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In memory of Andrzej Grzegorczyk

NON-FREGEAN LOGICS OF ANALYTIC EQUIVALENCE (II)

Abstract

This paper presents the main assumptions of Andrzej Grzegorczyk's last research project concerning the logic of synonymity. It shows that the basis of logic of analytic equivalence, presented in the first part of the work, fully corresponds with these assumptions.

1. Introduction

From the formal point of view, the sentential part of non-Fregean logic (NFL) is a collection of calculi formulated in the language $\mathcal{L}_{\mathsf{SCI}}$ of SCI (sentential calculus with identity) and containing the set of SCI theses. In 2011-2012, Andrzej Grzegorczyk designed (in [7] and [8]) a project of sentential logic with a synonymity connective, which may be included in NFL[1] Grzegorczyk's scientific activity in the last years of his life was focused on the implementation of this project.

This activity, and the works and discussions it inspired, gave a new character to certain research related with NFL. In particular, [6] presented the semantics (in the style of the algebraic semantics by R. Suszko) for the system marked by the symbol "LD" (Logic of Description) which is

[1] J. Golińska-Pilarek and T. Huuskonen also believe that the logic of synonymity according to Grzegorczyk can be classified as non-Fregean logic (see [5]).

the realization of Grzegorczyk's original metalogical conception. In turn, in [4] and [5] LD was compared with the system marked as "LE" (Logic of Equimeaning), which is the last of Grzegorczyk's metalogical proposals.

The languages of the LD and LE systems are identical; they have four primary connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and synonimity ($\equiv$). The same rules of inference apply in both systems:

$$(\mathrm{MP}_{LE}) \quad \frac{\alpha \equiv \beta, \alpha}{\beta}$$

$$(\wedge_1) \quad \frac{\alpha, \beta}{\alpha \wedge \beta}$$

$$(\wedge_1) \quad \frac{\alpha \wedge \beta}{\alpha, \beta}$$

and rule of substitution. The following are LE system axioms.

$$
\begin{array}{ll}
(\mathrm{Ax0})_{LE} & \neg(p \wedge \neg p) \\
(\mathrm{Ax1})_{LE} & p \equiv p \\
(\mathrm{Ax2})_{LE} & \neg(\neg p \equiv p) \\
(\mathrm{Ax3})_{LE} & p \equiv (p \wedge p) \\
(\mathrm{Ax4})_{LE} & p \equiv (p \vee p) \\
(\mathrm{Ax5})_{LE} & (p \wedge q) \equiv (q \wedge p) \\
(\mathrm{Ax6})_{LE} & (p \vee q) \equiv (q \vee p) \\
(\mathrm{Ax7})_{LE} & (p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r) \\
(\mathrm{Ax8})_{LE} & (p \vee (q \vee r)) \equiv ((p \vee q) \vee r) \\
(\mathrm{Ax9})_{LE} & (p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r)) \\
(\mathrm{Ax10})_{LE} & (p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r)) \\
(\mathrm{Ax11})_{LE} & \neg(p \wedge q) \equiv (\neg p \vee \neg q) \\
(\mathrm{Ax12})_{LE} & \neg(p \vee q) \equiv (\neg p \wedge \neg q) \\
(\mathrm{Ax13})_{LE} & (p \equiv q) \equiv (q \equiv p) \\
(\mathrm{Ax14})_{LE} & (p \equiv q) \equiv (\neg p \equiv \neg q) \\
(\mathrm{Ax15})_{LE} & ((p \equiv q) \wedge (p \equiv r)) \equiv ((p \equiv q) \wedge (q \equiv r)) \\
(\mathrm{Ax16})_{LE} & ((p \equiv q) \wedge (p \wedge r)) \equiv ((p \equiv q) \wedge (q \wedge r)) \\
(\mathrm{Ax17})_{LE} & ((p \equiv q) \wedge (p \vee r)) \equiv ((p \equiv q) \wedge (q \vee r))
\end{array}
$$

The analyses by Joanna Golińska-Pilarek and Taneli Huuskonen (in [5]) show the fact ("$At(\alpha)$" means below the collection of all sentential variables occurring in α):

FACT 1. If '$\alpha \equiv \beta$' $\in$ LE, then $At(\alpha) = At(\beta)$.

PROOF. Let v be any valuation function. Let h^v be the (unique) extension of this function on $\mathcal{L}_{\mathsf{SCI}}$, defined as follows:

(i) $h^v(p) = 1$ iff $v(p) = 1$,
(ii) $h^v(\neg\alpha) = 1$ iff $h^v(\alpha) = 0$,
(iii) $h^v(\alpha \wedge \beta) = 1$ iff $h^v(\alpha) = h^v(\beta) = 1$,
(iv) $h^v(\alpha \vee \beta) = 1$ iff $h^v(\alpha) = 1$ or $h^v(\beta) = 1$,
(v) $h^v(\alpha \equiv \beta) = 1$ iff $h^v(\alpha) = h^v(\beta)$ and $At(\alpha) = At(\beta)$.

Let us assume that W is a property (of some formulas) defined as follows: $W(\alpha) =_{df} h^v(\alpha) = 1$, for every valuation v. Taking into account (ii)-(iv), each axiom $(\text{Ax}0)_{LE}$–$(\text{Ax}17)_{LE}$ has the property W. In the addition all primitive rules of the LE system preserve this property. Consequently, the following condition is satisfied:

If $\alpha \in \mathsf{LE}$, then $h^v(\alpha) = 1$ for every $\alpha \in \mathcal{L}_{\mathsf{SCI}}$.

In particular:

If '$\alpha \equiv \beta$' $\in \mathsf{LE}$, then h^v('$\alpha \equiv \beta$') $= 1$,

where directly – by (v) – we obtain the Fact. ■

According to this fact, the LE system implements one of the key assumptions of the principle of analytic equivalence, whose detailed versions were formulated (for the given systems of NFL) in [3] and [11]. In this sense, LE is a system related to the non-Fregean logic of analytic equivalence.

For the LD system, no analogous fact occurs – some of its axioms have the form '$\alpha \equiv \beta$' such that $At(\alpha) \neq At(\beta)$. For this reason (and also considering that Grzegorczyk did not prefer this system in the last period of his activity) LD will not be further examined.

We will call the synonymity connective a connective representing a certain equivalence relation, whose formal properties are determined by the logical systems achieving the main assumptions of Grzegorczyk's project (which will be shortly further discussed). Within the intuitive interpretation of the symbol "$\equiv$" as a synonymity connective, we accept the rule:

(E) '$p \equiv q$' we read: that p means the same as q[2].

The purpose of the remaining part of this paper is to systematize the main assumptions of this project and to compare them with the basis of logic of analytic equivalence, described in [1], [2], [3], and [11][3].

2. The principles of the logic of synonymity

There is no counterpart to the synonymity connective in standard sentential logic. In particular, it is not the connective of material equivalence. In accordance with its meaning, any sentences with the same logical value are equivalent; this also applies to sentences devoid of any contents connection. Therefore, the following sentences are true: "$2+2=4$ *iff* Warsaw is on the Vistula" (Grzegorczyk's example), "$2+2=5$ *iff* there are six empty sets" and so on.

When Grzegorczyk drew attention to this fact, the idea came up to undertake research whose purpose was: (a) to replace classical sentential logic (CL) with the alternative logic of synonymity connective, which is a fragment of CL (plan "A"), or (b) to extend CL with the logic of such a connective (plan "B").

Plan "B" is closer to the idea of NFL (built, as we know, on the basis of CL).

Another, related motive of the work concerning this project was the need for a formalization of the general concept of description as a language text in which some properties or relations are assigned to described objects and that can be formulated using simple sentences in combination with connectives of negation, conjunction, and disjunction. According to Grzegorczyk, "human language primarily serves to create descriptions of reality and logical connectives are its tool" ([9], p. 2; cf. [10], p. 1–4). The philosophical significance of the mentioned connectives is clearly given by the term "Necessary Descriptive Operators" ([10], p. 5).

In turn, the synonymity operator "comprises a reflection of the human capacity of self-knowledge" which allows the transfer of certain semantic

[2]This way of reading was applied by Grzegorczyk. The term "perceptive equivalence" appears in [7] (to describe this connective). Later, Grzegorczyk also used the term "descriptive equivalence".

[3]A characterization of the philosophical and metalogical assumptions of Grzegorczyk's project is also included in the initial excerpts of [5] and [6].

information on the level of the object language ([8], p. 5). Consequently, philosophically grounded sentential logic must be "descriptive logic of synonymity". There are four connectives in its vocabulary: $\neg$, $\wedge$, $\vee$, and $\equiv$ ([10], p. 11).

The following postulate summarizes the basic assumption of Grzegorczyk's project.

PRINCIPLE 1 (the weak principle of primitivity of the descriptive connectives and the synonymity connective). Operators of negation, conjunction, disjunction, and synonymity are primitive constants of logic of synonymity.

The *strong version* of this principle states that there are no other primitive constants in the language of sentential logic of synonymity than connectives of negation, conjunction, disjunction and synonymity. This version (towards which Grzegorczyk probably was inclined) will not be applied here.

Grzegorczyk accepted that sentential *logic of synonymity*, in addition to the above principle, should meet two additional principles (the symbol "$\gamma(p/\alpha)$" indicates bellow the formula that results from substituting α for the variable p in the formula γ).

PRINCIPLE 2 (the weak principle of extensionality for the synonymity connective, cf. [7], p. 449 and [5], section 7). If formula '$(\alpha \equiv \beta)$' is thesis of logic of synonymity, the thesis of this logic is the formula $\gamma(p/\alpha) \equiv \gamma(p/\beta)$.

The above postulate is distinguished from the strong principle of extensionality (for the synonymity connective in a given system), claiming that the thesis of the system is the following formula:

$$(\alpha \equiv \beta) \to (\gamma(p/\alpha) \equiv \gamma(p/\beta))$$

We can easily notice that in any system with the strong principle of extensionality (and the Modus Ponens), the weak principle of extensionality is derived.

FACT 2. ([5], Theorem 7.6). The weak principle of extensionality does not hold for LE[4].

[4]However, this principle holds for LD ([5], Theorem 7.5).

It turns out that LE – the last of the logical systems proposed by Grzegorczyk – does not meet the requirements he sets.

Because of that the principle of extensionality is usually used in the proofs of the normal form theorems (e.g. for CL), LE is unlikely to meet the next adequacy postulate formulated by Grzegorczyk:

Principle 3 (theorem on the maximum distributive form of description, cf. [10], p. 15). For any $\alpha \in \mathsf{CL}$, formulated only with the help of connectives of negation, conjunction and disjunction, there is a formula β such that the following conditions are satisfied:

(i) '$\alpha \equiv \beta$' is a thesis of logic of synonymity,
(ii) β has the disjunctive normal form.

It is worth noting that in his last notes (cf. [10]), lectures and correspondence concerning his project, Grzegorczyk often expressed the belief that the theorem on the maximum distributive form of description is a key test for the adequacy of logic of synonymity.

3. The principle of the truth nature of descriptions

Plan "A" – thus the project of constructing a certain alternative logic in reference to classical logic - relies on seeking a system that does not require the so-called paradoxical theorems of CL. Grzegorczyk included among these the formulas:

(a) $(p \leftrightarrow q) \vee (p \leftrightarrow r) \vee (q \leftrightarrow r)$[5],
(b) $q \vee (p \vee \neg p)$[6].

However, attempts to satisfactorily implement this plan faced both significant formal and philosophical difficulties. It turned out, for example, that (b) is a thesis of LD. In addition, many other laws specific to CL (with

[5] "[Its] sense can be expressed (in metalogic) in the words: of any three sentences, two must always be logically equivalent to each other. We can say that this thesis discredits classical logic, as contrary to the common sense of the diversity of content (or sense) of human speech, which Roman Suszko once emphasized. It is obvious that we can not only indicate three sentences, the contents of which have no connection with each other, but we can construct as many sentences as we want and guessing their mutual entailment would be absurd." ([10], p. 9-10).

[6] This example was indicated by Grzegorczyk in a private correspondence (in 2013).

the law of contradiction and the law of excluded middle at the front) are generated by this logic.

The concept of description was defined by Grzegorczyk relatively independently from the concept of synonymity. According to the above, it is possible to determine the meaning of Necessary Descriptive Operators in a system where the only constants are these operators. This raises the question: what would this system be like? The following postulate provides a totally natural response to this question.

PRINCIPLE 4 (the principle of the truth nature of descriptions). The descriptive meaning of negation, conjunction and disjunction connectives is fully determined by CL.

At least two reasons favor supporting this postulate.

First, CL determines the purely truth meaning of these connectives. This meaning is best described by the following way of reading them: "it is not true that ..." ($\neg$), "it is true that ... and it is true that ..." ($\wedge$) and "at least one of two things is true: ... or ..." ($\vee$). Every well-defined description is true or false, that is it has one of two logical values. CL provides the simplest formalization of the concept of description, which takes into account these basic epistemic values.

Second, CL determines the Boolean system of meanings of the considered operators, which is standard in modern logic, mathematics and computer science. According to the commonly used in the sciences "principle of conservatism", the rejection of this standard would require strong justification. Such a justification is not evident in the philosophical context of the considered research project.

In particular, the strength of the arguments referring to the "paradoxical" statements such as (a) and (b) significantly weaken when we give them an appropriate reinterpretation according to their descriptive nature: ad (a) of any three sentences, two always have the same truth value; ad (b) at least one of two things is true: q or $p \vee \neg p$. In a similar way, we can minimize the counterintuitive sounding sentences (quoted at the beginning of the previous section) with their truth-value reinterpretation: "It is true that $2+2=4$ *iff* it is true that Warsaw is on the Vistula", "That $2+2=5$ is true *iff* it is true that there are six empty sets", and so on.

Grzegorczyk was sometimes inclined to recognize the principle of the truth nature of descriptions. This assumption clearly proves that: "in the area of ¬, ∧ and ∨, quite important due to computer technology, Perceptive Logic coincides with the classical logic" ([7], p. 450; the term "Perceptive Logic" is used there to denote the logic of synonymity), as well as the opinion that the operators of negation, conjunction and disjunction are clearly defined within CL (ibid., p. 446)[7].

4. AE and logic of synonymity

According to the findings of [2] (p. 84) and [3], we denote by the symbol "AE" (the smallest system of logic of analytical equivalence) which is the smallest NFL system (i.e. system of the non-Fregean logic) closed under the restricted quasi-Fregean rule:

(RQF) $\dfrac{\alpha \leftrightarrow \beta}{\alpha \equiv \beta}$ provided that $At(\alpha) = At(\beta)$[8].

The following basic fact occurs for the AE system.

Fact 3 (the analytic equivalence principle for AE, [3]). The following equivalence is satisfied, for any $\alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}$:

'$\alpha \equiv \beta$' $\in$ AE iff '$\alpha \leftrightarrow \beta$' $\in$ AE and $At(\alpha) = At(\beta)$.

Theorem 1. AE *meets the principle 1-4.*

Proof. Ad 1 and ad 4: obvious. Ad 2: the strong principle of extensionality is a derived rule in SCI. All the more, its weak version is derived rule in AE. Ad 3: according to the disjunctive normal form theorem for classical logic, for any formula α of the language of this logic, there is a formula β of this form such that '$\alpha \leftrightarrow \beta$' $\in$ CL and α and β contain the same variables. CL is a subsystem of SCI. Thus, under the RQF rule, the formula '$\alpha \equiv \beta$' is the thesis of AE. ■

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[7] This opinion was also expressed, inter alia, in the description of this research project: "In this project, we accept four basic connectives: three classical (underline A.B.) of negation, conjunction and disjunction, and one new: the descriptive equivalence (aka synonymity)." ([8], p. 5). The philosophical characterization of these operators given in [10] (p. 3-4) is fully consistent with the truth-value characterization. What's more, [9] presented an outline of the matrix semantics for the so-called Extended Boolean Logic, that is, for a certain CL extension by synonymity connective (this extension includes a certain matrix for this connective).

[8] "$At(\alpha)$" means the collection of all sentential variables occurring in α.

In accord with the above theorem, the logic of analytical equivalence is a version of the logic of synonymity according to Grzegorczyk.

The following theorem corresponds with this conclusion.

THEOREM 2. LE *is a subsystem of* AE.

PROOF. On the basis of the relevant theses of CL and the RQF rule, we conclude that all the substitutions of $(\mathrm{Ax}0)_{LE}$–$(\mathrm{Ax}12)_{LE}$ formulas are AE theses. It is therefore sufficient to show that substitutions of the remaining axioms are AE theses $(\mathrm{Ax}13)_{LE}$–$(\mathrm{Ax}17)_{LE}$, that is:

(a) $\alpha \equiv \beta) \equiv (\beta \equiv \alpha)$,
(b) $(\alpha \equiv \beta) \equiv (\neg\alpha \equiv \neg\beta)$,
(c) $((\alpha \equiv \beta) \wedge (\alpha \equiv \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \equiv \gamma))$,
(d) $((\alpha \equiv \beta) \wedge (\alpha \wedge \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \wedge \gamma))$,
(e) $((\alpha \equiv \beta) \wedge (\alpha \vee \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \vee \gamma))$.

1.	$((\alpha \equiv \beta) \leftrightarrow (\beta \equiv \alpha)$	SCI
2.	$\boldsymbol{((\alpha \equiv \beta) \equiv (\beta \equiv \alpha)}$	1, RQF
3.	$(\neg\alpha \equiv \neg\beta) \to ((\neg\neg\alpha \equiv \neg\neg\beta)$	SCI
4.	$\neg\neg\beta \equiv \beta$	CL, RQF
5.	$(\neg\alpha \equiv \neg\beta) \to (\neg\neg\alpha \equiv \beta)$	3, 4, SCI, CL
6.	$\neg\neg\alpha \equiv \alpha$	CL, RQF
7.	$(\neg\alpha \equiv \neg\beta) \to (\alpha \equiv \beta)$	5, 6, SCI, CL
8.	$(\alpha \equiv \beta) \leftrightarrow (\neg\alpha \equiv \neg\beta)$	SCI, 7, Ext
9.	$\boldsymbol{(\alpha \equiv \beta) \equiv (\neg\alpha \equiv \neg\beta)}$	8, RQF
10.	$((\alpha \equiv \beta) \wedge (\alpha \equiv \gamma)) \to ((\alpha \equiv \beta) \wedge (\beta \equiv \gamma))$	CL, SCI
11	$((\alpha \equiv \beta) \wedge (\beta \equiv \gamma)) \to ((\alpha \equiv \beta) \wedge (\alpha \equiv \gamma))$	CL, SCI
12.	$\boldsymbol{((\alpha \equiv \beta) \wedge (\alpha \equiv \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \equiv \gamma))}$	10, 11, CL, RQF
13.	$((\alpha \equiv \beta) \wedge (\alpha \equiv \gamma)) \leftrightarrow ((\alpha \equiv \beta) \wedge (\beta \wedge \gamma))$	CL, SCI
14.	$\boldsymbol{((\alpha \equiv \beta) \wedge (\alpha \wedge \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \wedge \gamma))}$	13, RQF
15.	$(\alpha \equiv \beta) \to ((\alpha \vee \gamma)) \to (\beta \vee \gamma))$	SCI, CL
16.	$(\alpha \equiv \beta) \wedge (\alpha \vee \gamma) \to (\alpha \equiv \beta) \wedge (\beta \vee \gamma))$	15, CL
17.	$(\alpha \equiv \beta) \wedge (\beta \vee \gamma)) \to (\alpha \equiv \beta) \wedge (\alpha \vee \gamma))$	CL, SCI
18.	$(\alpha \equiv \beta) \wedge (\alpha \vee \gamma) \leftrightarrow (\alpha \equiv \beta) \wedge (\beta \vee \gamma)$	16, 17, CL
19.	$\boldsymbol{((\alpha \equiv \beta) \wedge (\alpha \vee \gamma)) \equiv ((\alpha \equiv \beta) \wedge (\beta \vee \gamma))}$	18, RQF

Rules $(\wedge_1)$ and $(\wedge_2)$ are derived in CL and the $\mathrm{MP}_{\mathsf{LE}}$ rule is derived in SCI. Thus, all the rules of LE apply to AE. ■

Taking into consideration that LE does not meet some Grzegorczyk's principles (in accord to Fact 2), the following conclusion may seem to be interesting.

THEOREM 3. AE *is the smallest* NFL *system* S *containing* LE *that satisfies Grzegorczyk's principles 1-4 and the analytic equivalence principle:*

$$\text{`}\alpha \equiv \beta\text{'} \in S \textit{ iff } \text{`}\alpha \leftrightarrow \beta\text{'} \in S \textit{ and } At(\alpha) = At(\beta).$$

PROOF. From Theorem 1, 2, Fact 3, and the fact that AE is the smallest NFL system closed under the (RQF) rule. ■

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References

[1] A. Biłat, *Zasada Wittgensteina a logika niefregowska* (in Polish: Wittgenstein's principle versus non-Fregean logic), [in:] M. Omyła (ed.), **Szkice z semantyki i ontologii sytuacji**, BMS series, Warszawa 1991, pp. 63–68.
[2] A. Biłat, *O pewnym kryterium tożsamości sądów* (in Polish: On a criterion of propositional identity), [in:] W. Krysztofiak, H. Perkowska (eds.), **Szkice z fenomenologii i filozofii analitycznej**, Wyd. Naukowe USz, Szczecin 1993, pp. 81–85.
[3] A. Biłat, *Non-fregean logics of analytic equivalence (I)*, this **Bulletin**.
[4] J. Golińska-Pilarek, *On the minimal non-Fregean Grzegorczyk's logic*, submitted to **Studia Logica**, 2015.
[5] J. Golińska-Pilarek, T. Huuskonen, *Grzegorczyk's non-Fregean logics and their formal properties*, submitted to the monograph **Applications of Formal Philosophy**, series Logic, Argumentation and Reasoning, 2015.
[6] J. Golińska-Pilarek, T. Huuskonen, *Logic of descriptions. A new approach to the foundations of mathematics and science*, [in:] K. Trzęsicki, S. Krajewski, J. Woleński (eds.), **Papers on logic and rationality. Festschrift in honour of Andrzej Grzegorczyk, Studies in Logic, Grammar and Rhetoric** 27/40 (2012), pp. 63–94.

[7] A. Grzegorczyk, *Filozofia logiki i formalna logika niesymplifikacyjna* (in Polish: Philosophy of logic and non-simplification formal logic), **Zagadnienia Naukoznawstwa** 4/190 (2011), pp. 445–450.

[8] A. Grzegorczyk, **Opis projektu badawczego NCN 'Logika językowego doświadczenia'** (in Polish: Description of the NCN research project 'Logic and language experience'), 2011, manuscript

[9] A. Grzegorczyk, **LD od podstaw** (in Polish: LD from the bottom up), 2012, unpublished notes.

[10] A. Grzegorczyk, **Wstęp do logiki opisu** (in Polish: Introduction to the logic of description), 2013, unpublished notes.

[11] M. Nowak, *The logics of analytic equivalence*, **Bulletin of the Section of Logic** 37:3/4 (2008), pp. 265–272.

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