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MINIMAL SEQUENT CALCULI FOR ŁUKASIEWICZ’S FINITELY-VALUED LOGICS

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Abstract

The primary objective of this paper, which is an addendum to the author’s [8], is to apply the general study of the latter to Łukasiewicz’s $n$-valued logics [4]. The paper provides an analytical expression of a $2(n-1)$-place sequent calculus (in the sense of [10, 9]) with the cut-elimination property and a strong completeness with respect to the logic involved which is most compact among similar calculi in the sense of a complexity of systems of premises of introduction rules. This together with a quite effective procedure of construction of an equality determinant (in the sense of [5]) for the logics involved to be extracted from the constructive proof of Proposition 6.10 of [2] yields an equally effective procedure of construction of both Gentzen-style (i.e., 2-place) and Tait-style [11] (i.e., 1-place) minimal sequent calculi following the method of translations described in Subsection 4.2 of [7].

1. Introduction

Here we entirely follow the general study [8] extending it to Łukasiewicz’s finitely-valued logics [4] in addition to Dunn’s finitely-valued normal extensions of $RM$ [1] as well as Gödel’s finitely-valued logics [3] completely...
studied in [8]. Lukasiewicz’s logics do deserve a particular emphasis because, as opposed to Dunn’s and Gödel’s logics, they do all have both equality determinant (in the sense of [5]) and singularity determinant (in the sense of [7]) (cf. Proposition 6.10 of [6] and Corollary 6.2 of [7] for positive results as well as Propositions 6.5 and 6.8 therein for negative ones), in which case many-place sequent calculi (in the sense of [10, 9]) to be constructed following [8] for the former logics are naturally translated into both Gentzen-style (i.e., 2-place) and Tait-style (i.e., 1-place) sequent calculi according to Subsections 4.2.1 and 4.2.2 of [7].

2. Main results

$L = \{\neg, \land, \lor, \supset\}$. Take any $n \geq 2$. Here we deal with the matrix underlying algebra $\mathfrak{A}_n$ specified as follows. The carrier $A_n$ of $\mathfrak{A}_n$ is set to be $n$. Finally, operations of $\mathfrak{A}_n$ are defined as follows:

\begin{align*}
\neg^{\mathfrak{A}_n} a & \triangleq n - 1 - a, \\
a \land^{\mathfrak{A}_n} b & \triangleq \min(a, b), \\
a \lor^{\mathfrak{A}_n} b & \triangleq \max(a, b), \\
a \supset^{\mathfrak{A}_n} b & \triangleq \min(n - 1, n - 1 - a + b),
\end{align*}

for all $a, b \in A_n$.

**Lemma 2.1.** For any $i \in n \setminus \{0\}$ and any $j \in n \setminus \{n - 1\}$, we have the following introduction rules for $\mathcal{M}^{\mathfrak{A}_n}$:

\[
\frac{\{\{I_{n-1-i};p_0\}\} }{\{F_i;\neg;p_0\} } \\
\frac{\{\{F_i;p_0\}, \{F_i;p_1\}\} }{\{I_j;p_0, I_j;p_1\} } \\
\frac{\{\{F_i;p_0 \land p_1\}\} }{\{I_j;p_0 \land p_1\} } \\
\frac{\{\{F_i;p_0, F_i;p_1\}\} }{\{I_j;p_0 \lor p_1\} } \\
\frac{\{\{I_{n-2-k};p_0, F_{i-k};p_1\} | 0 \leq k < i\} }{\{F_i;(p_0 \supset p_1)\} } \\
\frac{\{\{F_{n-1-i};p_0, I_{j-i};p_1\} | 0 < l \leq j\} \cup \{\{F_{n-1-j};p_0\}, \{I_j;p_1\}\} }{\{I_j;(p_0 \supset p_1)\} }
\]
Proof: Let $i \in n \setminus \{0\}$ and $j \in n \setminus \{n - 1\}$. Checking (1) of \cite{8} for the introduction rules of types $s \gamma$, where $s \in \{F_i, I_j\}$ and $\gamma \in \{\neg, \land, \lor\}$, is trivial. As for those of types $s \supset$, where $s \in \{F_i, I_j\}$, take any $a, b \in n$. Remark that $(a \supset b) \in F_i \iff n - 1 - a + b \geq i$. Likewise, $(a \supset b) \in I_j \iff n - 1 - a + b \leq j$.

Suppose $n - 1 - a + b \geq i$, that is, $n - 1 - i + b \geq a$. Consider any $0 \leq k < i$. Suppose $a \in F_{n - 1 - k} = n \setminus I_{n - 2 - k}$, that is, $a \geq n - 1 - k$. Combining two inequalities, we get $k \geq i - b$, that is, $b \in F_{i - k}$.

Conversely, assume $n - 1 - a + b < i$, in which case $n - 1 - a < i$ too. As $0 \leq n - 1 - a$, we can choose $k \triangleq n - 1 - a$. If $a$ was in $I_{n - 2 - k}$, we would have $0 \leq -1$. Likewise, by the inequality under assumption, if $b$ was in $F_{i - k}$, we would have $b > b$. Thus, both $a \notin I_{n - 2 - k}$ and $b \notin F_{i - k}$.

Remark that (1) of \cite{8} for the introduction rule of type $I_j$: $\supset$ is equivalent to the following condition:

\[ n - 1 - a + b \leq j \iff \forall l \in (j + 2): a \leq n - l - 1 \implies b \leq j - l \] (2.1)

for all $a, b \in A_n$.

First, suppose $n - 1 - a + b \leq j$, that is, $n - 1 - j + b \leq a$. Consider any $l \in (j + 2)$. Assume $a \leq n - l - 1$. Combining two inequalities, we get $b \leq j - l$ as required.

Finally, assume $n - 1 - a + b > j$. Put $l \triangleq \min(n - 1 - a, j + 1)$. Then, $l \in (j + 2)$. Moreover, $a \leq n - l - 1$. If $b$ was not greater than $j - l$, we would have $l + b \leq j$, in which case $l \leq j$, and so $l = n - 1 - a$, in which case $n - 1 - a + b \leq j$. The contradiction with the inequality under assumption shows that $b > j - l$. Thus, (2.1) holds. This completes the argument. \(\square\)

Notice that each of the sets of premises of rules involved in the formulation of Lemma 2.1 consists of functional $S_n$-signed $\emptyset$-sequents of some type $V \subseteq \text{Var}$ and forms an anti-chain with respect to $\preceq$. Then, by Theorem 2.15(ii) of \cite{8}, Lemma 2.1 yields
Theorem 2.2. For any \( i \in n \setminus \{0\} \) and any \( j \in n \setminus \{n-1\} \):

\[
\begin{align*}
P_{n,\rightarrow}^{2n} & = \{\{I_{n-1-i} : p_0\}\}, \\
P_{n,\vdash}^{2n} & = \{\{F_{n-1-j} : p_0\}\}, \\
P_{n,\land}^{2n} & = \{\{F_i : p_0, F_i : p_1\}\}, \\
P_{n,\land}^{2n} & = \{\{I_j : p_0, I_j : p_1\}\}, \\
P_{n,\lor}^{2n} & = \{\{I_j : p_0\}\}, \\
P_{n,\lor}^{2n} & = \{\{F_i : p_0\}\}, \\
P_{n,\supset}^{2n} & = \{\{I_{n-2-k} : p_0, F_{n-k} : p_1\} \mid 0 \leq k < i\}, \\
P_{n,\supset}^{2n} & = \{\{F_{n-1-j} : p_0, I_{n-1-j} : p_1\}\} \cup \{\{F_{n-1-j} : p_0\}, \{I_j : p_1\}\}.
\end{align*}
\]

This provides the minimal 2\((n-1)\)-place sequent calculus for \( \mathbb{A}_n \). Notice that \( P_{n-2,\supset}^{2n} \) has exactly \( n \) elements. Remark that, in case \( n = 2 \), the resulted calculus coincides with Gentzen’s classical calculus \( LK \) \[2\].

References


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