Abstract

This is the second, out of two papers, in which we identify all logics between \( \textbf{C1} \) and \( \textbf{S5} \) having the same theses without iterated modalities. All these logics can be divided into certain groups. Each such group depends only on which of the following formulas are theses of all logics from this group: \( (\text{n}) \), \( (\text{T}) \), \( (\text{D}) \), \( \lceil (\text{T}) \lor \Box q \rceil \), and for any \( n > 0 \) a formula \( \lceil (\text{T}) \lor (\text{alt}_n) \rceil \), where \( (\text{T}) \) has not the atom ‘q’, and \( (\text{T}) \) and \( (\text{alt}_n) \) have no common atom. We generalize Pollack’s result from \[1\], where he proved that all modal logics between \( \textbf{S1} \) and \( \textbf{S5} \) have the same theses which does not involve iterated modalities (i.e., the same first-degree theses).

Keywords: first-degree theses of modal logics; theses without iterated modalities; Pollack’s theory of Basic Modal Logic; basic theories for modal logics between \( \textbf{C1} \) and \( \textbf{S5} \).

5. Auxiliary facts

The facts given in this section provide a basis for proofs of main theorems of the paper, given in the next section.

\textbf{Fact 5.1.} Let \( \Lambda \) be a modal logic such that \( \textbf{C1} \subseteq \Lambda \subseteq \textbf{S5} \) and \( ^*\Lambda \nsubseteq \textbf{S0.5}^{\text{alt}_0} \). Then either \( (\text{T}) \in \Lambda \) or \( (\text{D}) \in \Lambda \).

\textbf{Proof:} Suppose that \( ^*\Lambda \nsubseteq \textbf{S0.5}^{\text{alt}_0} \) and \( \Lambda \subseteq \textbf{S5} \). Then there is \( \varphi \in ^*\Lambda \) such that \( \varphi \nsubseteq \textbf{S0.5}^{\text{alt}_0} \). Hence, by Theorem 2.9, \( \varphi \) is false in some model from \( \textbf{M}^{\text{alt}} \cup \textbf{M}^{\text{NS}} \), but \( \varphi \) is true in all models from \( \textbf{M}^{\text{alt}} \), since \( \varphi \in ^*\Lambda, ^*\Lambda \subseteq ^*\textbf{S5} = ^*\textbf{S0.5} \). Therefore \( \varphi \) is false in some t-normal model \( \mathcal{M}^\varphi = \langle w^\varphi, A^\varphi, V^\varphi \rangle \) with \( A^\varphi = \emptyset \).
In MCNF (see p. 115 in Part 1) there is a formula \( \varphi^N := \bigwedge_{i=1}^c \kappa_i^\varphi \) such that \( \varphi^N \equiv \varphi \in C1 \) and every conjunct of \( \varphi^N \) belongs to \( 1A \) and has one of the forms (a)–(d) given in Lemma 2.8. Since \( \varphi^N \in 1A \) and \( M^p \not\models \varphi^N \), so there is \( \kappa_* \in \{\kappa_1^\varphi, \ldots, \kappa_c^\varphi\} \) such that \( \kappa_* \in 1A \) and \( M^p \not\models \kappa_* \). Now we show:

**Claim.** The conjunct \( \kappa_* \) satisfies the following conditions:

1. \( \kappa_* \not\in ForC1 \).
2. \( \kappa_* \) has no disjunct of the form \( \Box \gamma \).
3. \( \kappa_* \) has one of the following forms:
   - (i) \( \Diamond \beta \), where \( \beta \in \text{Taut} \).
   - (ii) \( \Diamond \alpha \lor \Diamond \beta \), where \( \Diamond \alpha \lor \Diamond \beta \in \text{Taut} \), but \( \alpha \not\in \text{Taut} \).

**Proof of Claim.** Ad 1. Since \( M^p \not\models \kappa_* \), so \( \kappa_* \not\in ForC1 \); but \( \kappa_* \in A \) and \( ForC1 \cap A = \text{Taut} \), by Corollary 2.15.

Ad 2. All formulas of the form \( \Box \gamma \) are true in \( M^p \), but \( M^p \not\models \kappa_* \).

Ad 3. By items 1 and 2 and Lemma 2.8, \( \kappa_* \) has one of two forms (b) or (c) with \( k = 0 \) given in this lemma. So we use Lemma 2.2(1,3). Moreover, in the case 3 we have \( \alpha \not\in \text{Taut} \), since \( \kappa_* \not\in \text{Taut} \).

Thus, by Claim, there are only two alternative forms of \( \kappa_* \) described in item 3.

In case 3, \( \kappa_* = \Diamond \beta \), for some \( \beta \in \text{Taut} \). So \( \langle D \rangle \in A \), since \( \langle D \rangle \equiv \Diamond \beta \in C1 \).

In case 3 we have \( \kappa_* = \Diamond \alpha \lor \Diamond \beta \), for some \( \alpha, \beta \in ForC1 \) such that \( \Diamond \alpha \lor \Diamond \beta \in \text{Taut} \) and \( \alpha \not\in \text{Taut} \). We consider three subcases.

The first case, when \( \not\exists \alpha \notin \text{Taut} \) and \( \beta \in \text{Taut} \). Then \( \Diamond \alpha \lor \Diamond \beta \in A \), since \( \not\exists \alpha \lor \Diamond \beta \in \text{PL} \). Moreover, \( \beta \in \text{Taut} \), since \( \Diamond \alpha \lor \Diamond \beta \in \text{Taut} \). So \( \langle D \rangle \in A \), since \( \langle D \rangle \equiv \Diamond \beta \in C1 \).

The second case, when \( \not\exists \alpha \notin \text{Taut} \) and \( \beta \notin \text{Taut} \). Then for some uniform substitution \( s \) both \( s(\alpha) \equiv q \) and \( s(\beta) \) belong to \( \text{Taut} \). Hence \( s(\kappa_*) \equiv (q \lor s(\beta)) \in C1 \). So \( q \lor s(\beta) \in A \), since \( s(\kappa_*) \in A \). Hence both \( q \lor \Diamond \top \) and \( \Diamond \top \lor q \lor \Diamond \top \) belong to \( A \). So also \( \Diamond \top \lor \Diamond \top \) and \( \langle D \rangle \in \text{Taut} \).

The third case, when \( \not\exists \alpha \notin \text{Taut} \) and \( \beta \notin \text{Taut} \). Then, by Lemma A.2, with \( k = 0 \) there is a uniform substitution \( s \) such that both \( s(\alpha) \equiv p \) and \( s(\beta) \equiv \neg p \) belong to \( \text{Taut} \). Hence \( s(\kappa_*) \equiv (p \lor \neg p) \in C1 \), i.e., \( s(\kappa_*) \equiv (p \lor \neg \Box \neg p) \in C1 \). So \( p \lor \neg \Box \neg p \) belongs to \( A \), since \( \Box \neg p \equiv \Box \neg p \) belongs to \( C1 \). Therefore \( \langle T \rangle \in A \).

1Lemma A.2 is proved in the Appendix on p. 215
FACT 5.2. Let $A$ be a modal logic such that $C1 \subseteq A \subseteq S5$ and $^1A \not\subseteq S0.5^5[D]$. Then either $(\ell) \in A$ or for some $n \geq 0$ we have $(\text{Taut}_n) \in A$.

PROOF: Suppose that $^1A \not\subseteq S0.5^5[D]$ and $A \subseteq S5$. Then there is $\varphi \in ^1A$ such that $\varphi \not\in S0.5^5$. Hence, by Theorem 2.9, $\varphi$ is false in some model from $M^*$, but $\varphi$ is true in all models from $M^{sa}$, since $\varphi \in ^1A$, $^1A \subseteq S5 = ^1S0.5$. Therefore $\varphi$ is false in some t-normal model $M^\varphi = \langle w^\varphi, A^\varphi, V^\varphi \rangle$ with $w^\varphi \not\in A^\varphi \neq \emptyset$.

In MCNF there is a formula $\varphi^N := \bigwedge_{i=1}^n \kappa_i^\varphi$ such that $\varphi^N \equiv \varphi^N \in C1$ and every conjunct of $\varphi^N$ belongs to $^1A$ and has one of the forms (a)–(d) given in Lemma 2.8. Since $\varphi^N \in ^1A$ and $M^\varphi \not\models \varphi^N$, so there is $\kappa_* \in \{\kappa_1^\varphi, \ldots, \kappa_n^\varphi\}$ such that $\kappa_* \in ^1A$ and $M^\varphi \not\models \kappa_*$. Now we show:

CLAIM. The conjunct $\kappa_*$ satisfies the following conditions:

1. $\kappa_* \notin \text{For}_{C1}$.
2. $\kappa_*$ has no disjunct of the form $\Box \gamma$ with $\gamma \in \text{Taut}$.
3. $\kappa_*$ has no disjunct of the form $\Diamond \beta$ with $\beta \in \text{Taut}$.
4. $\kappa_*$ has no disjunct of the form $\Diamond \beta \lor \Box \gamma$ with $\beta \lor \gamma \in \text{Taut}$.
5. $\kappa_*$ has one of the following forms:
   (i) $\alpha \lor \Box \beta$, where $\alpha \lor \beta \in \text{Taut}$, but $\alpha, \beta \notin \text{Taut}$,
   (ii) $\alpha \lor \Diamond \beta \lor \bigvee_{i=1}^k \Box \gamma_i$, where $k > 1$ and $\alpha \lor \beta \in \text{Taut}$, but $\alpha, \beta \notin \text{Taut}$ and $\beta \lor \gamma_j \notin \text{Taut}$, for any $j \in \{1, \ldots, k\}$.


Ad [2]. If $\kappa_*$ had a disjunct of the form $\Box \gamma$ (resp. $\Diamond \beta$, $\Diamond \beta \lor \Box \gamma$) with $\gamma \in \text{Taut}$ (resp. $\beta \in \text{Taut}$, $\beta \lor \gamma \in \text{Taut}$), then $\kappa_*$ would be true in $M^\varphi$, since $\Box \gamma$ (resp. $\Diamond \beta$, $\Diamond \beta \lor \Box \gamma$) would be true in $M^\varphi$. A contradiction.

Ad [3]. By Lemma 2.8, $\kappa_*$ has one of the forms (a)–(d) given in this lemma. First, by Lemma 2.2(2), if $\kappa_*$ had the form (a) then either $\alpha \in \text{Taut}$ or $\kappa_*$ would have some disjunct of the form $\Box \gamma_i$ with $\gamma_i \in \text{Taut}$. However, this is excluded due to items [1] and [2]. Second, by Lemma 2.2(3), if $\kappa_*$ had the form (b), then either $\beta \in \text{Taut}$ or $\kappa_*$ would have some disjunct of the form $\Box \beta \lor \Box \gamma_i$ with $\beta \lor \gamma_i \in \text{Taut}$; this is contrary to item [3] or [4]. Third, by Lemma 2.2(4), if $\kappa_*$ had the form (d) then $\kappa_*$ would have some disjunct of the form $\Box \gamma_i$ with $\gamma_i \in \text{Taut}$; what is contrary to the item [2]. Thus, $\kappa_*$ has the form (c) with $k = 0$ or $k > 0$. By Lemma 2.2(1) and the item [4] we obtain $\Diamond \alpha \lor \beta \in \text{Taut}$. Moreover, $\alpha, \beta \notin \text{Taut}$, by items [1] and [3]. Finally, in the case [5] we have $k > 1$, by the item [4].
Thus, by the claim, there are only two alternative forms of \( \kappa_s \) described in its item \(^{[5]}\).

In case \(^{[5]}\) we have \( \kappa_s = \gamma \alpha \lor \beta \gamma \) and \( \neg \alpha \gamma \not\in \text{Taut} \). Therefore we can prove that \((T) \in A\), as in the proof of Fact \(^{5.1}\) when we considered the third subcase of the case \(^{[3]}\) of the form of \( \kappa_s \).

In case \(^{[5]}\) when \( \gamma \alpha \lor \beta \lor \square \gamma \) where \( k > 1 \) and \( \gamma \alpha \lor \beta \gamma \in \text{Taut} \), but \( \alpha, \beta \not\in \text{Taut} \), we consider two subcases.

The first case \(^{[5](a)}\), when \( \gamma \beta \lor \gamma \beta \lor \gamma \gamma \in \text{Taut} \). Then, by Lemma \(^{A.2}\) for \( k > 0 \), there is a uniform substitution \( s \) such that both \( \gamma s(\alpha) \equiv p \) and \( \gamma s(\beta) \equiv \neg p \) \( \in \text{Taut} \), and for any \( i \in \{1, \ldots, n+1\} \) either \( \gamma s(\gamma_i) \equiv \neg \gamma \gamma \) \( \not\in \text{Taut} \) or \( \gamma s(\gamma_i) \gamma \equiv \text{Taut} \). Hence either \( \gamma s(\kappa_s) \equiv (p \lor \neg p \lor p) \gamma \not\in C1 \), or \( \gamma s(\kappa_s) \equiv (p \lor \neg p \lor p \lor p) \gamma \not\in C1 \), or \( \gamma s(\kappa_s) \equiv (p \lor \neg p \lor p) \gamma \not\in C1 \). Thus, since \( s(\kappa_s) \in A \) and \( C1 \subseteq A \), \( \square p \equiv (p \lor \square p) \gamma \not\in A \). Therefore \( (\text{Taut}_0) \in A \) (see Lemma \(^{2.6}\)).

The second case \(^{[5](b)}\), when \( \gamma \beta \lor \gamma \beta \lor \gamma \gamma \in \text{Taut} \). For the application of Lemma \(^{A.3}(1)\) notice that the following implications belong to \( ^{1A}\)

\[
\neg \alpha \lor \square \neg \beta \lor \square \gamma_i \\
\lor \square \gamma \land \lor \gamma_i \\
\lor \lor \land \\
\lor \lor \land \\
\lor \lor \land \\
\lor \lor \land \\
\lor \lor \land \\
\lor \lor \land 
\]

Hence \( \gamma \alpha \lor \beta \lor \gamma \lor (\square \gamma \land \gamma) \in 1A \). Thus, by Lemma \(^{A.3}(1)\), there are \( n \in \{1, \ldots, k\} \) and non-empty different subsets \( \Gamma_1, \ldots, \Gamma_{n+1} \) of \( \Gamma \) such that \( \Gamma = \bigcup_{i=1}^{n+1} \Gamma_i \) and for some uniform substitution \( s \) we have:

- \( \gamma s(\alpha) \equiv p \) and \( \gamma s(\beta) \equiv \neg p \) belong to \( \text{Taut} \);
- for any \( \gamma \in \Gamma_1 \): \( \gamma s(\neg \beta \land \gamma) \lor q \gamma \) belongs to \( \text{Taut} \);
- for all \( i \in \{1, \ldots, n\} \) and \( \gamma \in \Gamma_{i+1} \): \( \gamma s(\neg \beta \land \gamma) \lor (\bigwedge_{j=1}^{i} q_j \lor q_{i+1}) \gamma \) belongs to \( \text{Taut} \).

Therefore we also have:

- \( \gamma s(\beta) \equiv \neg p \) \( \in C1 \).
- For any \( \gamma \in \Gamma_1 \): \( \neg \gamma s(\neg \beta \land \gamma) \lor \square q \gamma \) \( \in C1 \).
- For all \( i \in \{1, \ldots, n\} \) and \( \gamma \in \Gamma_{i+1} \): \( \square \gamma s(\neg \beta \land \gamma) \lor (\bigwedge_{j=1}^{i} q_j \lor q_{i+1}) \gamma \) \( \in C1 \).

Thus, both \( \gamma p \lor \square p \lor q \lor q \lor \square \) \( \in A \). □

\(^{2}\)Lemma \(^{A.3}\) is proved in the Appendix on p. 216.
FACT 5.3. Let \( A \) be a modal logic such that \( C_1 \subseteq A \subseteq S_5 \) and \( ^1A \not\subseteq 1S_0.5^0[D,Talt_1] \). Then either \( (T) \in A \) or \( (Talt_0) \in A \).

PROOF: Suppose that \( ^1A \not\subseteq 1S_0.5^0[D,Talt_1] \) and \( A \subseteq S_5 \). Then there is \( \varphi \in ^1A \) such that \( \varphi \not\in 1S_0.5^0[D,Talt_1] \). Hence, by Corollary 2.17, \( \varphi \) is false in some model from \( M^a \cup (M^{S_5 \cap M^+}) \). But, by Theorem 2.9, \( \varphi \) is true in all models from \( M^a \), since \( \varphi \in ^1A \). \( ^1A \subseteq 1S_5 \). Therefore \( \varphi \) is false in some t-normal model \( M^\varphi = (w^\varphi, A^\varphi, V^\varphi) \) with \( \text{Card}A^\varphi = 1 \). Thus, we can repeat the proof of Fact 5.2. Hence there are only two alternative forms of \( \kappa_* \) described in the item (b) of the claim in that proof.

Now we show that either \( \kappa_* = \gamma \alpha \lor \beta \gamma \) or for some \( k > 0 \) we have \( \kappa_* = \gamma \alpha \lor \bigvee_{i=1}^{k} \beta \gamma_i \) and \( \gamma \beta \lor \bigvee_{i=1}^{k} \beta \gamma_i \not\in \text{Taut} \).

Indeed, if \( k > 0 \) and \( \gamma \beta \lor \bigvee_{i=1}^{k} \beta \gamma_i \in \text{Taut} \), then \( M^\varphi \models \gamma \alpha \lor \bigvee_{i=1}^{k} \beta \gamma_i \), since \( \text{Card}A^\varphi = 1 \). Hence also \( M^\varphi \not\models \kappa_* \). So we obtain a contradiction, because \( M^\varphi \not\models \kappa_* \).

Thus, as in the proof of Fact 5.2 we obtain that either \( (T) \in A \) or \( (Talt_0) \in A \).

FACT 5.4. Let \( A \) be a modal logic between \( C_1 \) and \( S_5 \). Then for any \( n > 0 \), if \( ^1A \subseteq 1S_0.5^0[D,Talt_n] \) and \( ^1A \not\subseteq 1S_0.5^0[D,Talt_{n+1}] \), then \( (Talt_n) \in A \).

PROOF: Let \( n > 0 \). Suppose that \( ^1A \not\subseteq 1S_0.5^0[D,Talt_{n+1}] \), \( ^1A \subseteq 1S_0.5^0[D,Talt_n] \), and \( A \subseteq S_5 \). Then there is \( \varphi \in ^1A \) such that \( \varphi \not\in 1S_0.5^0[D,Talt_1] \). Hence, by Corollary 2.17, \( \varphi \) is false in some model from \( M^a \cup (M^{S_5 \cap M^+}) \). But, by Theorem 2.9, \( \varphi \) is true in all models from \( M^a \cup (M^{S_5 \cap M^+}) \), since \( \varphi \in ^1A \). \( ^1A \subseteq 1S_5 \). Therefore \( \varphi \) is false in some t-normal model \( M^\varphi = (w^\varphi, A^\varphi, V^\varphi) \) with \( \text{Card}A^\varphi = n + 1 \). Thus, we can repeat the proof of Fact 5.2. Hence there are only two alternative forms of \( \kappa_* \) described in the item (b) of the claim in that proof.

However, since \( (T) \not\in A \) and \( (Talt_0) \not\in A \), so cases 3 and 4 of Claim in the proof of Fact 5.2 will not occur. So we have only case 5(b).

Let \( A^\varphi = \{a_1,\ldots,a_{n+1}\} \), where \( a_i \neq a_j \), if \( 1 \leq i < j \leq n + 1 \). Since \( M^\varphi \not\models \kappa_* \), so we have \( V^\varphi(a_i,\kappa_*) = 0 \). Therefore \( V^\varphi(a_1,\beta) = \cdots = V^\varphi(a_{n+1},\beta) = 0 \) and for any \( \gamma \in \Gamma := \{\gamma_1,\ldots,\gamma_k\} \) there is an \( i \in \{1,\ldots,n+1\} \) such that \( V^\varphi(a_i,\gamma) = 0 \). For any \( i \in \{1,\ldots,n+1\} \) we put \( \psi_i := \{\gamma \in \Gamma : V^\varphi(a_i,\gamma) = 0\} \). Of course, \( \Gamma = \bigcup_{i=1}^{n+1} \psi_i \). Since \( \kappa_* \) is true in all models from \( M^{\leq n} \cap M^+ \), so \( \psi_i \neq \emptyset \), for any \( i \in \{1,\ldots,n+1\} \). (Indeed, otherwise \( \kappa_* \) would be false in some \( n \)-element model.)
Theorem 6.1. For any modal logic $A$ between $C1$ and $S5$:

1. $A \not\preceq E1$, $A \not\preceq S0.5[Saltt_0]$ and $A \not\preceq S0.5[D, Saltt_1]$ if and only if $A = nB_1 = B$.

2. $A \not\preceq E1$, $A \preceq S0.5[Saltt_0]$ and $A \not\preceq S0.5[D, Saltt_1]$ if and only if $A = rB_1$.

6. Main theorems

In the light of lemmas from previous section we obtain the main results of this paper.

Fact 5.5. Let $A$ be a modal logic between $C1$ and $S5$. Then for any $n > 0$, if $A \not\preceq S0.5[D, Saltt_n]$ then either $(T) \in A$ or $(Saltt_k) \in A$, for some $k \in \{0, \ldots, n-1\}$.

Proof: Let $n > 0$. Suppose that $A \not\preceq S0.5[D, Saltt_n]$ and $A \subseteq S5$. This proof is done by induction on $n$. By Fact 5.3 the given fact holds for $n = 1$.

Inductive step. We prove that for any $n > 1$: if the given fact holds for $n - 1$, then it holds for $n$.

For $n > 0$ we suppose that $A \not\preceq S0.5[D, Saltt_n]$. We may also suppose that $A \subseteq S0.5[D, Saltt_{n-1}]$, since otherwise – by inductive hypothesis – either $(T) \in A$, or $(Saltt_k) \in A$, for some $k \in \{1, \ldots, n-2\}$ we have $(Saltt_k) \in A$. However, in such case, we have $(Saltt_n) \in A$, by Fact 5.4.

Fact 5.6. Let $A$ be a modal logic between $C1$ and $S5$. Then for any $n > 0$, if $A \subseteq S0.5[D, Saltt_n]$ and $A \not\preceq S0.5[D, Saltt_{n+1}]$, then $(Saltt_n) \in A$.

Proof: By Fact 5.5 either $(T) \in A$ or $(Saltt_k) \in A$, for some $k \in \{1, \ldots, n\}$. But $(T) \notin A$, $(Saltt_0) \notin A$, and $(Saltt_k) \notin A$, for any $k \in \{1, \ldots, n-1\}$. So $(Saltt_n) \in A$. \hfill $\square$
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$$1^A \subseteq 1^E_1, \ 1^A \subseteq 1^{S0.5^p}[Talt_0] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D, Talt_1] \iff 1^A = rB^0.$$ 

3. $$1^A \not\subseteq 1^E_1, \ 1^A \not\subseteq 1^{S0.5^p}[Talt_0], \ 1^A \subseteq 1^{S0.5^p}[D, Talt_1] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D] \iff (\exists n > 0) \ 1^A = nB^0.$$ 

$$1^A \subseteq 1^E_1, \ 1^A \not\subseteq 1^{S0.5^p}[Talt_0], \ 1^A \subseteq 1^{S0.5^p}[D, Talt_1] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D] \iff (\exists n > 0) \ 1^A = rB^0.$$ 

4. $$1^A \not\subseteq 1^E_1, \ 1^A \subseteq 1^{S0.5^p}[Talt_0] \cap 1^{S0.5^p}[D, Talt_1] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D] \iff (\exists n > 0) \ 1^A = nB^n.$$ 

$$1^A \subseteq 1^E_1 \cap 1^{S0.5^p}[Talt_0] \cap 1^{S0.5^p}[D, Talt_1] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D] \iff (\exists n > 0) \ 1^A = rB^n.$$ 

5. $$1^A \not\subseteq 1^E_1, \ 1^A \not\subseteq 1^{S0.5^p}[Talt_0] \text{ and } 1^A \subseteq 1^{S0.5^p}[D] \iff 1^A = nB^\infty.$$ 

$$1^A \subseteq 1^E_1, \ 1^A \not\subseteq 1^{S0.5^p}[Talt_0] \text{ and } 1^A \subseteq 1^{S0.5^p}[D] \iff 1^A = rB^\infty.$$ 

6. $$1^A \not\subseteq 1^E_1 \text{ and } 1^A \subseteq 1^{S0.5^p}[Talt_0] \cap 1^{S0.5^p}[D] \iff 1^A = nB^\infty.$$ 

$$1^A \subseteq 1^E_1 \cap 1^{S0.5^p}[Talt_0] \cap 1^{S0.5^p}[D] \iff 1^A = rB^\infty.$$ 

Thus, either $$1^A = nB^\infty,$$ or $$1^A = rB^\infty,$$ or $$1^A = nB^\infty,$$ or $$1^A = rB^\infty,$$ or for some $$n \geq 0$$ either $$1^A = nB^n,$$ $$1^A = rB^n,$$ or $$1^A = nB^\infty,$$ or $$1^A = rB^\infty.$$ 

For items 5 and 6 for any $$n > 0,$$ we have the following particular cases:

7. $$1^A = nB^n \iff 1^A \not\subseteq 1^E_1, \ 1^A \subseteq 1^{S0.5^p}[D, Talt_n], \ 1^A \not\subseteq 1^{S0.5^p}[D, Talt_{n+1}], \text{ and } 1^A \not\subseteq 1^{S0.5^p}[Talt_0].$$ 

$$1^A = rB^n \iff 1^A \subseteq 1^E_1 \cap 1^{S0.5^p}[D, Talt_n], \ 1^A \not\subseteq 1^{S0.5^p}[D, Talt_{n+1}], \text{ and } 1^A \not\subseteq 1^{S0.5^p}[Talt_0].$$ 

8. $$1^A = nB^n \iff 1^A \not\subseteq 1^E_1, \ 1^A \subseteq 1^{S0.5^p}[Talt_n] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D, Talt_{n+1}], \ 1^A \not\subseteq 1^{S0.5^p}[Talt_0].$$ 

$$1^A = rB^n \iff 1^A \subseteq 1^E_1 \cap 1^{S0.5^p}[Talt_n] \text{ and } 1^A \not\subseteq 1^{S0.5^p}[D, Talt_{n+1}], \text{ and } 1^A \not\subseteq 1^{S0.5^p}[Talt_0].$$ 

**Proof:** The proofs of all “$$\infty$$”-parts of items 5 and 6 are obvious. We shall only go through the “$$\Rightarrow$$”-parts.

Ad 1. Suppose that (i) $$1^A \not\subseteq 1^{S0.5^p}[Talt_0]$$ and (ii) $$1^A \not\subseteq 1^{S0.5^p}[D, Talt_1].$$ Then, by (i) and Fact 5.1 either (T) $$\in A$$ or (D) $$\in A.$$ Moreover, by (ii) and Fact 5.3, either (T) $$\in A$$ or (Talt_0) $$\in A.$$ So (T) $$\in A,$$ because $$S0.5^p[D, Talt_0] = S0.5^p[T].$$ Hence if $$1^A \not\subseteq 1^E_1$$ then $$1^{S0.5^p}[T] = 1^{S0.5} \subseteq 1^A \subseteq 1^{S5} = 1^{S0.5} = 1^E,$$ by Fact 2.19 and Theorem 4.1 (or Theorem 3.4). Moreover, if $$1^A \subseteq 1^E_1$$ then $$1^E_1 = 1^C_1[T] \subseteq 1^A \subseteq 1^C_1[D, Talt_0] = 1^E_1 = rB^0,$$ by Theorem 4.1.
Ad \(2\) Let (i) \(\mathcal{A} \subseteq S.0.5^0[Tal\tau_0]\) and (ii) \(\mathcal{A} \not\subseteq S.0.5^0[D,Tal\tau_1]\). Then, by (ii) and Fact 5.3 either \((T) \in \mathcal{A}\) or \((Tal\tau_0) \in \mathcal{A}\). But \((T) \notin \mathcal{A}\), by (i). So \((Tal\tau_0) \in \mathcal{A}\). Hence if \(\mathcal{A} \not\subseteq E1\) then \(S.0.5^0[Tal\tau_0] \subseteq \mathcal{A} \subseteq S.0.5^0[Tal\tau_0] = nB^0\), by Fact 2.19 and Theorem 4.1. Moreover, if \(\mathcal{A} \subseteq E1\) then \(C1[Tal\tau_0] \subseteq \mathcal{A} \subseteq E1 \cap S.0.5^0[Tal\tau_0] = C1[Tal\tau_0] = rB^0\), by Theorem 4.1.

Ad \(3\) Let (i) \(\mathcal{A} \not\subseteq S.0.5^0[Tal\tau_0]\), (ii) \(\mathcal{A} \subseteq S.0.5^0[D,Tal\tau_1]\), and (iii) \(\mathcal{A} \not\subseteq S.0.5^0[D]\). Then, by (i) and Fact 5.3 either \((T) \in \mathcal{A}\) or \((D) \in \mathcal{A}\). Moreover, by (iii) and Fact 5.2, either \((T) \in \mathcal{A}\) or \((Tal\tau_0) \in \mathcal{A}\), for some \(n \geq 0\). But, by (ii), \((Tal\tau_0) \not\in \mathcal{A}\) and \((T) \notin \mathcal{A}\). So \((D) \in \mathcal{A}\) and \((Tal\tau_0) \in \mathcal{A}\), for some \(n > 0\). We put \(n_* := \min\{n > 0: (Tal\tau_0) \in \mathcal{A}\}\). Note that \(\mathcal{A} \subseteq S.0.5^0[D,Tal\tau_n]\), since otherwise, by Fact 5.5 we obtain a contradiction: \((T) \in \mathcal{A}\) or \((Tal\tau_n) \in \mathcal{A}\), for some \(k \in \{0, \ldots, n_* - 1\}\). Hence, by Fact 2.19, if \(\mathcal{A} \not\subseteq E1\), then \(S.0.5^0[D,Tal\tau_n] \subseteq \mathcal{A}\). Thus, \(\mathcal{A} = S.0.5^0[D,Tal\tau_n] = nB^0\). Moreover, if \(\mathcal{A} \subseteq E1\) then \(C1[D,Tal\tau_n] \subseteq \mathcal{A} \subseteq S.0.5^0[D,Tal\tau_n] \cap E1 = C1[D,Tal\tau_n] = rB^0\).

Ad \(4\) Let (i) \(\mathcal{A} \subseteq S.0.5^0[Tal\tau_0]\), (ii) \(\mathcal{A} \subseteq S.0.5^0[D,Tal\tau_1]\), and (iii) \(\mathcal{A} \not\subseteq S.0.5^0[D]\). Then, by (iii) and Fact 5.2 either \((T) \in \mathcal{A}\) or \((Tal\tau_0) \in \mathcal{A}\), for some \(n \geq 0\). But \((T) \notin \mathcal{A}\) and \((Tal\tau_0) \notin \mathcal{A}\), by (i) and (ii), respectively. So \((Tal\tau_0) \in \mathcal{A}\), for some \(n > 0\). We put \(n_* := \min\{n > 0: (Tal\tau_0) \in \mathcal{A}\}\). Note that \(\mathcal{A} \subseteq S.0.5^0[Tal\tau_n]\), since otherwise, by Fact 5.5 we obtain a contradiction: \((T) \in \mathcal{A}\) or \((Tal\tau_n) \in \mathcal{A}\), for some \(k \in \{0, \ldots, n_* - 1\}\). Hence if \(\mathcal{A} \not\subseteq E1\) then \(S.0.5^0[Tal\tau_n] \subseteq \mathcal{A}\). Thus, \(\mathcal{A} = nB^0\). Moreover, if \(\mathcal{A} \subseteq E1\) then \(C1[Tal\tau_n] \subseteq \mathcal{A} \subseteq S.0.5^0[Tal\tau_n] \cap E1 = C1[Tal\tau_n]\). Thus, \(\mathcal{A} = rB^0\).

Ad \(5\) Let (i) \(\mathcal{A} \not\subseteq S.0.5^0[Tal\tau_0]\) and (ii) \(\mathcal{A} \subseteq S.0.5^0[D]\). Then, by (ii) and Fact 5.3 either \((T) \in \mathcal{A}\) or \((D) \in \mathcal{A}\). But \((T) \notin \mathcal{A}\), by (i). So \((D) \in \mathcal{A}\). Hence if \(\mathcal{A} \not\subseteq E1\) then \(S.0.5^0[D] \subseteq \mathcal{A} \subseteq S.0.5^0[D]\). So \(\mathcal{A} = nB^0\). Moreover, if \(\mathcal{A} \subseteq E1\) then \(C1[D] \subseteq \mathcal{A} \subseteq S.0.5^0[D] \cap E1 = C1[D]\). So \(\mathcal{A} = rB^0\).

Ad \(6\) If \(\mathcal{A} \not\subseteq E1\) and \(\mathcal{A} \subseteq S.0.5^0[Tal\tau_0] \cap S.0.5^0[D]\), then \(S.0.5^0 \subseteq \mathcal{A}\) and \(nB^0 = S.0.5^0 \subseteq \mathcal{A} \subseteq S.0.5^0[Tal\tau_0] \cap S.0.5^0[D] = nB^0 \cap nB^0 = nB^0\), by Fact 2.19 and theorems 4.1 and 4.2(5), respectively. Moreover, if \(\mathcal{A} \subseteq E1 \cap S.0.5^0[Tal\tau_0] \cap S.0.5^0[D]\), then, by theorems 4.1(2,4) and 4.2(5), \(rB^0 = C1 \subseteq \mathcal{A} \subseteq S.0.5^0[Tal\tau_0] \cap S.0.5^0[D] \cap E1 = C1[Tal\tau_0] \cap C1[D] = rB^0 \cap rB^0 = rB^0\).

The proofs of “⇒”-parts of items 7 and 8 are obvious. For “⇐”-parts we have:
Ad \[7\] Let (i) \(1A \subseteq \textit{S0}5\textit{D}[\text{ Talta}]\), (ii) \(1A \not\subseteq \textit{S0}5\textit{D}[\text{ Talta}+1]\), and (iii) \(1A \not\subseteq \textit{S0}5\textit{D}[\text{ Talta}]\). Then (T) \(\not\in A\) and (Taln) \(\in A\), by (i), (ii), and Fact \(54\). Hence (D) \(\in A\), by (iii) and Fact \(51\). So if \(1A \not\subseteq 1E1\) then \(1\textit{S0}5\textit{D}[\text{ Talta}] = 1A = nB^5\). If \(1A \subseteq 1E1\) then \(1\textit{C1}[\text{D, Talta}] \subseteq 1A \subseteq 1\textit{S0}5\textit{D}[\text{D, Talta}] \cap 1E1 = \textit{C1}[\text{D, Talta}] = rB^5\).

Ad \[8\] Let \(1A \subseteq \textit{S0}5\textit{D}[\text{ Talta}]\) and \(1A \not\subseteq \textit{S0}5\textit{D}[\text{ Talta}+1]\). Then (Taln) \(\in A\), by Fact \(55\). Hence if \(1A \not\subseteq 1E1\) then \(1\textit{S0}5\textit{D}[\text{ Talta}] = 1A = nB^5\). Moreover, \(\textit{S0}5\textit{D}[\text{ Talta}] \cap \textit{S0}5\textit{D}[\text{D, Talta}+1] = \textit{S0}5\textit{D}[\text{D, Talta}+1]\), by Corollary \(213\). Hence if \(1A \subseteq 1\textit{S0}5\textit{D}[\text{ Talta}]\) and \(1A \not\subseteq 1\textit{S0}5\textit{D}[\text{D, Talta}+1]\), then \(1A \not\subseteq 1\textit{S0}5\textit{D}[\text{D, Talta}+1]\).

If \(1A \subseteq 1E1\) then \(1\textit{C1}[\text{D, Talta}] \subseteq 1A \subseteq 1\textit{S0}5\textit{D}[\text{D, Talta}] \cap 1E1 = 1\textit{C1}[\text{D, Talta}] = rB^5\). Moreover, \(\textit{C1}[\text{D, Talta}] \cap \textit{C1}[\text{D, Talta}+1] = \textit{C1}[\text{D, Talta}+1]\), by Corollary \(218\). Hence if \(1A \subseteq 1\textit{C1}[\text{D, Talta}]\) and \(1A \not\subseteq 1\textit{C1}[\text{D, Talta}+1]\), then \(1A \not\subseteq 1\textit{C1}[\text{D, Talta}+1]\).

The following theorem shows that for any modal logic \(\Lambda\) between \(\textit{C1}\) and \(\textit{S5}\) we are able to indicate a basic theorem which corresponds to \(A\) (see figures \(13\)). The proof of this theorem we obtain by theorems \(34, 41, 42, 61\) and facts \(219, 51, 55\).

**Theorem 6.2.** For any modal logic \(\Lambda\) such that \(\textit{C1} \subseteq \Lambda \subseteq \textit{S5}\):

1. \(1A = nB^1\) iff \((N) \in A, \) (D) \(\in A, \) and (Taln) \(\in A\) iff \((N) \in A\) and (T) \(\in A\).
2. \(1A = rB^1\) iff \((N) \not\in A, \) (D) \(\in A\) and (Taln) \(\in A\) iff \((N) \not\in A\) and (T) \(\in A\).
3. For any \(n > 0\): \(1A = nB^n\) iff \((N) \not\in A, \) (D) \(\in A, \) (Taln) \(\in A\), and (Taln-1) \(\not\in A\).
4. For any \(n > 0\): \(1A = rB^n\) iff \((N) \not\in A, \) (D) \(\in A, \) (Taln) \(\in A\), and (Taln-1) \(\not\in A\).
5. \(1A = nB^\infty\) iff \((N) \in A, \) (D) \(\in A, \) and \((\forall n \geq 0)\) (Taln) \(\not\in A\).
6. \(1A = rB^\infty\) iff \((N) \not\in A, \) (D) \(\in A, \) and \((\forall n \geq 0)\) (Taln) \(\not\in A\).
Theorem 6.1 and Fact 2.19. Thus, $\Lambda_n(\mathcal{T}^0_1)$, for some $n > 0$. Hence, by Theorem 4.1.

Ad 2. Suppose that $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_2) \notin \mathcal{A}$. Then $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_2) \notin \mathcal{A}$. So $\mathcal{A} \subseteq 1^{\mathcal{S}_0}\mathcal{T}_0^0 \cap 1^{\mathcal{S}_0}\mathcal{T}_1^0$, by Fact 5.1. Thus, $\mathcal{A} = nB^\infty$, by Theorem 4.1. If $(\Sigma^0_1) \notin \mathcal{A}$ then $1^{\mathcal{S}_0}\mathcal{T}_0^0 \subseteq \mathcal{A}$ and $(\Sigma^0_2) \notin \mathcal{A}$. So $\mathcal{A} \subseteq 1^{\mathcal{S}_0}\mathcal{T}_0^0 \cap 1^{\mathcal{E}_1} = 1^{\mathcal{E}_1} = 1^{\mathcal{E}_1} = 1^{\mathcal{E}_1}$, by Theorem 4.1.

Ad 3. Let $n > 0$. Suppose that $(\Sigma^0_1) \notin \mathcal{A}$, $(\Sigma^0_2) \notin \mathcal{A}$, and $(\Sigma^0_{n-1}) \notin \mathcal{A}$. Then $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_2) \notin \mathcal{A}$. So $\mathcal{A} \subseteq 1^{\mathcal{S}_0}\mathcal{T}_0^0 \cap 1^{\mathcal{S}_0}\mathcal{T}_1^0$, by facts 5.1 and 5.3 respectively. Therefore, by Theorem 6.1, for some $n_0 > 0$ either $\mathcal{A} = nB^{n_0}$ or $\mathcal{A} = rB^{n_0}$. If $(\Sigma^0_1) \notin \mathcal{A}$ then $nB^{n_1} = 1^{\mathcal{S}_0}\mathcal{T}_0^0 \subseteq \mathcal{A}$, since $(\Sigma^0_1) \notin \mathcal{A}$. Moreover, $nB^{n_1} = 1^{\mathcal{S}_0}\mathcal{T}_0^0 \subseteq \mathcal{A} = nB^{n_0}$, since $(\Sigma^0_1) \notin \mathcal{A}$. So, by Theorem 4.2, $nB^{n_0} \subseteq nB^{n_1}$. Thus, $\mathcal{A} = nB^{n_0}$, by Theorem 4.1. Similarly, if $(\Sigma^0_1) \notin \mathcal{A}$, we obtain $\mathcal{A} = rB^{n_0}$.

Ad 4. Suppose that $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_{n-1}) \notin \mathcal{A}$, for any $n > 0$. Then $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_{n-1}) \notin \mathcal{A}$. So $\mathcal{A} \subseteq 1^{\mathcal{S}_0}\mathcal{T}_0^0 \cap 1^{\mathcal{S}_0}\mathcal{T}_1^0$, by fact 5.2. Thus, $\mathcal{A} = nB^{n_0}$, by Theorem 6.1. If $(\Sigma^0_1) \notin \mathcal{A}$ then $1^{\mathcal{S}_0}\mathcal{T}_0^0 \subseteq \mathcal{A} \subseteq 1^{\mathcal{S}_0}\mathcal{T}_1^0 \cap 1^{\mathcal{E}_1} = 1^{\mathcal{E}_1}$, by Fact 2.19. Thus, $\mathcal{A} = rB^{n_0}$, by Theorem 4.1.

Ad 5. Suppose that $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_{n-1}) \notin \mathcal{A}$, for any $n > 0$. Then $(\Sigma^0_1) \notin \mathcal{A}$ and $(\Sigma^0_{n-1}) \notin \mathcal{A}$. Hence, by theorems 4.1 and 6.1, either $\mathcal{A} = nB^{n_0}$ or $\mathcal{A} = rB^{n_0}$. Thus, if $(\Sigma^0_1) \notin \mathcal{A}$ (resp. $(\Sigma^0_1) \notin \mathcal{A}$) then $\mathcal{A} = nB^{n_0}$ (resp. $\mathcal{A} = rB^{n_0}$), by Theorem 6.1 and Fact 2.19.

In the light of theorems 4.1 and 6.2 there is a correspondence between all “normal basic theories” and well known normal logics included in S5. We present graphically this correlation in Figure 3, showing a comparison of very weak t-normal logic and normal logics. (Note that KB4 = KB5 = K5 + (Taln); see p. 120 in Part 1.)

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3For the cases 1.3 we can provide other proofs using Theorem 6.1.
Similarly – in the light of theorems 4.1 and 6.2 – we can assign all “regular basic theories” to respective properly regular logics included in $S5$. We can make the following exchanges in Figure 3:

- each of the very weak $t$-normal logics is replaced corresponding to its $t$-regular logic,
- any normal logic $A_n$ is replaced by the properly regular logic $CF \cap A_n$. 

Fig. 3. Location of $S0.5^\circ$, $S0.5^\circ[T_{alt}]$, $S0.5^\circ[T_{alt_0}]$, $S0.5^\circ[D]$, $S0.5^\circ[D,T_{alt}]$, $S0.5^\circ = S0.5^\circ[D,T_{alt_0}] = S0.5^\circ[T]$ among some normal logics.
A. Some auxiliary facts from classical logic

In the proof of the auxiliary facts from Section 5 we have used the following lemmas [A.2] and [A.3]; while in the proofs of these lemmas we will use Lemma [A.1].

Lemma A.1. Let \( n \geq 0 \) and \( V_0, \ldots, V_{n+1} \) be different valuations on \( \text{For}_{cl} \). Then there is a uniform substitution \( s \) such that for any \( \theta \in \text{For}_{cl} \) and any cl-valuation \( V \) on \( \text{For}_{cl} \) the following conditions \((C_0)-(C_{n+1})\) hold.

\[
\begin{align*}
(C_0) & \quad \text{If } V(p) = 0 \text{ then } V(s(\theta)) = V_0(\theta). \\
(C_1) & \quad \text{If } V(p) = 1 \text{ then } V(s(\theta)) = V_1(\theta). \\
(C_2) & \quad \text{If } V(p) = 1 \text{ and } V(q_1) = 0 \text{ then } V(s(\theta)) = V_1(\theta). \\
(C_3) & \quad \text{If } V(p) = 1 = V(q_1) \text{ then } V(s(\theta)) = V_2(\theta). \\
(C_4) & \quad \text{For any } i \in \{2, \ldots, n\}: \text{ if } V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1 \text{ and } V(q_i) = 0, \text{ then } V(s(\theta)) = V_i(\theta). \\
(C_{n+1}) & \quad \text{If } V(p) = V(q_1) = \cdots = V(q_n) = 1 \text{ then } V(s(\theta)) = V_{n+1}(\theta).
\end{align*}
\]

Proof: We make the following substitution \( s \) for atoms. For any \( a \in \text{At} \) the formula \( s(a) \) will be dependent on the values \( V_0(a), V_1(a), \ldots, V_{n+1}(a) \).

We will consider six classes of valuations.

1. \( V_0(a) = V_1(a) = \cdots = V_{n+1}(a) = 1 \): Then we put \( s_1(a) := p \lor \neg p \).
2. \( V_0(a) = V_1(a) = \cdots = V_{n+1}(a) = 0 \): Then we put \( s_2(a) := p \land \neg p \).
3. \( V_0(a) = 0 \) and \( V_{n+1}(a) = 1 \): Then inductively we construct the following sequence \( Q_1^3, \ldots, Q_n^3 \) of formulas or «blanks» (further for the «blank formula» we use the symbol ‘\( \psi \)’). First we put:

\[
Q_n^3 := \begin{cases} 
q_n & \text{if } V_n(a) = 0 \\
\emptyset & \text{if } V_n(a) = 1
\end{cases}
\]

Second, if \( n > 1 \) then for any \( i = 1, \ldots, n-1 \) we put inductively:

\[
Q_i^3 := \begin{cases} 
q_i \land Q_{i+1}^3 & \text{if } V_i(a) = 0 \\
\neg q_i \lor Q_{i+1}^3 & \text{if } V_i(a) = 1 \text{ and } Q_{i+1}^3 \neq \emptyset \\
\emptyset & \text{if } V_i(a) = 1 \text{ and } Q_{i+1}^3 = \emptyset
\end{cases}
\]

Finally, we put \( s_3(a) := \psi \land Q_1^3 \). So if \( V_1(a) = \cdots = V_n(a) = 1 \) then \( s_3(a) := \psi \).
4. \(V_0(a) = 1\) and \(V_{n+1}(a) = 0\): Then as \(s_4(a)\) we will put \(\lnot s_3(a)\) calculated for the values \(V_i(a) = 1 - V_i(a)\). Thus, inductively we construct the following sequence \(Q_4^1, \ldots, Q_4^n\) of formulas or «blanks». First we put:

\[
Q_4^n := \begin{cases} q_n & \text{if } V_n(a) = 1 \\ \emptyset & \text{if } V_n(a) = 0 \end{cases}
\]

Second, if \(n > 1\) then for any \(i = 1, \ldots, n - 1\) we put inductively:

\[
Q_i^4 := \begin{cases} q_i \land Q_{i+1}^4 & \text{if } V_i(a) = 1 \\ \lnot q_i \lor Q_{i+1}^4 & \text{if } V_i(a) = 0 \text{ and } Q_{i+1} \neq \emptyset \\ \emptyset & \text{if } V_i(a) = 0 \text{ and } Q_{i+1} = \emptyset \end{cases}
\]

Finally, we put \(s_4(a) := \lnot (p \land Q_1^4)\). So if \(V_1(a) = \cdots = V_n(a) = 0\) then \(s_4(a) := \lnot p\).

5. \(V_0(a) = 0 = V_{n+1}(a)\) and there is an \(i \in \{1, \ldots, n\}\) such that \(V_i(a) = 1\): If \(n = 1\) then we put \(s_1(a) := p \land \lnot q_1\). If \(n > 1\) then we construct inductively the following sequence \(Q_5^1, \ldots, Q_5^n\) of formulas or «blanks». First we put:

\[
Q_5^n := \begin{cases} \lnot q_n & \text{if } V_n(a) = 1 \\ \emptyset & \text{if } V_n(a) = 0 \end{cases}
\]

Second, if \(n > 1\) then for any \(i = 1, \ldots, n - 1\) we put inductively:

\[
Q_i^5 := \begin{cases} q_i \land Q_{i+1}^5 & \text{if } V_i(a) = 0 \\ \lnot q_i \lor Q_{i+1}^5 & \text{if } V_i(a) = 1 \end{cases}
\]

Finally, we put \(s_5(a) := p \land Q_1^5\).

---

4We see that for \(n = 0\) we obtain the following uniform substitution \(s\) for any \(a \in At:\)

\[
s(a) := \begin{cases} p \lor \lnot p & \text{if } V_0(a) = 1 = V_1(a) \\ p & \text{if } V_0(a) = 0 \text{ and } V_1(a) = 1 \\ \lnot p & \text{if } V_0(a) = 1 \text{ and } V_1(a) = 0 \\ p \land \lnot p & \text{if } V_0(a) = 0 = V_1(a) \end{cases}
\]

So for \(n = 0\) by induction on the complexity of formulas it is easy to show that (C_0) and (C_1) hold.
6. \( V_0(a) = 1 = V_{n+1}(a) \) and there is an \( i \in \{1, \ldots, n\} \) such that \( V_i(a) = 0 \). Then as \( s_0(a) \) we will put \( \neg s_0(a) \) calculated for the values \( V'_0(a) = 1 - V_i(a) \). Thus, if \( n = 1 \) then we put \( s_1(a) := \neg(p \wedge q_i) \). If \( n > 1 \) then we construct inductively the following sequence \( Q^0_1, \ldots, Q^6_n \) of formulas or «blanks». First we put:

\[
Q^6_n := \begin{cases} 
\neg q_n & \text{if } V_n(a) = 0 \\
0 & \text{if } V_n(a) = 1 
\end{cases}
\]

Second, if \( n > 1 \) then for any \( i = 1, \ldots, n - 1 \) we put inductively:

\[
Q^6_i := \begin{cases} 
q_i \wedge Q^6_{i+1} & \text{if } V_i(a) = 1 \\
\neg q_i \vee Q^6_{i+1} & \text{if } V_i(a) = 0 
\end{cases}
\]

Finally, we put \( s_0(a) := \neg(p \wedge Q^6_1) \).

Now as \( s(a) \) we take respectively \( s_1(a), \ldots, s_6(a) \), depending on to which of the classes 1–6 the atom \( a \) belongs.

By induction on the complexity of formulas we can prove that for any \( \theta \in \text{FOR}_a \) and any cl-valuation \( V \) the conditions \( (C_0)^\theta \)–(\( C_{n+1} \)) hold.

Now we show the inductive hypothesis for atoms. Let \( a \in \text{At} \). For classes 1 and 2 of valuations the conditions \( (C_0)^\theta \)–(\( C_{n+1} \)) are obviously met. Next, note that for some \( k \in \{3, 4, 5, 6\} \) and \( i \in \{1, \ldots, n\} \), \( Q^k_i \) may be \( \emptyset \), even if it is not explicitly determined.

For class 3, where \( V_0(a) = 0 \) and \( V_{n+1}(a) = 1 \), we have:

For \( (C_0) \): Suppose that \( V(p) = 0 \). Then \( V(s_3(a)) = V(p \wedge Q^3_1) = 0 \).

For \( (C_1) \): Suppose that \( V(p) = 1 \) and \( V(q_1) = 0 \). First, if \( V_1(a) = 0 \) then either \( Q^3_1 = q_1 \) or \( Q^3_1 = q_1 \wedge Q^3_2, \) if \( n > 1 \). So either \( V(s_3(a)) = V(p \wedge q_1) = 0 \) or \( V(s_3(a)) = V(p \wedge q_1 \wedge Q^3_2) = 0 \). Second, if \( V_1(a) = 1 \) then either \( Q^3_1 = q_1 \vee Q^3_2 \) or \( Q^3_1 = \emptyset \). So either \( V(s_3(a)) = V(p \vee q_1 \vee Q^3_2) = 1 \) or \( V(s_3(a)) = V(p) = 1 \).

For \( (C_{n+1}) \): Suppose that \( V(p) = V(q_1) = \cdots = V(q_n) = 1 \). Note that \( Q^3_n = \neg q_n \) or \( Q^3_n = \emptyset \). So, in the first case, \( V(Q^3_n) = 1 \). Moreover, if \( n = 1 \) then either \( V(s_3(a)) = V(p) = 1 \) or \( V(s_3(a)) = V(p \wedge Q^3_1) = 1 \). If \( n > 1 \) then for \( j = 1, \ldots, n - 1 \) either \( Q^3_j = \emptyset \), or \( Q^3_j = q_1 \), or \( Q^3_j = q_1 \wedge Q^3_{j+1} \), or \( Q^3_j = \neg q_j \vee Q^3_{j+1} \), where \( Q^3_{j+1} \neq \emptyset \). So, in the last two cases, we can show inductively that \( V(Q^3_2) = 1 \). Therefore either \( V(s_3(a)) = V(p) = 1 \) or \( V(s_3(a)) = V(p \wedge Q^3_1) = 1 \).
If \( n > 1 \) then we show inductively that \((C_n)\) holds. Indeed, assume that \( V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1 \) and \( V(q_n) = 0 \). First, if \( V_n(a) = 0 \) then \( Q^3_n = \neg q_n \). Hence \( Q^3_{n-1} = \neg q_{n-1} \land q_n \) or \( Q^4_{n-1} = \neg q_{n-1} \land q_n \). So \( V(Q^3_{n-1}) = 0 \). Moreover, if \( n = 2 \) then \( V(s_3(a)) = V(p \land Q^2_n) = 0 \). If \( n > 2 \) then for \( j = 1, \ldots, n-2 \) we can show that \( Q^j_{n+1} \neq \emptyset \), and either \( Q^j_1 = \neg q_j \land Q^3_{j+1} \) or \( Q_j^1 = \neg q_j \lor Q^3_{j+1} \), and \( V(Q^j_1) = 0 \). So \( V(s_3(a)) = V(p \land Q^2_n) = 0 \). Second, if \( V_n(a) = 1 \) then \( Q^1_n = \emptyset \). Hence \( Q^1_{n-1} = \emptyset \) or \( Q^1_{n-1} = \neg q_{n-1} \). So \( Q^1_{n-1} = \emptyset \) or \( V(Q^2_{n-1}) = 1 \). Moreover, if \( n = 2 \) then either \( V(s_3(a)) = V(p) = 1 \) or \( V(s_3(a)) = V(p \land Q^2_n) = 1 \). If \( n > 2 \) then for \( j = 1, \ldots, n-2 \) we can show that either \( Q^j_1 = \emptyset \), or \( Q^j_1 = \neg q_j \lor Q^3_{j+1} \), where \( Q^1_{j+1} \neq \emptyset \); so, in the last three cases, \( V(Q^j_1) = 1 \). Thus, either \( V(s_3(a)) = V(p) = 1 \) or \( V(s_3(a)) = V(p \land Q^2_n) = 1 \).

For class 4, where \( V_0(a) = 1 \) and \( V_{n+1}(a) = 0 \), we have:

For \((C_0)\): Suppose that \( V(p) = 0 \). Then \( V(s_4(a)) = V(\neg p \land Q^1_n) = 1 \).

For \((C_1)\): Suppose that \( V(p) = 1 \) and \( V(q) = 0 \). First, if \( V_1(a) = 0 \) then either \( Q^1_1 = \emptyset \) or \( Q^1_1 = \neg q_1 \lor Q^3_2 \). So either \( V(s_4(a)) = V(\neg p) = 0 \) or \( V(s_4(a)) = V(\neg p \land Q^3_2) = 0 \). Second, if \( V_1(a) = 1 \) then either \( Q^1_1 = q_1 \) or \( Q^1_1 = q_1 \land Q^3_2 \). So either \( V(s_4(a)) = V(\neg p \land q_1) = 1 \) or \( V(s_4(a)) = V(\neg p \land q_1 \land Q^3_2) = 1 \).
For \((C_{n+1})\): Suppose that \(V(p) = V(q_1) = \cdots = V(q_n) = 1\). Note that either \(Q^4_i = \neg q_i\) or \(Q^4_i = \emptyset\). So, in the first case, \(V(Q^4_i) = 1\). Moreover, if \(n = 1\) then either \(V(s_n(a)) = V(\neg p) = 0\) or \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 0\).

If \(n > 1\) then for \(j = 1, \ldots, n - 1\) either \(Q^4_j = \emptyset\), or \(Q^4_j = \neg q_j\), or \(Q^4_j = \neg q_j \land Q^4_{j+1}\), or \(Q^4_j = \neg q_j \lor Q^4_{j+1}\), where \(Q^4_{j+1} \neq \emptyset\). Therefore, in the last two cases, we can show inductively that \(V(Q^4_j) = 1\). So either \(V(s_n(a)) = V(\neg p) = 0\) or \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 0\).

If \(n > 1\) then we show inductively that \((C_n)\) holds. Indeed, assume that \(V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1\) and \(V(q_n) = 0\). First, if \(V_n(a) = 0\) then \(Q^4_n = \emptyset\). Hence \(Q^4_{n-1} = \emptyset\) or \(Q^4_{n-1} = \neg q_{n-1}\). So, in the last case, \(V(Q^4_{n-1}) = 1\). Moreover, if \(n = 2\) then either \(V(s_n(a)) = V(\neg p) = 0\) or \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 0\).

Thus, \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 1\). If \(n > 2\) then for \(j = 1, \ldots, n - 2\) we can show that either \(Q^4_j = \emptyset\), or \(Q^4_j = \neg q_j\), or \(Q^4_j = \neg q_j \land Q^4_{j+1}\), or \(Q^4_j = \neg q_j \lor Q^4_{j+1}\), where \(Q^4_{j+1} \neq \emptyset\); so, in the last three cases, \(V(Q^4_j) = 1\). Second, if \(V_n(a) = 1\) then \(Q^4_i = \neg q_i\). Hence either \(Q^4_{i-1} = \neg q_{i-1} \land q_i\), or \(Q^4_{i-1} = \neg q_{i-1} \lor q_i\). So \(V(Q^4_{i-1}) = 0\). Moreover, if \(n = 2\) then \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 1\). If \(n > 2\) then for \(j = 1, \ldots, n - 2\) we can show that either \(Q^4_j = \neg q_j \land Q^4_{j+1}\) or \(Q^4_j = \neg q_j \lor Q^4_{j+1}\); and \(V(Q^4_j) = 0\).

Thus, \(V(s_n(a)) = V(\neg(p \land Q^4_i)) = 1\). If \(n > 2\) then for \(i = 2, \ldots, n - 1\) we show inductively that \((C_i)\) holds.

Indeed, assume that \(V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1\) and \(V(q_i) = 0\). First, if \(V_i(a) = 0\) then either \(Q^4_i = \emptyset\) or \(Q^4_i = \neg q_i \lor Q^4_{i+1}\), where \(Q^4_{i+1} \neq \emptyset\). In the last case we have \(V(Q^4_i) = 1\). Moreover, either \(Q^4_{i-1} = \emptyset\), or \(Q^4_{i-1} = \neg q_{i-1}\), or \(Q^4_{i-1} = \neg q_{i-1} \land Q^4_i\), or \(Q^4_{i-1} = \neg q_{i-1} \lor Q^4_i\), where \(Q^4_i \neq \emptyset\); so, in the last three cases, \(V(Q^4_{i-1}) = 1\). If \(i = 2\) then either \(V(s_i(a)) = V(\neg p) = 0\) or \(V(s_i(a)) = V(\neg(p \land Q^4_i)) = 0\). Similarly, if \(i > 2\), then \(n > 3\) and for \(j = 1, \ldots, i - 2\) we can show that either \(Q^4_j = \emptyset\), or \(Q^4_j = \neg q_j\), or \(Q^4_j = \neg q_j \land Q^4_{j+1}\), or \(Q^4_j = \neg q_j \lor Q^4_{j+1}\), where \(Q^4_{j+1} \neq \emptyset\); so, in the last three cases, \(V(Q^4_j) = 1\). Thus, either \(V(s_i(a)) = V(\neg p) = 0\) or \(V(s_i(a)) = V(\neg(p \land Q^4_i)) = 0\). Second, if \(V_i(a) = 1\) then either \(Q^4_i = \neg q_i\) or \(Q^4_i = \neg q_i \land Q^4_{i+1}\). So \(V(Q^4_i) = 0\). Moreover, either \(Q^4_{i-1} = \neg q_{i-1} \land Q^4_i\) or \(Q^4_{i-1} = \neg q_{i-1} \lor Q^4_i\). So \(V(Q^4_{i-1}) = 0\). If \(i = 2\) then \(V(s_i(a)) = V(\neg(p \land Q^4_i)) = 1\). Similarly, if \(i > 2\), then \(n > 3\) and for \(j = 1, \ldots, i - 2\) we can show that either \(Q^4_j = \neg q_j \land Q^4_{j+1}\) or \(Q^4_j = \neg q_j \lor Q^4_{j+1}\); and \(V(Q^4_j) = 0\). Therefore \(V(s_i(a)) = V(\neg(p \land Q^4_i)) = 1\).
For class 5, where \( V_0(\alpha) = 0 = V_{n+1}(\alpha) \) and there is an \( i \in \{1, \ldots, n\} \) such that \( V_i(\alpha) = 1 \), we have:

For \((C_0)\): Suppose that \( V(p) = 0 \). Then \( V(s_5(\alpha)) = V(p \land Q_5^2) = 0 \).

For \((C_1)\): Suppose that \( V(p) = 1 \) and \( V(q_i) = 0 \). First, if \( V_1(\alpha) = 0 \), then \( n > 1 \) and \( Q_1^i = q_1 \land Q_5^2 \land \). So \( V(Q_1^3) = 0 \) and \( V(s_5(\alpha)) = V(p \land Q_5^2) = 0 \).

Second, if \( V_1(\alpha) = 1 \) then either \( Q_1^i = q_1 \land Q_5^2 \land \) or \( Q_1^i = q_1 \land Q_5^2 \land Q_5^{j+1} \). So \( V(s_5(\alpha)) = V(p \land \neg q_1) = 1 \) or \( V(s_5(\alpha)) = V(p \land (\neg q_1 \lor Q_5^{j+1})) = 1 \).

For \((C_n+1)\): Let \( V(p) = V(q_1) = \ldots = V(q_n) = 1 \). First, suppose that \( V_0(\alpha) = 1 \). Then \( Q_5^i = q_5 \land \) and \( V(Q_5^n) = 0 \). If \( n = 1 \) then \( V(s_5(\alpha)) = V(p \land \neg q_1) = 0 \). Moreover, if \( n > 1 \) then for \( j = 1, \ldots, n-1 \) either \( Q_j^i = q_j \land Q_5^{j+1} \lor Q_j^i = q_j \land Q_5^{j+1} \land \), where \( Q_5^{j+1} \neq \emptyset \), and in the last two cases we can show inductively that \( V(Q_5^2) = 0 \).

Therefore \( V(s_5(\alpha)) = V(p \land Q_5^2) = 0 \). Second, suppose that \( V_0(\alpha) = 0 \). Then \( n > 1 \) and \( Q_5^i = \emptyset \). Let \( i_0 \) be the largest \( i \in \{1, \ldots, n-1\} \) such that \( V_i(\alpha) = 1 \).

If \( i_0 = n-1 \), then \( Q_5^{n-1} = q_5 \land \) and \( V(Q_5^n) = 0 \). If \( n = 2 \) then \( V(s_5(\alpha)) = V(p \land q_1) = 0 \).

Moreover, if \( n > 2 \) then for \( j = 1, \ldots, n-2 \) either \( Q_j^i = q_j \land Q_5^{j+1} \lor Q_j^i = q_j \land Q_5^{j+1} \land \), where \( Q_5^{j+1} \neq \emptyset \), and we can show inductively that \( V(Q_5^2) = 0 \).

Therefore \( V(s_5(\alpha)) = V(p \land Q_5^2) = 0 \).

If \( n > 1 \) then we show inductively that \((C_n)\) holds. Indeed, assume that \( V(p) = V(q_1) = \ldots = V(q_{n-1}) = 1 \) and \( V(q_n) = 0 \). First, if \( V_{n+1}(\alpha) = 1 \) then \( Q_5^{n+1} = q_5 \land \) and \( V(Q_5^n) = 1 \). Hence \( Q_5^{n-1+1} = q_{n-1} \land q_5 \land \) or \( Q_5^{n-1} = q_{n-1} \land q_5 \land \). So \( V(Q_5^{n-1}) = 1 \). If \( n = 2 \) then \( V(s_5(\alpha)) = V(p \land Q_5^2) = 1 \).

Moreover, if \( n > 2 \) then for \( j = 1, \ldots, n-2 \) we can show that \( Q_j^{j+1} \neq \emptyset \) and either \( Q_j^i = q_j \land Q_5^{j+1} \lor Q_j^i = q_j \land Q_5^{j+1} \land \), and \( V(Q_5^2) = 1 \). So \( V(s_5(\alpha)) = V(p \land Q_5^2) = 1 \). Second, if \( V_n(\alpha) = 0 \), then \( n > 1 \) and \( Q_5^n = \emptyset \). Let \( i_0 \) be the largest \( i \in \{1, \ldots, n-1\} \) such that \( V_i(\alpha) = 1 \).

If \( i_0 = n-1 \), then \( Q_5^{n-1} = q_5 \land \) and \( V(Q_5^n) = 0 \). If \( n = 2 \) then \( V(s_5(\alpha)) = V(p \land q_1) = 0 \).

Moreover, if \( n > 2 \) then for \( j = 1, \ldots, n-2 \) either \( Q_j^i = q_j \land Q_5^{j+1} \lor Q_j^i = q_j \land Q_5^{j+1} \land \), where \( Q_5^{j+1} \neq \emptyset \), and we can show inductively that \( V(Q_5^2) = 0 \).

Therefore \( V(s_5(\alpha)) = V(p \land Q_5^2) = 0 \).

If \( i_0 < n-1 \), then \( n > 2 \) and \( Q_5^n = q_5 \land \), and \( V(Q_5^n) = 0 \). If \( n = 3 \), then \( i_0 = 1 \) and \( V(s_5(\alpha)) = V(p \land \neg q_1) = 0 \). Moreover, if \( n > 3 \) then...
for \( j = 1, \ldots, n - 3 \) either \( Q_j^6 = \neg q_j \lor Q_{j+1}^6 \) or \( Q_j^6 = q_j \land Q_{j+1}^6 \),
where \( Q_{j+1}^6 \neq \emptyset \); and we can show inductively that \( V(Q_j^6) = 0 \).
Therefore \( V(s_n(a)) = V(p \land Q_1^6) = 0 \).

If \( n > 2 \) then for \( i = 2, \ldots, n - 1 \) we show inductively that \((C_i)\) holds. Indeed, assume that \( V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1 \) and \( V(q_i) = 0 \). First, if \( V_i(a) = 1 \) then \( Q_i^5 = \neg q_i \lor Q_{i+1}^5 \) and \( V(Q_i^5) = 1 \).
Moreover, \( Q_{i-1}^5 = q_{i-1} \land Q_i^5 \lor Q_{i+1}^5 = \neg q_{i-1} \lor Q_i^5 \). So \( V(Q_{i-1}^5) = 1 \).
If \( i = 2 \) then \( V(s_5(a)) = V(p \land Q_1^5) = 1 \). Similarly, if \( i > 2 \), then \( n > 3 \) and for \( j = 1, \ldots, i - 2 \) we can show that either \( Q_j^5 = q_j \land Q_{j+1}^5 \lor Q_j^5 = \neg q_j \lor Q_{j+1}^5 \),
and so \( V(Q_j^5) = 1 \). Thus, \( V(s_5(a)) = V(p \land Q_1^5) = 1 \).
Second, if \( V_i(a) = 0 \) then \( Q_i^5 = q_i \land Q_{i+1}^5 \) and \( V(Q_i^5) = 0 \).
Moreover, \( Q_{i-1}^5 = q_{i-1} \land Q_i^5 \lor Q_{i-1}^5 = q_{i-1} \lor Q_i^5 \). So \( V(Q_{i-1}^5) = 0 \).
If \( i = 2 \) then \( V(s_5(a)) = V(p \land Q_1^5) = 0 \).
Similarly, if \( i > 2 \), then \( n > 3 \) and for \( j = 1, \ldots, i - 2 \) we can show that either \( Q_j^5 = q_j \land Q_{j+1}^5 \lor Q_j^5 = \neg q_j \lor Q_{j+1}^5 \),
and so \( V(Q_j^5) = 0 \).
Thus, \( V(s_5(a)) = V(p \land Q_1^5) = 0 \).

For class 6, where \( V_0(a) = 1 = V_{n+1}(a) \) and there is an \( i \in \{1, \ldots, n\} \) such that \( V_i(a) = 0 \), we have:

For \((C_0)\): Suppose that \( V(p) = 0 \). Then \( V(s_6(a)) = V(\neg(p \land Q_n^6)) = 1 \).
For \((C_1)\): Suppose that \( V(p) = 1 \) and \( V(q_1) = 0 \). First, if \( V_1(a) = 1 \), then \( n > 1 \) and \( Q_1^6 = q_1 \land Q_2^6 \).
So \( V(Q_1^6) = 0 \) and \( V(s_6(a)) = V(\neg(p \land Q_1^6)) = 1 \). Second, if \( V_1(a) = 0 \) then either \( Q_1^6 = \neg q_1 \lor Q_2^6 = q_1 \lor Q_2^6 \).
So \( V(s_6(a)) = V(\neg(p \land q_1)) = 0 \) or \( V(s_6(a)) = V(\neg(p \land q_1)) = 0 \).

For \((C_{n+1})\): Let \( V(p) = V(q_1) = \cdots = V(q_n) = 1 \). First, suppose that \( V_n(a) = 0 \). Then \( Q_n^6 = q_n \land V(Q_n^6) = 0 \). If \( n = 1 \) then \( V(s_6(a)) = V(\neg(p \land q_1)) = 1 \). Moreover, if \( n > 1 \) then for \( j = 1, \ldots, n - 1 \) either \( Q_j^6 = q_j \lor Q_{j+1}^6 \lor Q_j^6 = q_j \land Q_{j+1}^6 \),
and in the last two cases we can show inductively that \( V(Q_j^6) = 0 \).
Therefore \( V(s_6(a)) = V(\neg(p \land Q_j^6)) = 1 \). Second, suppose that \( V_n(a) = 1 \). Then \( n > 1 \) and \( Q_n^6 = \emptyset \).
Let \( i_0 \) be the largest \( i \in \{1, \ldots, n-1\} \) such that \( V_i(a) = 0 \). If \( i_0 = n - 1 \), then \( Q_{n-1}^6 = q_{n-1} \land V(Q_{n-1}^6) = 0 \).
If \( n = 2 \) then \( V(s_6(a)) = V(\neg(p \land q_1)) = 1 \). Moreover, if \( n > 2 \) then for \( j = 1, \ldots, n - 2 \) either \( Q_j^6 = q_j \lor Q_{j+1}^6 \lor Q_j^6 = q_j \land Q_{j+1}^6 \),
where \( Q_{j+1}^6 \neq \emptyset \); and we can show inductively that \( V(Q_j^6) = 0 \).
Therefore \( V(s_6(a)) = V(\neg(p \land Q_j^6)) = 1 \). If \( i_0 < n - 1 \), then \( n > 2 \), \( Q_i^6 = \neg q_i \land V(Q_i^6) = 0 \),
and \( V(Q_i^6) = 0 \). If \( n = 3 \), then \( i_0 = 1 \) and \( V(s_6(a)) = V(\neg(p \land q_1)) = 1 \). Moreover, if \( n > 3 \) then for \( j = 1, \ldots, n - 3 \) either \( Q_j^6 = q_j \lor Q_{j+1}^6 \).
or $Q^6_j = \neg q_j \land Q^6_{j+1} \land \gamma$, where $Q^6_{j+1} \neq \emptyset$; and we can show inductively that $V(Q^6_j) = 0$. Therefore $V(s_0(a)) = V(\neg(p \land Q^6_j)) = 1$.

If $n > 1$ then we show inductively that $(C_n)$ holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{n-1}) = 1$ and $V(q_n) = 0$. First, if $V_n(a) = 0$ then $Q^6_n = \neg q_n \land \gamma$ and $V(Q^6_n) = 1$. Hence $Q^6_{n-1} = \neg q_{n-1} \land \neg q_n \land \gamma$. So $V(Q^6_{n-1}) = 1$. If $n = 2$ then $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 0$. Moreover, if $n > 2$ then for $n > 1$, then $Q^6_n = \neg q_n \land \gamma$, and we can show inductively that $V(Q^6_n) = 0$. Therefore $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 1$. If $i_0 < n - 1$, then $n > 2$, $Q^6_{i_0} = \neg q_{i_0} \land \gamma$, and $V(Q^6_{i_0}) = 0$. If $n = 3$, then $i_0 = 1$ and $V(s_0(a)) = V(\neg(p \land q_1)) = 1$.

Moreover, if $n > 3$ then for $j = 1, \ldots, n - 3$ either $Q^6_j = \neg q_j \land Q^6_{j+1} \land \gamma$ or $Q^6_j = \neg q_j \land Q^6_{j+1} \land \gamma$, and $V(Q^6_j) = 0$. Therefore $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 1$.

If $n > 2$ then for $i = 2, \ldots, n - 1$ we show inductively that $(C_i)$ holds. Indeed, assume that $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$. First, if $V_i(a) = 0$ then $Q^6_i = \neg q_i \land Q^6_{i+1} \land \gamma$ and $V(Q^6_i) = 1$. Moreover, $Q^6_{i-1} = \neg q_{i-1} \land Q^6_i \land \gamma$ or $Q^6_{i-1} = \neg q_{i-1} \land Q^6_i \land \gamma$. So $V(Q^6_{i-1}) = 1$. If $i = 2$ then $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 0$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i - 2$ we can show that either $Q^6_j = q_j \land Q^6_{j+1} \land \gamma$ and $V(Q^6_j) = 1$. Thus, $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 0$.

Second, if $V_i(a) = 1$ then $Q^6_i = q_i \land Q^6_{i+1} \land \gamma$ and $V(Q^6_i) = 0$. Moreover, $Q^6_{i-1} = \neg q_{i-1} \land Q^6_i \land \gamma$ or $Q^6_{i-1} = \neg q_{i-1} \land Q^6_i \land \gamma$. So $V(Q^6_{i-1}) = 0$. If $i = 2$ then $V(s_0(a)) = V(\neg(p \land Q^6_1)) = 1$. Similarly, if $i > 2$, then $n > 3$ and for $j = 1, \ldots, i - 2$ we can show that either $Q^6_j = q_j \land Q^6_{j+1} \land \gamma$ or $Q^6_j = \neg q_j \land Q^6_{j+1} \land \gamma$; and so $V(Q^6_j) = 0$. Thus, $V(s_0(a)) = V(\neg(p \land Q^6_j)) = 1$.

The inductive steps for complex formulas are obvious.

**Lemma A.2.** Let $k \geq 0$ and $\alpha, \beta, \gamma_1, \ldots, \gamma_k \in \text{For}_k$. Suppose that:

- $\alpha \lor \beta \lor \gamma \in \text{Taut}$, but $\alpha \notin \text{Taut}$ and $\beta \lor \bigvee_{j=1}^{k} \gamma_j \notin \text{Taut}$. 


Then there is a uniform substitution $s$ such that $\Gamma s(\alpha) \equiv p^\gamma$ and $\Gamma s(\beta) \equiv \neg p^\gamma$ belong to $\text{Taut}$, and for any $i \in \{1, \ldots, k\}$, either $\Gamma s(\gamma_i) \equiv \neg p^\gamma$ or $\Gamma s(\gamma_i) \land \alpha \in \text{Taut}$. 

**Proof:** By both assumptions, there are two (different) cl-valuations $V_0$ and $V_1$ such that:

- $V_0(\alpha) = 0$ and $V_0(\beta) = 1$,
- $V_1(\beta) = V_1(\gamma_1) = \cdots = V_1(\gamma_k) = 0$ and $V_1(\alpha) = 1$.

By Lemma A.1, with $n = 0$, for the valuations $V_0$ and $V_1$ we make some substitution $s$ which for any $\theta \in \text{For}_3$ and any cl-valuation $V$ satisfies the conditions $(C_0)$ and $(C_1)$ from this lemma. In the light of these conditions we obtain:

- $\Gamma s(\alpha) \equiv p^\gamma \in \text{Taut}$.

Indeed, for any cl-valuation $V$: if $V(p) = 1$ then $V(s(\alpha)) = V(\alpha) = 1$, by $(C_1)$; if $V(p) = 0$ then $V(s(\alpha)) = V(\alpha) = 0$, by $(C_0)$.

- $\Gamma s(\beta) \equiv \neg p^\gamma \in \text{Taut}$.

Indeed, for any cl-valuation $V$: if $V(p) = 1$ then $V(s(\beta)) = V(\beta) = 0$, by $(C_1)$; if $V(p) = 0$ then $V(s(\beta)) = V(\beta) = 1$, by $(C_0)$.

- For any $i \in \{1, \ldots, k\}$ either $\Gamma s(\gamma_i) \equiv \neg p^\gamma \in \text{Taut}$ or $\Gamma s(\gamma_i) \land \alpha \in \text{Taut}$.

Indeed, for any cl-valuation $V$: if $V(p) = 1$ then $V(s(\gamma_i)) = V(\gamma_i) = 0$, by $(C_1)$. Hence $\Gamma p \supset \neg s(\gamma_i) \land \alpha \in \text{Taut}$. Moreover, since $\text{At}(s(\gamma_i)) = \{p\}$, so either $\Gamma \neg s(\gamma_i) \land \alpha \in \text{Taut}$ or $\Gamma s(\gamma_i) \equiv \neg p^\gamma \in \text{Taut}$.

**Lemma A.3.** Let $k > 1$ and $\alpha, \beta, \gamma_1, \ldots, \gamma_k$ belong to $\text{For}_3$. Suppose that:

- $\Gamma \alpha \lor \beta \land \gamma \in \text{Taut}$, but $\alpha \notin \text{Taut}$,
- for any $\gamma \in \Gamma := \{\gamma_1, \ldots, \gamma_k\}$ we have $\Gamma \beta \lor \gamma \land \alpha \notin \text{Taut}$.

1. Then for some $n \in \{1, \ldots, k - 1\}$ there are non-empty different subsets $\Gamma_1, \ldots, \Gamma_{n+1}$ of the set $\Gamma$ such that $\Gamma = \bigcup_{i=1}^{n+1} \Gamma_i$ and for some uniform substitution $s$ we have:

- $\Gamma s(\alpha) \equiv p^\gamma$ and $\Gamma s(\beta) \equiv \neg p^\gamma$ belong to $\text{Taut}$;
- for any $\gamma \in \Gamma_1$: $\Gamma s(\beta \lor \gamma) \lor q_1 \land \alpha \in \text{Taut}$;
- for all $i \in \{1, \ldots, n\}$ and $\gamma \in \Gamma_{i+1}$: $\Gamma s(\beta \land \gamma) \lor \bigwedge_{j=i}^{i+1} q_j \lor q_{i+1} \land \alpha \in \text{Taut}$.
2. Moreover, for any subset $\Psi$ of $\Gamma$ such that $^r$ $\beta \lor \bigvee \Psi \blacklozenge \in \text{Taut}$ we can take $n = \text{Card} \Psi - 1$.

**Proof:** Ad 1. By assumptions, there are cl-valuations $A_0, \ldots, A_k$ such that:

- $A_0(\alpha) = 0$ and $A_0(\beta) = 1$,
- for any $i \in \{1, \ldots, k\}$: $A_i(\gamma_i) = 0 = A_i(\beta)$ and $A_i(\alpha) = 1$.

For any $i \in \{1, \ldots, k\}$ both $A_0(\beta) \neq A_i(\beta)$ and $A_0(\alpha) \neq A_i(\alpha)$, and there is a $j \in \{1, \ldots, k\}$ such that $A_i(\gamma_j) = 1$; so $A_i(\gamma_j) \neq A_j(\gamma_j)$. Hence among $A_1, \ldots, A_k$ we have at least two valuations which are different on the set $\Gamma$.

Let $m$ be the number of all such valuations. We put $n := m - 1$. Note that $m > 1$; so $n > 0$. We choose $n + 1$ such valuations $V_1, \ldots, V_{n+1}$ which are different on $\Gamma$.

Now for any $i \in \{1, \ldots, n + 1\}$ we put:

$$\Gamma_i := \{ \gamma \in \Gamma : V_i(\gamma) = 0 \}.$$ 

The sets $\Gamma_1, \ldots, \Gamma_{n+1}$ are non-empty and pairwise different and $\Gamma = \bigcup_{i=1}^{n+1} \Gamma_i$.

By Lemma $A.1$ with $n > 0$, for the valuations $V_0, \ldots, V_{n+1}$ we make some substitution $s$ which for any $\theta \in \text{For}_3$ and any cl-valuation $V$ satisfies the conditions $(C_0) - (C_{n+1})$ from the lemma. In the light of these conditions we obtain:

- $^r s(\alpha) \equiv p^\top \in \text{Taut}$.

Let $V$ be any cl-valuation. First, if $V(p) = 0$ then $V(s(\alpha)) = V_0(\alpha) = 0$, by $(C_0)$. Second, for any $i \in \{1, \ldots, n\}$: if $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\alpha)) = V_i(\alpha) = 1$, by $(C_i)$. Thirdly, if $V(p) = V(q_1) = \cdots = V(q_n) = 1$, then $V(s(\alpha)) = V_{n+1}(\alpha) = 1$, by $(C_{n+1})$.

- $^r s(\beta) \equiv \neg p^\top \in \text{Taut}$.

Let $V$ be any cl-valuation. First, if $V(p) = 0$ then $V(s(\beta)) = V_0(\beta) = 1$, by $(C_0)$. Second, for any $i \in \{1, \ldots, n\}$: if $V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1$ and $V(q_i) = 0$, then $V(s(\beta)) = V_i(\beta) = 0$, by $(C_i)$. Thirdly, if $V(p) = V(q_1) = \cdots = V(q_n) = 1$, then $V(s(\beta)) = V_{n+1}(\beta) = 0$, by $(C_{n+1})$.

- For any $\gamma \in \Gamma_1$: $^r s(\neg \beta \land \gamma) \supset q_1^\top \in \text{Taut}$.

Let $V$ be any cl-valuation and $\gamma \in \Gamma_1$. If $V(s(\neg \beta)) = V(p) = 1$ and $V(q_1) = 0$, then $V(s(\gamma)) = V_1(\gamma) = 0$, by $(C_1)$.
If \( n > 1 \) then for any \( i \in \{2, \ldots, n\} \): \[ s(\neg \beta \land \gamma_i) \supset (\bigwedge_{j=1}^{i-1} q_j \supset q_i)^n \in \text{Taut}. \]

Let \( V \) be any cl-valuation, \( n > 1, i \in \{2, \ldots, n\} \), and \( \gamma \in \Gamma_i \). If \( V(s(\neg \beta)) = V(p) = V(q_1) = \cdots = V(q_{i-1}) = 1 \) and \( V(q_i) = 0 \), then \( V(s(\gamma)) = V_i(\gamma) = 0 \), by \((C_i)\).

For any \( \gamma \in \Gamma_{n+1} \): \[ s(\neg \beta \land \gamma) \supset (\bigwedge_{j=1}^{n} q_j \supset q_{n+1}) \in \text{Taut}. \]

Let \( V \) be any cl-valuation and \( \gamma \in \Gamma_{n+1} \). If \( V(s(\neg \beta)) = V(p) = V(q_1) = \cdots = V(q_n) = 1 \), then \( V(s(\gamma)) = V_{n+1}(\gamma) = 0 \), by \((C_{n+1})\).

Ad 2. Let \( \Psi \) be any subset of \( \Gamma \) such that \( \gamma \beta \lor \bigvee \Psi \in \text{Taut} \). We put \( m := \text{Card} \Psi, m > 1 \). Suppose that and \( \Psi = \{ \psi_1, \ldots, \psi_m \} \). By assumption there are different cl-valuations \( V_0, \ldots, V_m \) such that:

1. \( V_0(\alpha) = 0 \) and \( V_0(\beta) = 1 \),
2. for any \( i \in \{1, \ldots, m\} \): \( V_i(\beta) = V_i(\psi_1) = \cdots = V_i(\psi_{i-1}) = V_i(\psi_{i+1}) = \cdots = V_i(\psi_m) = 0 \) and \( V_i(\alpha) = 1 = V_i(\gamma_\iota) \).

Of course all valuations \( V_1, \ldots, V_m \) are pairwise different on the set \( \Gamma \). We put \( n := m - 1 \). So for the valuations \( V_0, V_1, \ldots, V_{n+1} \) we can repeat the proof of the item 1.

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