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NON-FREGEAN LOGICS OF ANALYTIC EQUIVALENCE (I)

Abstract

The identity connective is usually interpreted in non-Fregean logic as an operator representing the identity of situations. This interpretation is related to the modal criterion of the identity of sentence correlates, characteristic of the WT system and some stronger systems. However, this connective can also be interpreted in a different way – as an operator representing the identity of propositions. The "propositional" interpretation is in turn associated with the modal-contents criterion of the identity of sentence correlates. This begs the question of whether there is a system of non-Fregean logic, providing an adequate formalization of this criterion. The aim of the paper is to systematize the metalogical and philosophical context of the issue and to point to a system that provides its solution.

1. Introduction

The basic intuitive sense of the specific connective of non-Fregean logic may be explained by the following semantic rule:

'$p \equiv q$' we read: *that p is the same as q.*

For example, if p represents "John is the husband of Jane" and q represents "Jane is the wife of John", then '$p \equiv q$' represents "That John is the husband of Jane is the same as Jane is the wife of John." The most general properties of this connective identity are defined by the smallest (in the sense of inclusion of sets of theses) non-Fregean calculus, or SCI (sentential calculus with identity). As is known, the SCI language ($\mathcal{L}_{\mathsf{SCI}}$) is obtained from the language of the classical sentential logic by adding the primitive symbol "$\equiv$". The consequence relation is defined on this language by the Modus Ponens rule and the following axioms:

(CL) a set of axioms defining classical sentential logic;
(SCI1) $\alpha \equiv \alpha$;
(SCI2) $(\alpha \equiv \beta) \rightarrow (\neg\alpha \equiv \neg\beta)$;
(SCI3$_F$) $(\alpha \equiv \beta) \wedge (\gamma \equiv \delta) \rightarrow ((\alpha F \gamma) \equiv (\beta F \delta))$, for $F \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$;
(SCI4) $(\alpha \equiv \beta) \rightarrow (\alpha \leftrightarrow \beta)$.

Non-Fregean logic in the narrower sense is the set of calculi including the SCI and every calculus which is the result of strengthening the SCI. *Non-Fregean logic in the broader sense* (NFL) is the set of calculi formulated in $\mathcal{L}_{\mathsf{SCI}}$ and containing the set of all SCI theses (of course, these definitions only apply to the sentential fragment of non-Fregean logic).

SCI was constructed by Roman Suszko as the basis for building stronger deductive systems, philosophically interpreted as theories of situations[1]. According to this interpretation:

(S) '$p \equiv q$' we read: *that p is the same situation as q*.

According to Suszko, the smallest calculus being the philosophically adequate theory of situations is the WT system. This theory is the result of strengthening the SCI by the so-called *quasi-Fregean rule*:

(QF) $\dfrac{\alpha \leftrightarrow \beta}{\alpha \equiv \beta}$

Suszko showed (in [12]) that there is a close relationship between the WT and the system of modal logic S4. Let g be a function from $\mathcal{L}_{\mathsf{S4}}$ (more precisely: from the set of $\mathcal{L}_{\mathsf{S4}}$ formulas) to the $\mathcal{L}_{\mathsf{SCI}}$, defined as follows:

$f(\alpha) = \alpha$, if connective "$\Box$" does not occur in α,
$g(`\Box\alpha') = `\alpha \equiv (\alpha \equiv \alpha)'$,

and the remaining recursive conditions for truth-functional connectives leave them unchanged. According to one of Suszko's theorems (concerning the above WT and S4 relationship), for each formula α of the $\mathcal{L}_{\mathsf{S4}}$, the equivalence holds:

[1] "Roman Suszko claimed that non-Fregean logic is primarily a tool for formulating and studying formal aspects of the ontology of situations" ([11], p. 96). Every theory in SCI closed under the substitution was treated by Suszko as a *theory of situations* (ibid.).

(1) $\alpha \in \mathsf{S4}$ iff $g(\alpha) \in \mathsf{WT}$.

In other words, the operator "$\Box$" defined into the WT system:

(2) $\Box\alpha =_{df} \alpha \equiv (\alpha \equiv \alpha)$

represents the specific $\mathsf{S4}$ connective.

One of the theses of the WT system is the formula according to which the situations described by the sentences are identical if and only if these sentences necessarily have the same truth value (i.e. they are *strictly equivalent*):

(WT) $(\alpha \equiv \beta) \leftrightarrow \Box(\alpha \leftrightarrow \beta)$.

The strict equivalence (in the $\mathsf{S4}$ sense) is often interpreted as a relation of mutual entailment. With this interpretation, (WT) expresses in the object language the so-called *Wittgensteins principle*[2].

In accord with (WT), the identity connective in the WT system is indistinguishable from the strict equivalence connective. Consequently, the situations described by sentences are identical when these sentences are strictly equivalent. From now on, we will call this condition the *modal criterion* of situational identity[3].

In recent years, it was pointed out that some NFL calculi can be – and perhaps should be – intuitively interpreted according to the rule (cf. [8], pp. 18–22, [9], p. 265, and [7], p. 71):

(P) '$p \equiv q$' we read: that p is the same proposition as q.

Formal differences between interpretations (S) and (P) result from different criteria used for identifying situations and propositions formulated at the level of their sentence representation. Although the situations are identified by the modal criterion, the propositions are identified with the assistance of the modal-contents criterion: propositions expressed by the sentences are identical if and only if these sentences are strictly equivalent,

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[2] "5.122 If p follows from q, the sense of 'p' is contained in the sense of 'q' [...] 5.141 If p follows from q and q from p, then they are one and the same proposition" ([13]). The term "Wittgenstein's principle" originates from [16]. The term "WT" comes from Wittgenstein's name.

[3] The modal criterion is the assumption of B. Wolniewicz's ontology of situations (cf. [15], p. 20). The main source of inspiration for Suszko when constructing non-Fregean logic was reading Wolniewicz's work on the topic of Wittgenstein's philosophy ([14]).

and furthermore, there is a certain contents connection between them. For example, that $2+2=4$ is the same (abstract) situation as that the empty set is included in each set – but it is not the same proposition: there is no contents connection between them.

We accept the natural condition: two sentences are in a contents connection, if they are composed of exactly the same terms. In a similar way, two formulas of propositional calculus are in this connection, if they contain exactly the same variables.

This raises the question whether there is any NFL calculus providing an adequate formalization of the modal-contents criterion. The aim of this work is to sketch the metalogical and philosophical context of this issue, to systematize its most important assumptions and to indicate a system that meets adequacy conditions of the formulated task.

2. AE as the smallest logic of analytic equivalence

At the beginning of the eighties of the last century, Jon Barwise and John Perry drew attention (in [1] and [2]) to the following possibility of clarifying the modal-contents criterion: sentences express the same proposition ("describe the same situation" in its original terms) when they are logically equivalent and contain the same extra-logical terms. Ryszard Wójcicki proposed (in [17] and [18]) to use this condition in the construction of a system of non-Fregean predicate logic.

A simple application of this criterion in sentential calculus is the following principle: if formulas α and β are logically equivalent and contain the same sentential variables, then α and β represent the same proposition. This principle has become (in [10], [19], and independently in [3], [4]) the main assumption of the idea of the so-called logic of analytical equivalence. The smallest system of this logic, designated (in [4]) by the symbol "AE" is the result of strengthening the SCI by the following rule of inference:

(RQF) $\dfrac{\alpha \leftrightarrow \beta}{\alpha \equiv \beta}$ provided that $At(\alpha) = At(\beta)$[4].

(RQF) will be further called the *restricted quasi-Fregean rule.*

The modal-contents criterion is generally expressed in the AE metasystem by the theorem:

[4] "$At(\alpha)$" means the collection of all sentential variables occurring in α.

Theorem 1 (the analytic equivalence principle for AE). *The following equivalence is satisfied, for any* $\alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}$:

$$\text{‘}\alpha \equiv \beta\text{’} \in \mathsf{AE} \textit{ iff } \text{‘}\alpha \leftrightarrow \beta\text{’} \in \mathsf{AE} \textit{ and } At(\alpha) = At(\beta).$$

Proof. Taking into account (RQF), the proof of the implication from right to left is obvious.

In turn, in virtue of (SCI4) and (MP), we hold that if ‘$\alpha \equiv \beta$’ $\in \mathsf{AE}$, then ‘$\alpha \leftrightarrow \beta$’ $\in \mathsf{AE}$. It is therefore sufficient to show that the following implication holds:

(3) If ‘$\alpha \equiv \beta$’ $\in \mathsf{AE}$, then $At(\alpha) = At(\beta)$.

For this purpose, let us assume that v is any valuation function and h^v is the (unique) extension of this function on the $\mathcal{L}_{\mathsf{SCI}}$ language, defined as follows:

(i) $h^v(p) = 1$ iff $v(p) = 1$,
(ii) $h^v(\neg\alpha) = 1$ iff $h^v(\alpha) = 0$,
(iii) $h^v(\alpha \wedge \beta) = 1$ iff$h^v(\alpha) = h^v(\beta) = 1$,
(iv) $h^v(\alpha \vee \beta) = 1$ iff $h^v(\alpha) = 1$ or $h^v(\beta) = 1$,
(v) $h^v(\alpha \rightarrow \beta) = 1$ *iff* $h^v(\alpha) = 0$ or $h^v(\beta) = 1$,
(vi) $h^v(\alpha \leftrightarrow \beta) = 1$ iff $h^v(\alpha) = h^v(\beta)$,
(vii) $h^v(\alpha \equiv \beta) = 1$ iff $h^v(\alpha) = h^v(\beta)$ & $At(\alpha) = At(\beta)$.

Let us assume that W is a property owned by some formulas defined as follows: $W(\alpha) =_{df} h^v(\alpha) = 1$, for every valuation v. On the one hand, considering (ii)-(vi), each axiom of the (CL) has the property W. Taking into account (vii), in turn, the axioms under schemes (SCI1)-(SCI4) possess it. On the other hand, the two primitive rules of the AE system (i.e., (MP) and (QF)) preserve the property W. Consequently, the following condition is satisfied:

If $\alpha \in \mathsf{AE}$, then $h^v(\alpha) = 1$, for every $\alpha \in \mathcal{L}_{\mathsf{SCI}}$, for every valuation v.

In particular:

If ‘$\alpha \equiv \beta$’ $\in \mathsf{AE}$, then $h^v(\text{‘}\alpha \equiv \beta\text{’}) = 1$,

where directly – by (vii) – we obtain (3). ■

The first segment on the right side of the proved theorem stating a kind of "logical" necessity of equivalence for '$\alpha \leftrightarrow \beta$' (in the sense of necessity of the AE theses) is the metalogical counterpart of the modal criterion. In turn, two formulas possessing exactly the same variables reflect the fact that there is a direct contents connection between sentences represented by these formulas. Thus, the analytic equivalence theorem states the fact that the equality '$\alpha \equiv \beta$' is a logical thesis if and only if α and β are logically equivalent (on the basis of AE), and they represent sentences which are in a contents connection. There is no doubt that this theorem comprises the metalogical counterpart of the modal-contents criterion.

3. The Nowak's logic of analytic equivalence

Is the modal-contents criterion expressible in the $\mathcal{L}_{\mathsf{SCI}}$ (object) language? Analyses in [9] bring us closer to answering this question. Marek Nowak introduced – with the help of the metalogical definition – a connective marked by the symbol "$\|$" (ibid., p. 268):

(4) $\alpha \parallel \beta =_{df} (\alpha \equiv \alpha) \equiv (\beta \equiv \beta)$.

The connective can be – according to the intentions of the author – interpreted as an operator representing the occurrence of the contents connection between sentences represented by the arguments of the operator. Regardless of its formal and semantical analysis (such an analysis is included in Nowak's work), we can justify the accuracy of the interpretation already based on a general and intuitive level. The reasoning is as follows.

According to the modal-contents criterion, the definiens of the above definition is true if and only if the equalities '$(\alpha \equiv \alpha)$' and '$(\beta \equiv \beta)$' are strictly equivalent, and are contents connected. Under the axioms (CL), (SCI1) and the (QF) rule, they are strictly equivalent. Thus, the question of the truthfulness of the definiens boils down to the question of the existence of contents connection between them. Let's accept the rule: for every sentence γ, γ is related in contents with '$\gamma \equiv \gamma$'. Assuming that the relevant relationship (i.e. the contents connection) is transitive and symmetric, we conclude that the definiens is true if and only if sentences α and β are contents connected to each other.

We will call such interpreted symbol "$\|$" the *contents connective.*

Let us note the following fact.

FACT 1. The contents connective is indistinguishable in WT from the verum connective:

(5) '$\alpha \parallel \beta$' $\in \mathsf{WT}$, for every $\alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}$.

PROOF. Taking into consideration (CL) and (SCI1), the following formula is a thesis of WT:

$$(\alpha \equiv \alpha) \leftrightarrow (\beta \equiv \beta), \text{ for any } \alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}.$$

Hence, under (QF), we obtain a formula identical to the definiens of (4).■

This fact shows that WT is not an adequate basis for the NFL system design with the contents connective.

In the cited paper, Nowak presented a certain NFL system, providing an axiomatic characterization of this connective. This system is the result of the extension of the SCI by the list of axioms (its only primitive rule of inference is (MP)):

(N1) $\alpha \parallel \neg\alpha$,
(N2$_f$) $(\alpha f \beta) \parallel (\alpha \equiv \beta)$, where $f \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
(N3) $(\alpha \parallel \beta) \wedge (\gamma \parallel \delta) \rightarrow ((\alpha \equiv \gamma) \parallel (\beta \equiv \delta))$,
(N4) $(\alpha \equiv \alpha) \parallel \alpha$,
(N5) $(\alpha \equiv \beta) \parallel (\beta \equiv \alpha)$,
(N6) $(\alpha \equiv (\beta \equiv \gamma)) \parallel ((\alpha \equiv \beta) \equiv \gamma)$,
(N7) $(\alpha \equiv \beta) \rightarrow ((\alpha \leftrightarrow \beta) \wedge (\alpha \parallel \beta))$,
(N8) $((\alpha \leftrightarrow \beta) \wedge (\alpha \parallel \beta)) \rightarrow (\alpha \equiv \beta)$.

In that paper, the proofs of the completeness theorem and the semantic version of the analytic equivalence theorem were given. According to them the following version of the analytic equivalence theorem holds:

THEOREM 2 (the analytic equivalence principle for Nowak's system; cf. [9], p. 269). *'$\alpha \equiv \beta$' is a thesis of Nowak's system iff '$\alpha \leftrightarrow \beta$' is a thesis of Nowak's system and $At(\alpha) = At(\beta)$, for any $\alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}$.*

Thus, this system meets the metalogical version of the modal-contents criterion for propositional identity. Does it also meet its objectual version? More specifically: is there a thesis of Nowak's system which adequately expresses this criterion?

The answer to this question is negative. Actually, the thesis of the system which is of interest to us is the following formula (which is quite easy to check by using semantical techniques introduced by Nowak):

(6) $(p \equiv q) \leftrightarrow (\Box(p \leftrightarrow q) \wedge (p \parallel q))$,

However, the symbol "$\Box$" (introduced by definition (2)) does not represent in this system the necessity connective but the assertion connective. It results from the fact:

FACT 2. The thesis of Nowak's system is the following formula:

$(*)$ $\Box\alpha \equiv \alpha$.

PROOF. The formula '$\Box\alpha \rightarrow \alpha$' is the thesis of SCI (among others, under (SCI1) and (SCI4)). The reverse implication is obtained by substituting $\beta/$'$\alpha \equiv \alpha$' in the N8 (and by using CL and N4). As a result, we obtain the equivalence:

$\Box\alpha \leftrightarrow \alpha$.

Hence, based on the analytical equivalence theorem for the considered system, we derive the conclusion that the formula $(*)$ is its thesis. ■

This fact proves that Nowak's system is not an adequate foundation for constructing such a theory in SCI, in which the modal-contents criterion of propositional identity would be expressible.

The conjunction of (N7) and (N8) directly provides the criterion of identity of propositions, according to which the propositions expressed by sentences are identical if and only if these sentences have the same truth value, and there is a contents connection between them, formally:

(N7,8) $(p \equiv q) \leftrightarrow ((p \leftrightarrow q) \wedge (p \parallel q))$.

At this point in our considerations, the following doubt could occur: since we have a complete system providing us with the "truth-contents" criterion of propositional identity (N7,8), is it not enough to accept this criterion (abandoning the modal-contents criterion)?

There are counterexamples for (N7,8). Let us replace $q/$'$p \equiv p$' in (N7,8). Hence and from N4 (and CL) we can derive:

$p \rightarrow (p \equiv (p \equiv p))$.

Let p represents the sentence "Warsaw is the capitol of Poland". This sentence is true, so we obtain:

(a) *Warsaw is the capital of Poland*

is the same proposition as

(b) *that Warsaw is the capital of Poland is the same proposition as Warsaw is the capital of Poland.*

But these sentences do not express the same propositions: (a) is an empirical truth and (b) is a logical one (as we can see, it is a propositional version of the Morning Star Paradox).

The troubles increase, if we extend (in a natural way) the SCI language to the language containing individual names and predicates. Let us replace in (N8): $p/$'$Rab \wedge Qcd$' and $q/$'$Rcb \wedge Qab$' (e.g. $Rab \wedge Qcd$ may represent "John is older than Jane and Peter is taller than she is," and $Rcb \wedge Qab$ may represent "Peter is older than Jane and John is taller than she is."). It is possible to have a situation in which both of these statements are true. In addition, there is a contents connection between them (they are composed of exactly the same terms). Thus, according to the (N7,8), these sentence should express the same proposition; but they do not express it.

As we can see, the modal-contents criterion cannot be replaced – it is in danger of being in conflict with our basic intuitions – through the "truth-contents" criterion.

4. The problem of strengthening of the logic

In order to compare the two recently considered systems of philosophical logic, we note the following fact:

FACT 3. (N1), (N2), (N4), (N5), (N6), (N7) $\in$ AE.

PROOF. Ad (N1), (N2), (N4)-(N6): the proof is obvious, taking into account the appropriate laws of SCI and the rule (RQF). Ad (N7): from (SCI4) and (SCI3$_{\equiv}$). ■

In contrast to Nowak's system, the necessity connective is distinguishable in AE from the classical assertion connective.

FACT 4. '$\Box\alpha \leftrightarrow \alpha$' $\notin$ AE.

Proof. The basis of the proof is the mentioned (in section 1) Suszko's theorem on the interpretability of WT in S4. Taking into account the fact that the RQF rule is weaker than QF (and the fact that the set of AE axioms is identical to the set of SCI axioms), we obtain the conclusion that all AE theses are WT theses. As is known, the formula '$\Box\alpha \leftrightarrow \alpha$' is not an S4 thesis. Thus, the corresponding (via the function g defined in section 1) formula '$(\alpha \equiv (\alpha \equiv \alpha)) \leftrightarrow \alpha$' is also not a WT thesis and – what is more – it is not an AE thesis. ■

As we can see, both the contents connective and the necessity connective are nontrivially definable in the AE system. Thus, it seems that the AE or its strengthening may provide convenient tools to formalize the modal-contents criterion. This begs the question of what such strengthening looks like. Before analyzing this issue, we turn our attention to a fundamental question that might arise in this context.

5. The Golińska-Pilarek's question

One of the thesis of the SCI is (taking into account (CL) and (SCI4)) the following formula:

(7) $((p \equiv q) \wedge p) \leftrightarrow ((p \equiv q) \wedge q)$.

Hence (in virtue of (RQF)) we obtain as a thesis of the AE:

(8) $((p \equiv q) \wedge p) \equiv ((p \equiv q) \wedge q)$.

The comments [5] and paper [6] expressed some doubt associated with an intuitive interpretation of the theses (8). Assuming that p represents "$2 + 2 = 4$" and q represents "$2 + 2 = 5$", we obtain the sentence:

(a) $(2 + 2 = 4 \equiv 2 + 2 = 5)$ and $2 + 2 = 4$

is the same proposition as:

(b) $(2 + 2 = 4 \equiv 2 + 2 = 5)$ and $2 + 2 = 5$.

Are not this consequence paradoxical and, consequently, is the AE system not an overly strong basis for formalizing the modal-contents criterion?

The source of this doubt is the belief that two complex sentences, if they are synonymous and have the same structure, they also have the same "intensional structure" in which the corresponding simple sentences are synonymous. In the case of (a) and (b), this assumption takes the form:

$$(*) \quad ((r \wedge p) \equiv (r \wedge q)) \rightarrow (p \equiv q).$$

It is easy to point to counterexamples for $(*)$. Let us make a substitution: $r/$'$p \wedge q$'. Following a few simple transformations of $(*)$, we obtain the formula:

$$(**) \quad ((p \wedge q) \equiv (p \wedge q)) \rightarrow (p \equiv q).$$

The implication $(**)$ is obviously false: it directly leads to the consequence that all sentences are synonymous. A general intuition about the "intensional isomorphism" of complex sentences at the same time turns out to be inaccurate. Nor are there any particular reasons to accept a similar intuition in relation to sentences such as (a) and (b).

The above analysis may raise the question of whether there is any equivalence condition analogous to $(*)$, but weaker. It turns out that a positive answer to this question coincides with the main theme of our research.

6. The AE^+ system

Let us suppose that r represents a sentence that is necessarily true, and p and q are contents connected with each other. Under these assumptions, the thesis $(*)$ does not raise intuitive objections. Therefore, it seems that this kind of a modification of this thesis may be a new axiom for the considered strengthening of the AE system.

Let AE^+ be the least system containing SCI, the axiom (N3) (i.e. Nowak's axiom, see section 3) and the formula:

$$(+) \quad (\Box\alpha \wedge (\beta \parallel \gamma)) \wedge ((\alpha \wedge \beta) \equiv (\alpha \wedge \gamma)) \rightarrow (\beta \equiv \gamma)$$

closed on Modus Ponens and (RQF). Obviously, AE is a subsystem of AE^+.

THEOREM 3 (the analytic equivalence theorem for AE^+). *For any $\alpha, \beta \in \mathcal{L}_{\mathsf{SCI}}$:*

(9) '$\alpha \equiv \beta$' $\in \mathsf{AE}^+$ *iff* '$\alpha \leftrightarrow \beta$' $\in \mathsf{AE}^+$ *and* $At(\alpha) = At(\beta)$.

PROOF. The proof is analogous to the proof of Theorem 1. ■

THEOREM 4. AE^+ *is a subsystem of* WT.

Proof. The essence of the proof is to demonstrate the relevance of two conditions: (a) the formula (N3) is the WT thesis, and (b) the formula $(+)$ is the WT thesis.

Ad (a): the proof is obvious, taking into account Fact 1.

Ad (b): According to Fact 1, the proof of this condition boils down to deriving this formula from system WT:

$(+')$ $\quad (\Box\alpha \wedge ((\alpha \wedge \beta) \equiv (\alpha \wedge \gamma))) \to (\beta \equiv \gamma).$

To this end, it suffices to note that the formula:

$(+'')$ $\quad (\Box\alpha \wedge \Box((\alpha \wedge \beta) \leftrightarrow (\alpha \wedge \gamma))) \to \Box(\beta \leftrightarrow \gamma).$

is a tautology of the $\mathsf{S4}$ system. By virtue of Suszko's theorem, WT is interpretable in $\mathsf{S4}$ due to the translation of f (from $\mathcal{L}_{\mathsf{SCI}}$ into $\mathcal{L}_{\mathsf{S4}}$), whose specific condition is the equality:

$f(\text{‘}(\alpha \equiv \beta)\text{’}) = \text{‘}\Box(\alpha \leftrightarrow \beta)\text{’}.$

More specifically, in accord with the Suszko's theorem, the following equivalence occurs ([12], cf. [11], p. 108):

(10) $\quad \alpha \in \mathsf{WT}$ iff $f(\alpha) \in \mathsf{S4}$.

Using this equivalence in the formula $(+'')$, we conclude that $(+')$ is the thesis of WT. ■

Theorem 5.
(a) ‘$\Box\alpha \leftrightarrow \alpha$’ $\notin \mathsf{AE}^+$.
(b) ‘$(\alpha \equiv \beta) \leftrightarrow (\alpha \leftrightarrow \beta)$’ $\notin \mathsf{AE}^+$.
(c) If $At(\alpha) \notin At(\beta)$, then ‘$\alpha \parallel \beta$’ $\notin \mathsf{AE}^+$.

Proof. None of the formulas (a) and (b) is a counterpart of the $\mathsf{S4}$ thesis, therefore, none of them is - by virtue of Suszko's theorems - the WT thesis. What is more, taking into account Theorem 4, none of them is thesis AE^+. The condition (c) is derived from the analytical equivalence theorem for AE^+ (i.e. Theorem 3) by substitutions: $\alpha/$‘$\alpha \equiv \alpha$’, $\beta/$‘$\beta \equiv \beta$’. ■

7. Modal characterization of AE^+

In the final section, we will derive two theorems characterizing some modal features of the AE^+ system.

THEOREM 6. *If g is the function defined as in section 1, then the following condition is satisfied:*

(11) $`\alpha' \in \mathsf{S4}$ iff $g(\alpha) \in \mathsf{AE}^+$.

PROOF. The implication from right to left follows from Theorem 4 and Suszko's theorem:

$`\alpha' \in \mathsf{S4}$ iff $g(\alpha) \in \mathsf{WT}$.

In order to obtain the reverse implication, we will derive in AE^+ all counterparts of the specific $\mathsf{S4}$ axioms and a counterpart of Gödel's rule. The proofs will often use the rule of extensionality (standardly derivable in SCI based on (CL), (SCI2) and ($\mathrm{SCI3}_F$)):

(Ext) $\dfrac{(\alpha \equiv \beta) \to \gamma}{(\alpha \equiv \beta) \to \gamma(\alpha//\beta)}$

In the remainder of the proof, we will derive further counterparts of $\mathsf{S4}$ axioms:

(a) $\Box(\alpha \to \beta) \wedge \Box\alpha \to \Box\beta$,
(b) $\Box\alpha \to \alpha$,
(c) $\Box\alpha \to \Box\Box\alpha$,

and the counterpart of Gödel's rule:

(G) $\dfrac{\alpha}{\Box\alpha}$

Ad (a)
1. $((\alpha\equiv(\alpha\equiv\alpha))\wedge(\beta\,\|(\beta\equiv\beta)))\wedge((\alpha\wedge\beta)\equiv(\alpha\wedge(\beta\equiv\beta)))\to(\beta\equiv(\beta\equiv\beta))$ (+)
2. $\beta \parallel (\beta \equiv \beta)$ SCI1, CL, RQF
3. $((\alpha \equiv (\alpha \equiv \alpha)) \to [((\alpha \wedge \beta) \equiv (\alpha \wedge (\beta \equiv \beta))) \to \Box\beta]$ 1, 2, CL, Def.
4. $((\alpha\equiv(\alpha\equiv\alpha))\to[(((\alpha\equiv\alpha)\wedge\beta)\equiv((\alpha \equiv \alpha)\wedge(\beta \equiv \beta))) \to \Box\beta]$ 3, Ext
5. $((\alpha \equiv \alpha) \wedge \beta) \equiv ((\alpha \equiv \alpha) \to \beta)$ SCI1, CL, RQF
6. $((\alpha \equiv \alpha) \wedge (\beta \equiv \beta)) \equiv ((\alpha \to \beta) \equiv (\alpha \to \beta))$ SCI1, CL, RQF
7. $((\alpha \equiv (\alpha \equiv \alpha)) \to [(((\alpha \equiv \alpha) \to \beta) \equiv ((\alpha \to \beta) \equiv (\alpha \to \beta))) \to \Box\beta]$ 4, 5, 6, Ext, CL
8. $((\alpha \equiv (\alpha \equiv \alpha)) \to [((\alpha \to \beta) \equiv ((\alpha \to \beta) \equiv (\alpha \to \beta))) \to \Box\beta]$ 7, Ext
9. $\Box\alpha \to (\Box(\alpha \to \beta) \to \Box\beta)$ 8, Def.
10. $\Box(\alpha \to \beta) \wedge \Box\alpha \to \Box\beta$ 9, CL

Ad (b)

1. $(\alpha \equiv (\alpha \equiv \alpha)) \to ((\alpha \equiv \alpha) \to \alpha)$ SCI4, CL
2. $(\alpha \equiv \alpha) \to (\Box\alpha \to \alpha)$ 1, CL, Def.
3. $\Box\alpha \to \alpha$ 2, SCI1

Ad (c)

1. $(\alpha \equiv \alpha) \equiv ((\alpha \equiv \alpha) \equiv (\alpha \equiv \alpha))$ SCI1, CL, RQF
2. $(\alpha \equiv (\alpha \equiv \alpha)) \to [(\alpha \equiv \alpha) \equiv ((\alpha \equiv \alpha) \equiv (\alpha \equiv \alpha)))]$ 1, CL
3. $(\alpha \equiv (\alpha \equiv \alpha)) \to [(\alpha \equiv (\alpha \equiv \alpha)) \equiv ((\alpha \equiv (\alpha \equiv \alpha)) \equiv (\alpha \equiv (\alpha)))]$ 2, Ext
4. $\Box\alpha \to \Box\Box\alpha$ 3, Def.

Ad (G) Let α be a thesis of AE^+. Hence and from (SCI1), (CL) and (RQF), formula '$\alpha \equiv (\alpha \equiv \alpha)$' is a thesis of AE^+. ■

As expected, the theorem concerning the expressibility of the modal-contents criterion in AE^+ applies here.

Theorem 7. '$(\alpha \equiv \beta) \equiv (\Box(\alpha \leftrightarrow \beta) \land (\alpha \parallel \beta))$' $\in \mathsf{AE}^+$.

Proof.

1. $(\alpha \equiv \beta) \to [(\alpha \leftrightarrow \alpha) \equiv ((\alpha \leftrightarrow \alpha) \equiv (\alpha \leftrightarrow \alpha))]$ CL, SCI1, RQF
2. $(\alpha \equiv \beta) \to [(\alpha \leftrightarrow \beta) \equiv ((\alpha \leftrightarrow \beta) \equiv (\alpha \leftrightarrow \beta) \equiv (\alpha \leftrightarrow \beta))]$ 1, Ext
3. $(\alpha \equiv \beta) \to \Box(\alpha \leftrightarrow \beta)$ 2, Def.
4. $(\alpha \equiv \beta) \to ((\alpha \equiv \alpha) \equiv (\alpha \equiv \alpha))$ SCI1, CL
5. $(\alpha \equiv \beta) \to ((\alpha \equiv \alpha) \equiv (\beta \equiv \beta))$ 4, Ext
6. $(\alpha \equiv \beta) \to (\alpha \parallel \beta)$ 5, Def.
7. $(\Box(\alpha \leftrightarrow \beta) \land (\alpha \parallel \beta)) \land [((\alpha \leftrightarrow \beta) \land \alpha) \equiv ((\alpha \leftrightarrow \beta) \land \beta)] \to (\alpha \equiv \beta)$ (+)
8. $((\alpha \leftrightarrow \beta) \land \alpha) \equiv ((\alpha \leftrightarrow \beta) \land \beta)$ CL, RQF
9. $\Box(\alpha \leftrightarrow \beta) \land (\alpha \parallel \beta) \to (\alpha \equiv \beta)$ 7, 8, CL
10. $(\alpha \equiv \beta) \leftrightarrow (\Box(\alpha \leftrightarrow \beta) \land (\alpha \parallel \beta))$ 3, 6, 9, CL
11. $(\alpha \equiv \beta) \equiv (\Box(\alpha \leftrightarrow \beta) \land (\alpha \parallel \beta))$ 10, RQF

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