Part 1.)

[3] For the cases 1–3 we can provide other proofs using Theorem 6.1.

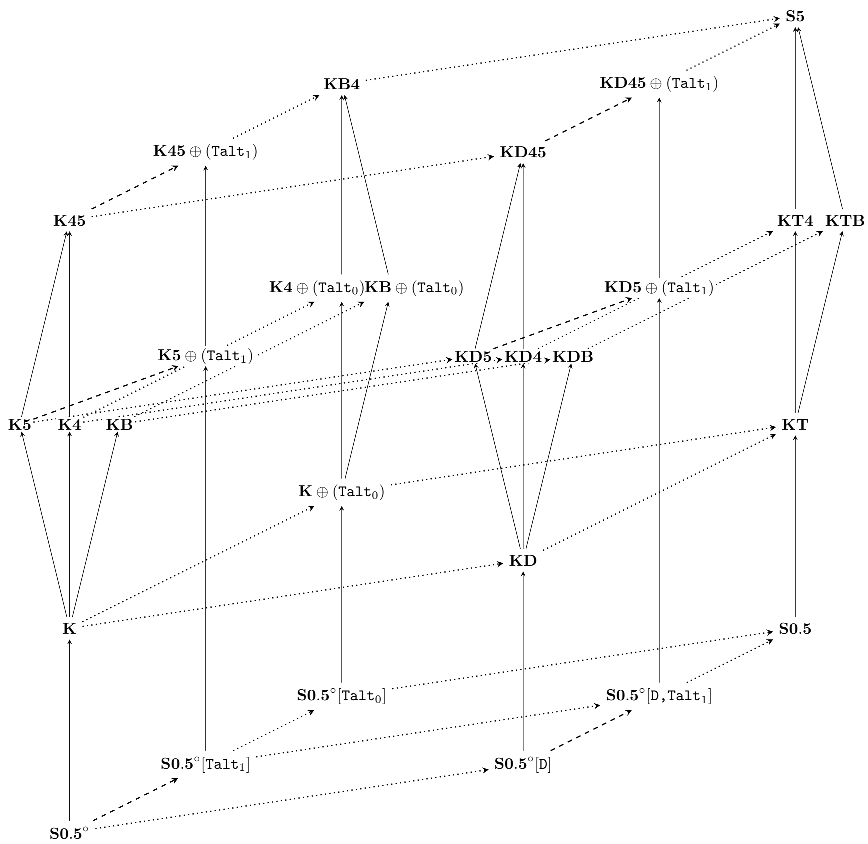

Fig. 3. Location of $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{Talt}_1]$, $\mathbf{S0.5}^\circ[\mathtt{Talt}_0]$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{D},\mathtt{Talt}_1]$, $\mathbf{S0.5}$ $(= \mathbf{S0.5}^\circ[\mathtt{D},\mathtt{Talt}_0] = \mathbf{S0.5}^\circ[\mathtt{T}])$ among some normal logics.

Similarly – in the light of theorems 4.1 and 6.2 – we can assign all "regular basic theories" to respective properly regular logics included in **S5**. We can make the following exchanges in Figure 3:

- each of the very weak t-normal logics is replaced corresponding to its t-regular logic,
- any normal logic $\boldsymbol{\Lambda}_{\rm n}$ is replaced by the properly regular logic $\mathbf{CF}\cap\boldsymbol{\Lambda}_{\rm n}$.

A. Some auxiliary facts from classical logic

In the proof of the auxiliary facts from Section 5 we have used the following lemmas A.2 and A.3, while in the proofs of these lemmas we will use Lemma A.1.

LEMMA A.1. *Let $n \geqslant 0$ and $V_0, \ldots, V_{n+1}$ be different valuations on $\mathbf{For_{cl}}$. Then there is a uniform substitution s such that for any $\theta \in \mathbf{For_{cl}}$ and any cl-valuation V on $\mathbf{For_{cl}}$ the following conditions* (C_0)–(C_{n+1}) *hold.*

(C_0) *If $V(p) = 0$ then $V(s(\theta)) = V_0(\theta)$.*

If $n = 0$ then:

(C_1) *If $V(p) = 1$ then $V(s(\theta)) = V_1(\theta)$.*

If $n > 0$ then:

(C_1) *If $V(p) = 1$ and $V(q_1) = 0$ then $V(s(\theta)) = V_1(\theta)$.*

If $n = 1$ then:

(C_2) *If $V(p) = 1 = V(q_1)$ then $V(s(\theta)) = V_2(\theta)$.*

If $n \geqslant 2$ then:

(C_i) *For any $i \in \{2, \ldots, n\}$: if $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\theta)) = V_i(\theta)$.*

(C_{n+1}) *If $V(p) = V(q_1) = \cdots = V(q_n) = 1$ then $V(s(\theta)) = V_{n+1}(\theta)$.*

PROOF: We make the following substitution s for atoms. For any $a \in \mathrm{At}$ the formula $s(a)$ will be dependent on the values $V_0(a), V_1(a), \ldots, V_{n+1}(a)$. We will consider six classes of valuations.

1. $V_0(a) = V_1(a) = \cdots = V_{n+1}(a) = 1$: Then we put $s_1(a) := p \vee \neg p$.
2. $V_0(a) = V_1(a) = \cdots = V_{n+1}(a) = 0$: Then we put $s_2(a) := p \wedge \neg p$.
3. $V_0(a) = 0$ and $V_{n+1}(a) = 1$: Then inductively we construct the following sequence $Q^3_1, \ldots, Q^3_n$ of formulas or «blanks» (further for the «blank formula» we use the symbol '$\emptyset$'). First we put:

$$Q^3_n := \begin{cases} q_n & \text{if } V_n(a) = 0 \\ \emptyset & \text{if } V_n(a) = 1 \end{cases}$$

Second, if $n > 1$ then for any $i = 1, \ldots, n-1$ we put inductively:

$$Q^3_i := \begin{cases} q_i \wedge Q^3_{i+1} & \text{if } V_i(a) = 0 \\ \neg q_i \vee Q^3_{i+1} & \text{if } V_i(a) = 1 \text{ and } Q^3_{i+1} \neq \emptyset \\ \emptyset & \text{if } V_i(a) = 1 \text{ and } Q^3_{i+1} = \emptyset \end{cases}$$

Finally, we put $s_3(a) := \ulcorner p \wedge Q^3_1 \urcorner$. So if $V_1(a) = \cdots = V_n(a) = 1$ then $s_3(a) :=$ 'p'.

4. $V_0(a) = 1$ and $V_{n+1}(a) = 0$: Then as $s_4(a)$ we will put $\ulcorner \neg s_3(a) \urcorner$ calculated for the values $V'_i(a) = 1 - V_i(a)$. Thus, inductively we construct the following sequence $Q^4_1, \ldots, Q^4_n$ of formulas or «blanks». First we put:

$$Q^4_n := \begin{cases} q_n & \text{if } V_n(a) = 1 \\ \emptyset & \text{if } V_n(a) = 0 \end{cases}$$

Second, if $n > 1$ then for any $i = 1, \ldots, n-1$ we put inductively:

$$Q^4_i := \begin{cases} q_i \wedge Q^4_{i+1} & \text{if } V_i(a) = 1 \\ \neg q_i \vee Q^4_{i+1} & \text{if } V_i(a) = 0 \text{ and } Q_{i+1} \neq \emptyset \\ \emptyset & \text{if } V_i(a) = 0 \text{ and } Q_{i+1} = \emptyset \end{cases}$$

Finally, we put $s_4(a) := \ulcorner \neg(p \wedge Q^4_1) \urcorner$. So if $V_1(a) = \cdots = V_n(a) = 0$ then $s_4(a) :=$ '$\neg p$'.[4]

5. $V_0(a) = 0 = V_{n+1}(a)$ and there is an $i \in \{1, \ldots, n\}$ such that $V_i(a) = 1$: If $n = 1$ then we put $s_1(a) := p \wedge \neg q_1$. If $n > 1$ then we construct inductively the following sequence $Q^5_1, \ldots, Q^5_n$ of formulas or «blanks». First we put:

$$Q^5_n := \begin{cases} \neg q_n & \text{if } V_n(a) = 1 \\ \emptyset & \text{if } V_n(a) = 0 \end{cases}$$

Second, if $n > 1$ then for any $i = 1, \ldots, n-1$ we put inductively:

$$Q^5_i := \begin{cases} q_i \wedge Q^5_{i+1} & \text{if } V_i(a) = 0 \\ \neg q_i \vee Q^5_{i+1} & \text{if } V_i(a) = 1 \end{cases}$$

Finally, we put $s_5(a) := \ulcorner p \wedge Q^5_1 \urcorner$.

[4] We see that for $n = 0$ we obtain the following uniform substitution s for any $a \in \mathrm{At}$:

$$s(a) := \begin{cases} p \vee \neg p & \text{if } V_0(a) = 1 = V_1(a) \\ p & \text{if } V_0(a) = 0 \text{ and } V_1(a) = 1 \\ \neg p & \text{if } V_0(a) = 1 \text{ and } V_1(a) = 0 \\ p \wedge \neg p & \text{if } V_0(a) = 0 = V_1(a) \end{cases}$$

So for $n = 0$ by induction on the complexity of formulas it is easy to show that (C_0) and (C_1) hold.

6. $V_0(a) = 1 = V_{n+1}(a)$ and there is an $i \in \{1, \dots, n\}$ such that $V_i(a) = 0$: Then as $s_6(a)$ we will put $\ulcorner \neg s_5(a) \urcorner$ calculated for the values $V_i'(a) = 1 - V_i(a)$. Thus, if $n = 1$ then we put $s_1(a) := \neg(p \wedge \neg q_1)$. If $n > 1$ then we construct inductively the following sequence $Q_1^6, \dots, Q_n^6$ of formulas or «blanks». First we put:

$$Q_n^6 := \begin{cases} \neg q_n & \text{if } V_n(a) = 0 \\ \emptyset & \text{if } V_n(a) = 1 \end{cases}$$

Second, if $n > 1$ then for any $i = 1, \dots, n-1$ we put inductively:

$$Q_i^6 := \begin{cases} q_i \wedge Q_{i+1}^6 & \text{if } V_i(a) = 1 \\ \neg q_i \vee Q_{i+1}^6 & \text{if } V_i(a) = 0 \end{cases}$$

Finally, we put $s_6(a) := \ulcorner \neg(p \wedge Q_1^6) \urcorner$.

Now as $s(a)$ we take respectively $s_1(a), \dots, s_6(a)$, depending on to which of the classes 1–6 the atom a belongs.

By induction on the complexity of formulas we can prove that for any $\theta \in \mathbf{For_{cl}}$ and any cl-valuation V the conditions (C_0)–(C_{n+1}) hold.

Now we show the inductive hypothesis for atoms. Let $a \in \mathrm{At}$. For classes 1 and 2 of valuations the conditions (C_0)–(C_{n+1}) are obviously met. Next, note that for some $k \in \{3, 4, 5, 6\}$ and $i \in \{1, \dots, n\}$, Q_i^k may be $\emptyset$, even if it is not explicitly determined.

For class 3, where $V_0(a) = 0$ and $V_{n+1}(a) = 1$, we have:

For (C_0): Suppose that $V(p) = 0$. Then $V(s_3(a)) = V(p \wedge Q_1^3) = 0$.

For (C_1): Suppose that $V(p) = 1$ and $V(q_1) = 0$. First, if $V_1(a) = 0$ then either $Q_1^3 =$ ‘q_1’ or $Q_1^3 = \ulcorner q_1 \wedge Q_2^3 \urcorner$, if $n > 1$. So either $V(s_3(a)) = V(p \wedge q_1) = 0$ or $V(s_3(a)) = V(p \wedge q_1 \wedge Q_2^3) = 0$. Second, if $V_1(a) = 1$ then either $Q_1^3 = \ulcorner q_1 \vee Q_2^3 \urcorner$ or $Q_1^3 = \emptyset$. So either $V(s_3(a)) = V(p \wedge (\neg q_1 \vee Q_2^3)) = 1$ or $V(s_3(a)) = V(p) = 1$.

For (C_{n+1}): Suppose that $V(p) = V(q_1) = \dots = V(q_n) = 1$. Note that $Q_n^3 = \ulcorner q_n \urcorner$ or $Q_n^3 = \emptyset$. So, in the first case, $V(Q_n^3) = 1$. Moreover, if $n = 1$ then either $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q_1^3) = 1$. If $n > 1$ then for $j = 1, \dots, n-1$ either $Q_j^3 = \emptyset$, or $Q_j^3 = \ulcorner q_j \urcorner$, or $Q_j^3 = \ulcorner q_j \wedge Q_{j+1}^3 \urcorner$, or $Q_j^3 = \ulcorner \neg q_j \vee Q_{j+1}^3 \urcorner$, where $Q_{j+1}^3 \neq \emptyset$. So, in the last two cases, we can show inductively that $V(Q_j^3) = 1$. Therefore either $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q_1^3) = 1$.

If $n > 1$ then we show inductively that (C_n) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1$ and $V(q_n) = 0$. First, if $V_n(a) = 0$ then $Q^3_n = \ulcorner q_n \urcorner$. Hence $Q^3_{n-1} = \ulcorner q_{n-1} \wedge q_n \urcorner$ or $Q^3_{n-1} = \ulcorner \neg q_{n-1} \vee q_n \urcorner$. So $V(Q^3_{n-1}) = 0$. Moreover, if $n = 2$ then $V(s_3(a)) = V(p \wedge Q^3_1) = 0$. If $n > 2$ then for $j = 1, \ldots, n-2$ we can show that $Q^3_{j+1} \neq \emptyset$, and either $Q^3_j = \ulcorner q_j \wedge Q^3_{j+1} \urcorner$ or $Q^3_j = \ulcorner \neg q_j \vee Q^3_{j+1} \urcorner$, and $V(Q^3_j) = 0$. So $V(s_3(a)) = V(p \wedge Q^3_1) = 0$. Second, if $V_n(a) = 1$ then $Q^3_n = \emptyset$. Hence $Q^3_{n-1} = \emptyset$ or $Q^3_{n-1} = \ulcorner q_{n-1} \urcorner$. So $Q^3_{n-1} = \emptyset$ or $V(Q^3_{n-1}) = 1$. Moreover, if $n = 2$ then either $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q^3_1) = 1$. If $n > 2$ then for $j = 1, \ldots, n-2$ we can show that either $Q^3_j = \emptyset$, or $Q^3_j = \ulcorner q_j \urcorner$, or $Q^3_j = \ulcorner q_j \wedge Q^3_{j+1} \urcorner$, or $Q^3_j = \ulcorner \neg q_j \vee Q^3_{j+1} \urcorner$, where $Q^3_{j+1} \neq \emptyset$; so, in the last three cases, $V(Q^3_j) = 1$. Thus, either $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q^3_1) = 1$.

If $n > 2$ then for $i = 2, \ldots, n-1$ we show inductively that (C_i) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$. First, if $V_i(a) = 0$ then either $Q^3_i = \ulcorner q_i \urcorner$ or $Q^3_i = \ulcorner q_i \wedge Q^3_{i+1} \urcorner$. So $V(Q^3_i) = 0$. Moreover, $Q^3_{i-1} = \ulcorner q_{i-1} \wedge Q^3_i \urcorner$ or $Q^3_{i-1} = \ulcorner \neg q_{i-1} \vee Q^3_i \urcorner$. So $V(Q^3_{i-1}) = 0$. If $i = 2$ then $V(s_3(a)) = V(p \wedge Q^3_1) = 0$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots i-2$ we can show that either $Q^3_j = \ulcorner q_j \wedge Q^3_{j+1} \urcorner$ or $Q^3_j = \ulcorner \neg q_j \vee Q^3_{j+1} \urcorner$; and $V(Q^3_j) = 0$. Therefore, $V(s_3(a)) = V(p \wedge Q^3_1) = 0$. Second, if $V_i(a) = 1$ then either $Q^3_i = \emptyset$ or $Q^3_i = \ulcorner \neg q_i \vee Q^3_{i+1} \urcorner$, where $Q^3_{i+1} \neq \emptyset$. In the last case we have $V(Q^3_i) = 1$. Moreover, either $Q^3_{i-1} = \emptyset$, or $Q^3_{i-1} = \ulcorner q_{i-1} \urcorner$, or $Q^3_{i-1} = \ulcorner q_{i-1} \wedge Q^3_i \urcorner$, or $Q^3_{i-1} = \ulcorner \neg q_{i-1} \vee Q^3_i \urcorner$, where $Q^3_i \neq \emptyset$. So, in the last three cases, $V(Q^3_{i-1}) = 1$. If $i = 2$ then $V(s_3(a)) = V(p \wedge Q^3_1) = 0$. If $i = 2$, then $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q^3_1) = 1$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q^3_j = \emptyset$, or $Q^3_j = \ulcorner q_j \urcorner$, or $Q^3_j = \ulcorner q_j \wedge Q^3_{j+1} \urcorner$, or $Q^3_j = \ulcorner \neg q_j \vee Q^3_{j+1} \urcorner$, where $Q^3_{j+1} \neq \emptyset$; so in the last three cases $V(Q^3_j) = 1$. Thus, $V(s_3(a)) = V(p) = 1$ or $V(s_3(a)) = V(p \wedge Q^3_1) = 1$.

For class 4, where $V_0(a) = 1$ and $V_{n+1}(a) = 0$, we have:

For (C_0): Suppose that $V(p) = 0$. Then $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 1$.

For (C_1): Suppose that $V(p) = 1$ and $V(q_1) = 0$. First, if $V_1(a) = 0$ then either $Q^4_1(a) = \emptyset$ or $Q^4_1 = \ulcorner \neg q_1 \vee Q^4_2 \urcorner$. So either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge (\neg q_1 \vee Q^4_2))) = 0$. Second, if $V_1(a) = 1$ then either $Q^4_1(a) =$ 'q_1' or $Q^4_1 = \ulcorner q_1 \wedge Q^4_2 \urcorner$. So either $V(s_4(a)) = V(\neg(p \wedge q_1)) = 1$ or $V(s_4(a)) = V(\neg(p \wedge q_1 \wedge Q^4_2)) = 1$.

For (C_{n+1}): Suppose that $V(p) = V(q_1) = \cdots = V(q_n) = 1$. Note that either $Q^4_n = \ulcorner q_n \urcorner$ or $Q^4_n = \emptyset$. So, in the first case, $V(Q^4_n) = 1$. Moreover, if $n = 1$ then either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$. If $n > 1$ then for $j = 1, \ldots, n-1$ either $Q^4_j = \emptyset$, or $Q^4_j = \ulcorner q_j \urcorner$, or $Q^4_j = \ulcorner q_j \wedge Q^4_{j+1} \urcorner$, or $Q^4_j = \ulcorner \neg q_j \vee Q^4_{j+1} \urcorner$, where $Q^4_{j+1} \neq \emptyset$. Therefore, in the last two cases, we can show inductively that $V(Q^4_j) = 1$. So either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$.

If $n > 1$ then we show inductively that (C_n) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1$ and $V(q_n) = 0$. First, if $V_n(a) = 0$ then $Q^4_n = \emptyset$. Hence $Q^4_{n-1} = \emptyset$ or $Q^4_{n-1} = \ulcorner q_{n-1} \urcorner$. So, in the last case, $V(Q^4_{n-1}) = 1$. Moreover, if $n = 2$ then either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$. If $n > 2$ then for $j = 1, \ldots, n-2$ we can show that either $Q^4_j = \emptyset$, or $Q^4_j = \ulcorner q_j \urcorner$, or $Q^4_j = \ulcorner q_j \wedge Q^4_{j+1} \urcorner$, or $Q^4_j = \ulcorner \neg q_j \vee Q^4_{j+1} \urcorner$, where $Q^4_{j+1} \neq \emptyset$; so, in the last three cases, $V(Q^4_j) = 1$. Thus, either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$. Second, if $V_n(a) = 1$ then $Q^4_i = \ulcorner q_n \urcorner$. Hence either $Q^4_{n-1} = \ulcorner q_{n-1} \wedge q_n \urcorner$ or $Q^4_{i-1} = \ulcorner \neg q_{i-1} \vee q_n \urcorner$. So $V(Q^4_{n-1}) = 0$. Moreover, if $n = 2$ then $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 1$. If $n > 2$ then for $j = 1, \ldots, n-2$ we can show that either $Q^4_j = \ulcorner q_j \wedge Q^4_{j+1} \urcorner$ or $Q^4_j = \ulcorner \neg q_j \vee Q^4_{j+1} \urcorner$; and $V(Q^4_j) = 0$. Thus, $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 1$.

If $n > 2$ then for $i = 2, \ldots, n-1$ we show inductively that (C_i) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$. First, if $V_i(a) = 0$ then either $Q^4_i = \emptyset$ or $Q^4_i = \ulcorner \neg q_i \vee Q^4_{i+1} \urcorner$, where $Q^4_{i+1} \neq \emptyset$. In the last case we have $V(Q^4_i) = 1$. Moreover, either $Q^4_{i-1} = \emptyset$, or $Q^4_{i-1} = \ulcorner q_{i-1} \urcorner$, or $Q^4_{i-1} = \ulcorner q_{i-1} \wedge Q^4_i \urcorner$, or $Q^4_{i-1} = \ulcorner \neg q_{i-1} \vee Q^4_i \urcorner$, where $Q^4_i \neq \emptyset$; so, in the last three cases, $V(Q^4_{i-1}) = 1$. If $i = 2$ then either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q^4_j = \emptyset$, or $Q^4_j = \ulcorner q_j \urcorner$, or $Q^4_j = \ulcorner q_j \wedge Q^4_{j+1} \urcorner$, or $Q^4_j = \ulcorner \neg q_j \vee Q^4_{j+1} \urcorner$, where $Q^4_{j+1} \neq \emptyset$; so, in the last three cases, $V(Q^4_j) = 1$. Thus, either $V(s_4(a)) = V(\neg p) = 0$ or $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 0$. Second, if $V_i(a) = 1$ then either $Q^4_i = \ulcorner q_i \urcorner$ or $Q^4_i = \ulcorner q_i \wedge Q^4_{i+1} \urcorner$. So $V(Q^4_i) = 0$. Moreover, either $Q^4_{i-1} = \ulcorner q_{i-1} \wedge Q^4_i \urcorner$ or $Q^4_{i-1} = \ulcorner \neg q_{i-1} \vee Q^4_i \urcorner$. So $V(Q^4_{i-1}) = 0$. If $i = 2$ then $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 1$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q^4_j = \ulcorner q_j \wedge Q^4_{j+1} \urcorner$ or $Q^4_j = \ulcorner \neg q_j \vee Q^4_{j+1} \urcorner$; and $V(Q^4_j) = 0$. Therefore $V(s_4(a)) = V(\neg(p \wedge Q^4_1)) = 1$.

For class 5, where $V_0(a) = 0 = V_{n+1}(a)$ and there is an $i \in \{1, \ldots n\}$ such that $V_i(a) = 1$, we have:

For (C_0): Suppose that $V(p) = 0$. Then $V(s_5(a)) = V(p \wedge Q^5_1) = 0$.

For (C_1): Suppose that $V(p) = 1$ and $V(q_1) = 0$. First, if $V_1(a) = 0$, then $n > 1$ and $Q^5_1 = \ulcorner q_1 \wedge Q^5_2 \urcorner$. So $V(Q^5_1) = 0$ and $V(s_5(a)) = V(p \wedge Q^5_1) = 0$. Second, if $V_1(a) = 1$ then either $Q^5_1 =$ '$\neg q_1$' or $Q^5_1 = \ulcorner \neg q_1 \vee Q^5_2 \urcorner$. So $V(s_5(a)) = V(p \wedge \neg q_1) = 1$ or $V(s_5(a)) = V(p \wedge (\neg q_1 \vee Q^5_2)) = 1$.

For (C_{n+1}): Let $V(p) = V(q_1) = \cdots = V(q_n) = 1$. First, suppose that $V_n(a) = 1$. Then $Q^5_n = \ulcorner \neg q_n \urcorner$ and $V(Q^5_n) = 0$. If $n = 1$ then $V(s_5(a)) = V(p \wedge \neg q_1) = 0$. Moreover, if $n > 1$ then for $j = 1, \ldots, n-1$ either $Q^5_j = \ulcorner \neg q_j \vee Q^5_{j+1} \urcorner$ or $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1} \urcorner$, where $Q^5_{j+1} \neq \emptyset$; and in the last two cases we can show inductively that $V(Q^5_j) = 0$. Therefore $V(s_5(a)) = V(p \wedge Q^5_1) = 0$. Second, suppose that $V_n(a) = 0$. Then $n > 1$ and $Q^5_n = \emptyset$. Let i_0 be the largest $i \in \{1, \ldots, n-1\}$ such that $V_i(a) = 1$. If $i_0 = n-1$, then $Q^5_{n-1} = \ulcorner \neg q_{n-1} \urcorner$ and $V(Q^5_{n-1}) = 0$. If $n = 2$ then $V(s_5(a)) = V(p \wedge \neg q_1) = 0$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ either $Q^5_j = \ulcorner \neg q_j \vee Q^5_{j+1} \urcorner$ or $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1} \urcorner$, where $Q^5_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^5_j) = 0$. Therefore $V(s_5(a)) = V(p \wedge Q^5_1) = 0$. If $i_0 < n-1$, then $n > 2$, $Q^5_{i_0} = \ulcorner \neg q_{i_0} \urcorner$, and $V(Q^5_{i_0}) = 0$. If $n = 3$, then $i_0 = 1$ and $V(s_5(a)) = V(p \wedge \neg q_1) = 0$. Moreover, if $n > 3$ then for $j = 1, \ldots, n-3$ either $Q^5_j = \ulcorner \neg q_j \vee Q^5_{j+1} \urcorner$ or $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1} \urcorner$, where $Q^5_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^5_j) = 0$. Therefore $V(s_5(a)) = V(p \wedge Q^5_1) = 0$.

If $n > 1$ then we show inductively that (C_n) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1$ and $V(q_n) = 0$. First, if $V_n(a) = 1$ then $Q^5_n = \ulcorner \neg q_n \urcorner$ and $V(Q^5_n) = 1$. Hence $Q^5_{n-1} = \ulcorner q_{n-1} \wedge \neg q_n \urcorner$ or $Q^5_{n-1} = \ulcorner \neg q_{n-1} \vee \neg q_n \urcorner$. So $V(Q^5_{n-1}) = 1$. So if $n = 2$ then $V(s_5(a)) = V(p \wedge Q^5_1) = 1$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ we can show that $Q^5_{j+1} \neq \emptyset$ and either $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1} \urcorner$ or $Q^5_j = \ulcorner \neg q_j \vee Q^5_{j+1} \urcorner$, and $V(Q^5_j) = 1$. So $V(s_5(a)) = V(p \wedge Q^5_1) = 1$. Second, if $V_n(a) = 0$, then $n > 1$ and $Q^5_n = \emptyset$. Let i_0 be the largest $i \in \{1, \ldots, n-1\}$ such that $V_i(a) = 1$. If $i_0 = n-1$, then $Q^5_{n-1} = \ulcorner \neg q_{n-1} \urcorner$ and $V(Q^5_{n-1}) = 0$. If $n = 2$ then $V(s_5(a)) = V(p \wedge \neg q_1) = 0$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ either $Q^5_j = \ulcorner \neg q_j \vee Q^5_{j+1} \urcorner$ or $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1} \urcorner$, where $Q^5_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^5_j) = 0$. Therefore $V(s_5(a)) = V(p \wedge Q^5_1) = 0$. If $i_0 < n-1$, then $n > 2$, $Q^5_{i_0} = \ulcorner \neg q_{i_0} \urcorner$, and $V(Q^5_{i_0}) = 0$. If $n = 3$, then $i_0 = 1$ and $V(s_5(a)) = V(p \wedge \neg q_1) = 0$. Moreover, if $n > 3$ then for $j = 1, \ldots, n-3$ either $Q^5_j = \ulcorner\neg q_j \vee Q^5_{j+1}\urcorner$ or $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1}\urcorner$, where $Q^5_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^5_j) = 0$. Therefore $V(s_5(a)) = V(p \wedge Q^5_1) = 0$.

If $n > 2$ then for $i = 2, \ldots, n-1$ we show inductively that (C_i) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$. First, if $V_i(a) = 1$ then $Q^5_i = \ulcorner\neg q_i \vee Q^5_{i+1}\urcorner$ and $V(Q^5_i) = 1$. Moreover, $Q^5_{i-1} = \ulcorner q_{i-1} \wedge Q^5_i\urcorner$ or $Q^5_{i-1} = \ulcorner\neg q_{i-1} \vee Q^5_i\urcorner$. So $V(Q^5_{i-1}) = 1$. If $i = 2$ then $V(s_5(a)) = V(p \wedge Q^5_1) = 1$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1}\urcorner$ or $Q^5_j = \ulcorner\neg q_j \vee Q^5_{j+1}\urcorner$; and so $V(Q^5_j) = 1$. Thus, $V(s_5(a)) = V(p \wedge Q^5_1) = 1$. Second, if $V_i(a) = 0$ then $Q^5_i = \ulcorner q_i \wedge Q^5_{i+1}\urcorner$ and $V(Q^5_i) = 0$. Moreover, $Q^5_{i-1} = \ulcorner q_{i-1} \wedge Q^5_i\urcorner$ or $Q^5_{i-1} = \ulcorner\neg q_{i-1} \vee Q^5_i\urcorner$. So $V(Q^5_{i-1}) = 0$. If $i = 2$ then $V(s_5(a)) = V(p \wedge Q^5_1) = 0$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q^5_j = \ulcorner q_j \wedge Q^5_{j+1}\urcorner$ or $Q^5_j = \ulcorner\neg q_j \vee Q^5_{j+1}\urcorner$; and so $V(Q^5_j) = 0$. Thus, $V(s_5(a)) = V(p \wedge Q^5_1) = 0$.

For class 6, where $V_0(a) = 1 = V_{n+1}(a)$ and there is an $i \in \{1, \ldots n\}$ such that $V_i(a) = 0$, we have:

For (C_0): Suppose that $V(p) = 0$. Then $V(s_6(a)) = V(\neg(p \wedge Q^6_1)) = 1$.

For (C_1): Suppose that $V(p) = 1$ and $V(q_1) = 0$. First, if $V_1(a) = 1$, then $n > 1$ and $Q^6_1 = \ulcorner q_1 \wedge Q^6_2\urcorner$. So $V(Q^6_1) = 0$ and $V(s_6(a)) = V(\neg(p \wedge Q^6_1)) = 1$. Second, if $V_1(a) = 0$ then either $Q^6_1 =$ '$\neg q_1$' or $Q^6_1 = \ulcorner\neg q_1 \vee Q^6_2\urcorner$. So $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 0$ or $V(s_6(a)) = V(\neg(p \wedge (\neg q_1 \vee Q^6_2))) = 0$.

For (C_{n+1}): Let $V(p) = V(q_1) = \cdots = V(q_n) = 1$. First, suppose that $V_n(a) = 0$. Then $Q^6_n = \ulcorner\neg q_n\urcorner$ and $V(Q^6_n) = 0$. If $n = 1$ then $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 1$. Moreover, if $n > 1$ then for $j = 1, \ldots, n-1$ either $Q^6_j = \ulcorner\neg q_j \vee Q^6_{j+1}\urcorner$ or $Q^6_j = \ulcorner q_j \wedge Q^6_{j+1}\urcorner$, where $Q^6_{j+1} \neq \emptyset$; and in the last two cases we can show inductively that $V(Q^6_j) = 0$. Therefore $V(s_6(a)) = V(\neg(p \wedge Q^6_1)) = 1$. Second, suppose that $V_n(a) = 1$. Then $n > 1$ and $Q^6_n = \emptyset$. Let i_0 be the largest $i \in \{1, \ldots, n-1\}$ such that $V_i(a) = 0$. If $i_0 = n-1$, then $Q^6_{n-1} = \ulcorner\neg q_{n-1}\urcorner$ and $V(Q^6_{n-1}) = 0$. If $n = 2$ then $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 1$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ either $Q^6_j = \ulcorner\neg q_j \vee Q^6_{j+1}\urcorner$ or $Q^6_j = \ulcorner q_j \wedge Q^6_{j+1}\urcorner$, where $Q^6_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^6_j) = 0$. Therefore $V(s_6(a)) = V(\neg(p \wedge Q^6_1)) = 1$. If $i_0 < n-1$, then $n > 2$, $Q^6_{i_0} = \ulcorner\neg q_{i_0}\urcorner$, and $V(Q^6_{i_0}) = 0$. If $n = 3$, then $i_0 = 1$ and $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 1$. Moreover, if $n > 3$ then for $j = 1, \ldots, n-3$ either $Q^6_j = \ulcorner\neg q_j \vee Q^6_{j+1}\urcorner$

or $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$, where $Q_{j+1}^6 \neq \emptyset$; and we can show inductively that $V(Q_j^6) = 0$. Therefore $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 1$.

If $n > 1$ then we show inductively that (C_n) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1$ and $V(q_n) = 0$. First, if $V_n(a) = 0$ then $Q_n^6 = \ulcorner \neg q_n \urcorner$ and $V(Q_n^6) = 1$. Hence $Q_{n-1}^6 = \ulcorner q_{n-1} \wedge \neg q_n \urcorner$ or $Q_{n-1}^6 = \ulcorner \neg q_{n-1} \vee \neg q_n \urcorner$. So $V(Q_{n-1}^6) = 1$. So if $n = 2$ then $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 0$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ we can show that $Q_{j+1}^6 \neq \emptyset$ and either $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$ or $Q_j^6 = \ulcorner \neg q_j \vee Q_{j+1}^6 \urcorner$, and $V(Q_j^6) = 1$. So $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 0$. Second, if $V_n(a) = 1$, then $n > 1$ and $Q_n^6 = \emptyset$. Let i_0 be the largest $i \in \{1, \ldots, n-1\}$ such that $V_i(a) = 1$. If $i_0 = n-1$, then $Q_{n-1}^6 = \ulcorner \neg q_{n-1} \urcorner$ and $V(Q_{n-1}^6) = 0$. If $n = 2$ then $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 1$. Moreover, if $n > 2$ then for $j = 1, \ldots, n-2$ either $Q_j^6 = \ulcorner \neg q_j \vee Q_{j+1}^6 \urcorner$ or $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$, where $Q_{j+1}^6 \neq \emptyset$; and we can show inductively that $V(Q_j^6) = 0$. Therefore $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 1$. If $i_0 < n-1$, then $n > 2$, $Q_{i_0}^6 = \ulcorner \neg q_{i_0} \urcorner$, and $V(Q_{i_0}^6) = 0$. If $n = 3$, then $i_0 = 1$ and $V(s_6(a)) = V(\neg(p \wedge \neg q_1)) = 1$. Moreover, if $n > 3$ then for $j = 1, \ldots, n-3$ either $Q_j^6 = \ulcorner \neg q_j \vee Q_{j+1}^6 \urcorner$ or $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$, where $Q_{j+1}^6 \neq \emptyset$; and we can show inductively that $V(Q_j^6) = 0$. Therefore $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 1$.

If $n > 2$ then for $i = 2, \ldots, n-1$ we show inductively that (C_i) holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$. First, if $V_i(a) = 0$ then $Q_i^6 = \ulcorner \neg q_i \vee Q_{i+1}^6 \urcorner$ and $V(Q_i^6) = 1$. Moreover, $Q_{i-1}^6 = \ulcorner q_{i-1} \wedge Q_i^6 \urcorner$ or $Q_{i-1}^6 = \ulcorner \neg q_{i-1} \vee Q_i^6 \urcorner$. So $V(Q_{i-1}^6) = 1$. If $i = 2$ then $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 0$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$ or $Q_j^6 = \ulcorner \neg q_j \vee Q_{j+1}^6 \urcorner$; and so $V(Q_j^6) = 1$. Thus, $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 0$. Second, if $V_i(a) = 1$ then $Q_i^6 = \ulcorner q_i \wedge Q_{i+1}^6 \urcorner$ and $V(Q_i^6) = 0$. Moreover, $Q_{i-1}^6 = \ulcorner q_{i-1} \wedge Q_i^6 \urcorner$ or $Q_{i-1}^6 = \ulcorner \neg q_{i-1} \vee Q_i^6 \urcorner$. So $V(Q_{i-1}^6) = 0$. If $i = 2$ then $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 1$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i-2$ we can show that either $Q_j^6 = \ulcorner q_j \wedge Q_{j+1}^6 \urcorner$ or $Q_j^6 = \ulcorner \neg q_j \vee Q_{j+1}^6 \urcorner$; and so $V(Q_j^6) = 0$. Thus, $V(s_6(a)) = V(\neg(p \wedge Q_1^6)) = 1$.

The inductive steps for complex formulas are obvious. □

LEMMA A.2. *Let* $k \geqslant 0$ *and* $\alpha, \beta, \gamma_1, \ldots, \gamma_k \in \mathbf{For_{cl}}$. *Suppose that*:

- $\ulcorner \alpha \vee \beta \urcorner \in \mathbf{Taut}$, *but* $\alpha \notin \mathbf{Taut}$ *and* $\ulcorner \beta \vee \bigvee_{j=1}^{k} \gamma_j \urcorner \notin \mathbf{Taut}$.

Then there is a uniform substitution s *such that* $\ulcorner s(\alpha) \equiv p \urcorner$ *and* $\ulcorner s(\beta) \equiv \neg p \urcorner$ *belong to* **Taut**, *and for any* $i \in \{1, \dots, k\}$, *either* $\ulcorner s(\gamma_i) \equiv \neg p \urcorner$ *or* $\ulcorner \neg s(\gamma_i) \urcorner$ *belongs to* **Taut**.

Proof: By both assumptions, there are two (different) cl-valuations V_0 and V_1 such that:

- $V_0(\alpha) = 0$ and $V_0(\beta) = 1$,
- $V_1(\beta) = V_1(\gamma_1) = \dots = V_1(\gamma_k) = 0$ and $V_1(\alpha) = 1$.

By Lemma A.1, with $n = 0$, for the valuations V_0 and V_1 we make some substitution s which for any $\theta \in \mathbf{For_{cl}}$ and any cl-valuation V satisfies the conditions (C_0) and (C_1) from this lemma. In the light of these conditions we obtain:

- $\ulcorner s(\alpha) \equiv p \urcorner \in$ **Taut**.

Indeed, for any cl-valuation V: if $V(p) = 1$ then $V(s(\alpha)) = V_1(\alpha) = 1$, by (C_1); if $V(p) = 0$ then $V(s(\alpha)) = V_0(\alpha) = 0$, by (C_0).

- $\ulcorner s(\beta) \equiv \neg p \urcorner \in$ **Taut**.

Indeed, for any cl-valuation V: if $V(p) = 1$ then $V(s(\beta)) = V_1(\beta) = 0$, by (C_1); if $V(p) = 0$ then $V(s(\beta)) = V_0(\beta) = 1$, by (C_0).

- For any $i \in \{1, \dots, k\}$ either $\ulcorner s(\gamma_i) \equiv \neg p \urcorner \in$ **Taut** or $\ulcorner \neg s(\gamma_i) \urcorner \in$ **Taut**.

Indeed, for any cl-valuation V: if $V(p) = 1$ then $V(s(\gamma_i)) = V_1(\gamma_i) = 0$, by (C_1). Hence $\ulcorner p \supset \neg s(\gamma_i) \urcorner \in$ **Taut**. Moreover, since $\mathrm{At}(s(\gamma_i)) = \{p\}$, so either $\ulcorner \neg s(\gamma_i) \urcorner \in$ **Taut** or $\ulcorner s(\gamma_i) \equiv \neg p \urcorner \in$ **Taut**. □

Lemma A.3. *Let* $k > 1$ *and* α, β, γ_1, ..., γ_k *belong to* $\mathbf{For_{cl}}$. *Suppose that*:

- $\ulcorner \alpha \vee \beta \urcorner \in$ **Taut**, *but* $\alpha \notin$ **Taut**,
- *for any* $\gamma \in \Gamma := \{\gamma_1, \dots, \gamma_k\}$ *we have* $\ulcorner \beta \vee \gamma \urcorner \notin$ **Taut**.

1. *Then for some* $n \in \{1, \dots, k-1\}$ *there are non-empty different subsets* Γ_1, ..., Γ_{n+1} *of the set* Γ *such that* $\Gamma = \bigcup_{i=1}^{n+1} \Gamma_i$ *and for some uniform substitution* s *we have*:
   - $\ulcorner s(\alpha) \equiv p \urcorner$ *and* $\ulcorner s(\beta) \equiv \neg p \urcorner$ *belong to* **Taut**;
   - *for any* $\gamma \in \Gamma_1$: $\ulcorner s(\neg\beta \wedge \gamma) \supset q_1 \urcorner$ *belongs to* **Taut**;
   - *for all* $i \in \{1, \dots, n\}$ *and* $\gamma \in \Gamma_{i+1}$: $\ulcorner s(\neg\beta \wedge \gamma) \supset (\bigwedge_{j=1}^{i} q_j \supset q_{i+1}) \urcorner$ *belongs to* **Taut**.

2. *Moreover, for any subset Ψ of Γ such that* $\ulcorner\beta \vee \bigvee\Psi\urcorner \in \mathbf{Taut}$ *we can take* $n = \mathrm{Card}\Psi - 1$.

PROOF: *Ad* 1. By assumptions, there are cl-valuations $A_0, \ldots, A_k$ such that:

- $A_0(\alpha) = 0$ and $A_0(\beta) = 1$,
- for any $i \in \{1, \ldots, k\}$: $A_i(\gamma_i) = 0 = A_i(\beta)$ and $A_i(\alpha) = 1$.

For any $i \in \{1, \ldots, k\}$ both $A_0(\beta) \neq A_i(\beta)$ and $A_0(\alpha) \neq A_i(\alpha)$, and there is a $j \in \{1, \ldots, k\}$ such that $A_i(\gamma_j) = 1$; so $A_i(\gamma_j) \neq A_j(\gamma_j)$. Hence among $A_1, \ldots, A_k$ we have at least two valuations which are different on the set Γ. Let m be the number of all such valuations. We put $n := m - 1$. Note that $m > 1$; so $n > 0$. We choose $n + 1$ such valuations $V_1, \ldots, V_{n+1}$ which are different on Γ.

Now for any $i \in \{1, \ldots, n + 1\}$ we put:

$$\Gamma_i := \{\gamma \in \Gamma : V_i(\gamma) = 0\}.$$

The sets $\Gamma_1, \ldots, \Gamma_{n+1}$ are non-empty and pairwise different and $\Gamma = \bigcup_{i=1}^{n+1} \Gamma_i$.

By Lemma A.1, with $n > 0$, for the valuations $V_0, \ldots, V_{n+1}$ we make some substitution s which for any $\theta \in \mathbf{For_{cl}}$ and any cl-valuation V satisfies the conditions (C_0)–(C_{n+1}) from the lemma. In the light of these conditions we obtain.

- $\ulcorner s(\alpha) \equiv p\urcorner \in \mathbf{Taut}$.

Let V be any cl-valuation. First, if $V(p) = 0$ then $V(s(\alpha)) = V_0(\alpha) = 0$, by ($\mathrm{C}_0$). Second, for any $i \in \{1, \ldots, n\}$: if $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\alpha)) = V_i(\alpha) = 1$, by ($\mathrm{C}_i$). Thirdly, if $V(p) = V(q_1) = \cdots = V(q_n) = 1$, then $V(s(\alpha)) = V_{n+1}(\alpha) = 1$, by ($\mathrm{C}_{n+1}$).

- $\ulcorner s(\beta) \equiv \neg p\urcorner \in \mathbf{Taut}$.

Let V be any cl-valuation. First, if $V(p) = 0$ then $V(s(\beta)) = V_0(\beta) = 1$, by ($\mathrm{C}_0$). Second, for any $i \in \{1, \ldots, n\}$: if $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\beta)) = V_i(\beta) = 0$, by ($\mathrm{C}_i$). Thirdly, if $V(p) = V(q_1) = \cdots = V(q_n) = 1$, then $V(s(\beta)) = V_{n+1}(\beta) = 0$, by ($\mathrm{C}_{n+1}$).

- For any $\gamma \in \Gamma_1$: $\ulcorner s(\neg\beta \wedge \gamma) \supset q_1\urcorner \in \mathbf{Taut}$.

Let V be any cl-valuation and $\gamma \in \Gamma_1$. If $V(s(\neg\beta)) = V(p) = 1$ and $V(q_1) = 0$, then $V(s(\gamma)) = V_1(\gamma) = 0$, by ($\mathrm{C}_1$).

- If $n > 1$ then for any $i \in \{2, \dots, n\}$: $\ulcorner s(\neg\beta \wedge \gamma_i) \supset (\bigwedge_{j=1}^{i-1} q_j \supset q_i)\urcorner \in$ **Taut**.

Let V be any cl-valuation, $n > 1$, $i \in \{2, \dots, n\}$, and $\gamma \in \Gamma_i$. If $V(s(\neg\beta)) = V(p) = V(q_1) = \dots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\gamma)) = V_i(\gamma) = 0$, by (C_i).

- For any $\gamma \in \Gamma_{n+1}$: $\ulcorner s(\neg\beta \wedge \gamma) \supset (\bigwedge_{j=1}^{n} q_j \supset q_{n+1})\urcorner \in$ **Taut**.

Let V be any cl-valuation and $\gamma \in \Gamma_{n+1}$. If $V(s(\neg\beta)) = V(p) = V(q_1) = \dots = V(q_n) = 1$, then $V(s(\gamma)) = V_{n+1}(\gamma) = 0$, by (C_{n+1}).

Ad 2. Let Ψ be any subset of Γ such that $\ulcorner \beta \vee \bigvee \Psi \urcorner \in$ **Taut**. We put $m := \mathrm{Card}\Psi$, $m > 1$. Suppose that and $\Psi = \{\psi_1, \dots, \psi_m\}$. By assumption there are different cl-valuations $V_0, \dots, V_m$ such that:

- $V_0(\alpha) = 0$ and $V_0(\beta) = 1$,
- for any $i \in \{1, \dots, m\}$: $V_i(\beta) = V_i(\psi_1) = \dots = V_i(\psi_{i-1}) = V_i(\psi_{i+1}) = \dots = V_i(\psi_m) = 0$ and $V_i(\alpha) = 1 = V_i(\gamma_i)$.

Of course all valuations $V_1, \dots, V_m$ are pairwise different on the set Γ. We put $n := m - 1$. So for the valuations $V_0, V_1, \dots, V_{n+1}$ we can repeat the proof of the item 1. $\square$

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