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ON SOME APPLIED FIRST-ORDER THEORIES WHICH CAN BE REPRESENTED BY DEFINITIONS

Abstract

In the paper we formulate a sufficient criterion in order for the first order theory with finite set of axioms to be represented by definitions in predicate calculus. We prove the corresponding theorem. According to this criterion such theories as the theory of equivalence relation, the theory of partial order and many theories based on the equality relation with finite set of functional and predicate symbols are represented by definitions in the first-order predicate calculus without equality.

Keywords: theory, definition, predicate calculus

The study was inspired by the work of my teacher, V. Smirnov*.

We assume that the language of first-order predicate calculus is defined in the standard way as the set of terms and formulas over signature Σ, which consists of nonlogical relational and functional symbols. We write $L(\Sigma)$ for the first-order language with signature Σ. Models are pairs $M = \langle D, I \rangle$, where D is a non-empty set of individuals, and I is an interpretation of function and predicate symbols in the domain D. The relation *"formula A is true in the model M"* is defined as usually and is written as $M \models A$.

A first-order theory in a language $L(\Sigma)$ is a set of logical axioms and non-logical postulates closed by derivability. Predicate calculus is the first-order theory with the empty set of non-logical postulates. We consider equality axioms as non-logical postulates.

*V. A. Smirnov, *Logical Relations between Theories*, **Synthese** 66/1 (1986), pp. 71–87.

1. Defining new predicate symbols

We can extend the language of a theory by definitions of new predicate symbols, which have the following form:

$$P(x_1, .., x_n) \equiv A(x_1, .., x_n)$$

The definition must satisfy the conditions:

1. P is a new symbol.
2. The variables $x_1, .., x_n$ are pairwise distinct.
3. The formula $A(x_1, .., x_n)$ does not contain the predicate symbol P.
4. The set of free variables of $A(x_1, .., x_n)$ is included into $\{x_1, .., x_n\}$.

In the language of the first order predicate calculus, we can define the universal n-ary predicate U^n by the following definition:

(DU) $\quad U^n x_1, .., x_n \equiv Px_1 \vee \neg Px_1$

The definition allows us to prove $DU \vdash \forall x_1..x_n U^n x_1, .., x_n$.

This example is interesting because in the right part of the definition we use an arbitrary predicate symbol of the language of the first order predicate calculus. As another example, we can give a definition of a symmetric relation. Let B be an arbitrary predicate symbol of the language. We accept the following definition:

(DS_1) $\quad S_1 xy \equiv \forall uv(Buv \supset Bvu) \supset Bxy$

Let us show that $DS_1 \vdash \forall xy(S_1 xy \supset S_1 yx)$.

1. $S_1 xy$ - hyp
2. $\forall uv(Buv \supset Bvu) \supset Bxy$ - from 1 by DS_1
3. $\forall uv(Buv \supset Bvu)$ - hyp
4. Bxy - from 2, 3
5. $Bxy \supset Byx$ - from 3
6. Byx - from 4, 5
7. $\forall uv(Buv \supset Bvu) \supset Byx$ - from 3-6
8. $S_1 yx$ - from 7 by DS_1
9. $S_1 xy \supset S_1 yx$ - from 1-8

There is another way to define a symmetric relation.

$(DS_2) \qquad S_2xy \equiv \forall uv(Buv \supset Bvu) \& Bxy$

We can show that $DS_2 \vdash \forall xy(S_2xy \supset S_2yx)$.

1. S_2xy	- hyp
2. $\forall uv(Buv \supset Bvu) \& Bxy$	- from 1 by DS_2
3. $\forall uv(Buv \supset Bvu)$	- from 2
4. Bxy	- from 2
5. $Bxy \supset Byx$	- from 3
6. Byx	- from 4, 5
7. $\forall uv(Buv \supset Bvu) \& Byx$	- from 3, 6
8. S_2yx	- from 7 by DS_2
9. $S_2xy \supset S_2yx$	- from 1-8

These examples motivate us to find the criterion for definability of non-logical predicates in predicate calculus.

DEFINITION 1. Theory T_2 in a language $L(\Sigma_2)$ is a conservative extension of T_1 in a language $L(\Sigma_1)$, if and only if the following conditions are met:

1. $L(\Sigma_1) \subseteq L(\Sigma_2)$.
2. If $A \in L(\Sigma_1)$ and $T_1 \vdash A$, then $T_2 \vdash A$.
3. If $A \in L(\Sigma_1)$ and $T_2 \vdash A$, then $T_1 \vdash A$.

2. Theorem on definitional representation

We need to define function π, which translates formulas of the first-order theories into formulas of the propositional logic.

DEFINITION 2.

1. $\pi(P(t_1, .., t_n)) = P$.
2. $\pi(\neg A) = \neg \pi(A)$.
3. $\pi(A \bigtriangledown B) = \pi(A) \bigtriangledown \pi(B)$, where $\bigtriangledown \in \{\&, \vee, \supset, \equiv\}$.
4. $\pi(\Sigma x A) = \pi(A)$, where $\Sigma \in \{\forall, \exists\}$.

Now we are ready to formulate and to prove our theorem.

THEOREM. *Let T be a first-order theory in a language $L(\Sigma)$ with a finite set of closed non-logical postulates $Ax = \{A_1, .., A_k\}$. If the set of formulas $\{\pi(A_1), .., \pi(A_k)\}$ is logically consistent, then there are a signature Σ' disjoint from Σ, and definitions DT of the relational symbols of Σ by formulas* *of* $L(\Sigma')$*, such that in the language* $L(\Sigma \cup \Sigma')$*,* DT *is conservative extension of* T*.*

PROOF. Let $\{P_1, .., P_m\}$ be the set of predicate symbols, which occur in non-logical postulates $A_1, .., A_k$. The logical consistency of $\{\pi(A_1), .., \pi(A_k)\}$ means that there exists at least one truth-value assignment v to propositional letters $\pi(P_1), .., \pi(P_m)$ with property $v(\pi(A_1)) = True, .., v(\pi(A_k)) = True$. Let us fix some such assignment v.

Take the signature Σ', which is disjoint from Σ and which for each predicate symbol $P_i \in \{P_1, .., P_m\}$ contains a predicate symbol R_i of the corresponding arity.

Let A be the conjunction of all postulates and let $A[R/P]$ denote the result of renaming all occurrences of symbols $P_1, .., P_m$ into $R_1, .., R_m$.

We associate the definition with each predicate symbol $P_i \in \{P_1, .., P_m\}$ by the following rule:

1) If $v(\pi(P_i)) = True$, then

$$P_i(x_1, .., x_r) \equiv A[R/P] \supset R_i(x_1, .., x_r).$$

2) If $v(\pi(P_i)) = False$, then

$$P_i(x_1, .., x_r) \equiv A[R/P] \& R_i(x_1, .., x_r).$$

Let $DT = \{D_1, .., D_m\}$ be the set of all definitions.

A. We must show that if $B \in L_T$ and $Ax \vdash B$, then $DT \vdash B$. By the properties of the deducibility relation it suffices to show $DT \vdash A$, where A is the conjunction of all non-logical postulates. By the completeness theorem of the first-order predicate calculus it is equivalent to $DT \models A$.

Let $M =< D, I >$ be a model in which all formulas of DT are true. Since the formula $A[R/P]$ is closed we have either $M \models A[R/P]$ or M $\models \neg A[R/P]$.

CASE 1. $M \models A[R/P]$. For each P_i we have one of the following two subcases:

SUBCASE 1.1 $v(\pi(P_i)) = True$

$M \models P_i(t_1, .., t_r) \Leftrightarrow$

$M \models A[R/P] \supset R_i(t_1, .., t_r) \Leftrightarrow$
$M \models R_i(t_1, .., t_r)$

Subcase 1.2 $v(\pi(P_i)) = False$

$M \models P_i(t_1, .., t_r) \Leftrightarrow$
$M \models A[R/P] \& R_i(t_1, .., t_r) \Leftrightarrow$
$M \models R_i(t_1, .., t_r)$

In each case P_i is interpreted as R_i and therefore $M \models A$.

Case 2. $M \models \neg A[R/P]$. For each P_i we have one of the following two subcases:

Subcase 2.1 $v(\pi(P_i)) = True$

$M \models P_i(t_1, .., t_r) \Leftrightarrow$
$M \models A[R/P] \supset R_i(t_1, .., t_r) \Leftrightarrow$
$M \models A[R/P] \& R_i(t_1, .., t_r) \vee \neg A[R/P] \& (R_i(t_1, .., t_r) \vee \neg R_i(t_1, .., t_r)) \Leftrightarrow$
$M \models \neg A[R/P] \& (R_i(t_1, .., t_r) \vee \neg R_i(t_1, .., t_r)) \Leftrightarrow$
$M \models R_i(t_1, .., t_r) \vee \neg R_i(t_1, .., t_r) \Leftrightarrow$
$M \models True \Leftrightarrow$
$M \models v(\pi(P_i))$

Subcase 2.2 $v(\pi(P_i)) = False$

$M \models P_i(t_1, .., t_r) \Leftrightarrow$
$M \models A[R/P] \& R_i(t_1, .., t_r) \Leftrightarrow$
$M \models A[R/P] \& R_i(t_1, .., t_r) \vee \neg A[R/P] \& (R_i(t_1, .., t_r) \& \neg R_i(t_1, .., t_r)) \Leftrightarrow$
$M \models \neg A[R/P] \& (R_i(t_1, .., t_r) \& \neg R_i(t_1, .., t_r)) \Leftrightarrow$
$M \models R_i(t_1, .., t_r) \& \neg R_i(t_1, .., t_r) \Leftrightarrow$
$M \models False \Leftrightarrow$
$M \models v(\pi(P_i))$

For all atomic formulas $P_i(t_1, .., t_r)$ we have $M \models P_i(t_1, .., t_r) \Leftrightarrow M \models v(\pi(P_i))$. The value of the atomic formula $P_i(t_1, .., t_r)$ doesn't depend on the particular assignments of values to individual variables. It can be used as basis step for the simple induction on the truth-functional structure of the formula A. Then we have $M \models A \Leftrightarrow M \models v(\pi(A))$. But according to the properties of the function v it holds $v(\pi(A_1)) = True, .., v(\pi(A_k)) = True$, and A is conjunction of $A_1, .., A_k$. Hence $v(\pi(A)) = True$ and $M \models A$.

With the help of the completeness theorem of first-order predicate calculus, we obtain $DT \vdash A$.

B. We must show that if $B \in L_T$ and $DT \vdash B$, then $Ax \vdash B$. By the completeness theorem of the first-order predicate calculus it is equivalent to show, that if $DT \models B$, then $Ax \models B$.

Let us assume, that $DT \models B$ but not $Ax \models B$. Then there exists such a model $M =< D, I >$ of the theory T, that $M \models A$ and $M \models \neg B$, where A is the conjunction of all postulates Ax.

We can extend the model $M =< D, I >$ to a model $M' =< D, I' >$, in which all the formulas of DT will be true. It is sufficient to expand the domain of the function I so that the new function of interpretation I' ascribed value $I'(R_i) = I(P_i)$ to predicate symbol R_i, and for all other functional and predicate symbols retained the same values as I.

Since $M \models A$, then in the model $M' =< D, I' >$ by definition of I' we will have $M' \models A[R/P]$, and hence, $M' \models P_i(x_1, \ldots, x_r) \equiv A[R/P] \& R_i(x_1, \ldots, x_r)$ for each R_i. It follows that all the formulas DT are true in the model M'. Therefore by our assumption $DT \models B$ it must be $M' \models B$. However, the formula B doesnt contain symbols $R_1, \ldots, R_m$, while all the other descriptive symbols are interpreted in the same way as in the model M, and by assumption it must be $M' \models \neg B$. We have obtained a contradiction. Therefore, the assumption, that $Ax \models B$ does not hold, is false.

Q.E.D.

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