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# UNIFIABILITY AND STRUCTURAL COMPLETENESS IN RELATION ALGEBRAS AND IN PRODUCTS OF MODAL LOGIC **S5**

**Abstract**

Unifiability of terms (and formulas) and structural completeness in the variety of relation algebras *RA* and in the products of modal logic **S5** is investigated. Non-unifiable terms (formulas) which are satisfiable in varieties (in logics) are exhibited. Consequently, *RA* and products of **S5** as well as representable diagonal-free $n$-dimensional cylindric algebras, $\mathrm{RDf_n}$, are almost structurally complete but not structurally complete. In case of $\mathbf{S5^n}$ a basis for admissible rules and the form of all passive rules are provided.

*Keywords and phrases:* admissible rules, passive rules, unification, projective unification, almost structural completeness, $n$-modal logic $\mathbf{S5^n}$, relation algebras, representable diagonal-free cylindric algebras.

## 0. Introduction

Unification and $E$-unification of terms is a fundamental tool in Automated Deduction and Term Rewriting Systems (see e.g. [3]). It has important applications in logic, especially in the problem of admissibility of rules. Let $E$ be an equational theory and $t_1, t_2$ two terms (called a "unification problem"). A substitution $\sigma$ is called a *unifier for* $t_1, t_2$ in $E$, if $\vdash_E \sigma(t_1) = \sigma(t_2)$. The terms $t_1$ and $t_2$ are *unifiable* if there is a unifier for them.

A substitution $\sigma$ is *more general* than a substitution $\tau$, $\tau \preceq \sigma$, if there is a substitution $\theta$ such that $\vdash_E \theta \circ \sigma = \tau$.

A *mgu*, a most general unifier, for $t_1, t_2$, is a unifier that is more general than any unifier for $t_1, t_2$. An theory $E$ has *unitary unification* if for every unifiable terms there is a mgu for them. Roughly speaking, a number of $\preceq$-maximal unifiers for unifiable terms determines the unification type. Unification types can be also finitary (a finite number of $\preceq$-maximal unifiers), infinitary (an infinite number of $\preceq$-maximal unifiers) or nullary ($\preceq$-maximal unifiers do not exist for some unifiable terms) see [3],[10].

Unification is studied in equational classes, or varieties, of algebras, corresponding to theories. Unification is also translated from varieties to the corresponding logics as follows (cf. [10], [11], [2]): a unification problem $t_1, t_2$ is reduced to a single formula $\varphi$ and a *unifier for a formula* $\varphi$ in a logic $L$ is a substitution $\sigma$ such that $\vdash_L \sigma(\varphi)$. A *formula* $\varphi$ *is unifiable* in $L$, if such $\sigma$ exists. If $\tau$, $\sigma$ are substitutions, than $\sigma$ is more general than $\tau$, $\tau \preceq \sigma$, if there is a substitution $\theta$ such that $\vdash_L \theta(\sigma(x)) \leftrightarrow \tau(x)$.

Classical propositional logic has unitary unification, every unifiable (= consistent) formula has a mgu. But unification in intuitionistic logic and some modal logics is finitary, not unitary; see S. Ghilardi [11], [12]. In his studies [11], [12],[10] Ghilardi introduced and successfully applied *projective formulas* and *projective unifiers.* A formula $\varphi$ is *projective* in a logic $L$ if there is a unifier $\sigma$ for $\varphi$ in $L$ such that, for each $x \in \mathrm{Var}(\varphi)$,

$$\varphi \vdash_L \sigma(x) \leftrightarrow x.$$

and $\sigma$, in this case, is called a *projective unifier* for $\varphi$ in $L$, see [2]. Note that $\sigma$ is a mgu. If every unifiable formula is projective in a logic, then we say that *unification is projective* in $L$ (and, hence, unitary). Projective unifiers are useful in recognizing admissible rules. If unification in $L$ is projective, then $L$ is *(almost) structurally complete*, that is, every admissible rule (with unifiable premises) is derivable in $L$, see e.g. [7], [8], [16]. Formulas which are not unifiable but consistent give rise to passive (hence admissible) rules which are not derivable. In [6], by a modification of the proof of S. Burris [4], it is observed that unification is projective in discriminator varieties.

Section 3 contains results for products for modal logic **S5**: a criterion for non-unifiability in $\mathbf{S5^n}$, description of passive rules, a basis for admissible rules in $\mathbf{S5^n}$ and almost structural completeness of $\mathbf{S5^n}$. As a corollary we get analogous results for representable diagonal-free $n$-dimensional cylindric algebras, $\mathrm{RDf_n}$, which are an algebraic face of $\mathbf{S5^n}$, see [9], [13], [14]. In Section 4 non-unifiable (but satisfiable) terms in relation algebras are given. It is shown that the variety of relation algebras are almost structurally complete but not structurally complete.

## 1. Algebraic Preliminaries

We use the basic notions of universal algebra, see for instance [4]. $\mathcal{V}(K)$ denotes the variety generated by a class $K$, $\mathcal{V}(K) = HSP(K)$. The class of subdirectly irreducible algebras in a variety $\mathcal{V}$ is denoted by $\mathcal{V}_{SI}$.

Given an algebra $\mathfrak{A}$, a term $t(x, y, z)$ is a *discriminator term* for $\mathfrak{A}$ if, for every $a, b, c \in A$,

$$t(a, b, c) = \begin{cases} c, & \text{if} \quad a = b, \\ a, & \text{if} \quad a \neq b. \end{cases}$$

A variety $\mathcal{V}$ is a *discriminator variety* if there is a class $K$ of algebras which generates $\mathcal{V}$ such that there is a term $t(x, y, z)$ which is a discriminator term for every algebra from $K$; in particular for $K = \mathcal{V}_{SI}$.

Let $\mathcal{V}$ be a variety. Given two terms $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$, a substitution $\tau$, $\tau(x_i) = t_i$ for $i \leq n$ is called a *unifier* of $p$ and $q$ in $\mathcal{V}$ if the equation $p(t_1, \ldots, t_n) = q(t_1, \ldots, t_n)$ holds in $\mathcal{V}$, i.e.

$$\models_{\mathcal{V}} \; p(t_1, \ldots, t_n) = q(t_1, \ldots, t_n).$$

If such $\tau$ exists, then the terms $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$ are *unifiable* in $\mathcal{V}$. $\sigma$ is *more general* than $\tau$, if $\models_{\mathcal{V}} \; \varepsilon \circ \sigma = \tau$, for some substitution $\varepsilon$.

The *semantic entailment* $\models_{\mathcal{V}}$ determined by $\mathcal{V}$ is defined, for two equations $p_i(x_1, \ldots, x_n) = q_i(x_1, \ldots, x_n), i = 1, 2$, as follows

$p_1(x_1, \ldots, x_n) = q_1(x_1, \ldots, x_n) \models_{\mathcal{V}} p_2(x_1, \ldots, x_n) = q_2(x_1, \ldots, x_n)$ iff for any $\mathfrak{A} \in \mathcal{V}$ and any $a_1, \ldots, a_n \in A$,
whenever $p_1(a_1, \ldots, a_n) = q_1(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$, then
$p_2(a_1, \ldots, a_n) = q_2(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$.
A unifier $\varepsilon$ for $p = p(x_1, \ldots, x_n)$ and $q = q(x_1, \ldots, x_n)$ is *projective* in $\mathcal{V}$ if

$$(p = q) \; \models_{\mathcal{V}} \; \varepsilon(x_i) = x_i, \text{ for all } i \leq n.$$

A variety $\mathcal{V}$ (or a logic $L$) has *projective unification* if for every two unifiable terms (for every formula) a projective unifier exists. From [4], [6] we get

Theorem 1. *Discriminator varieties have projective unification.*

Corollary 2. *Discriminator varieties are almost structurally complete.*

## 2. Unifiability, passive rules and a basis for admissible rules in products of S5 logics.

We find an "upper bound" for formulas that are not unifiable in products of logic **S5**. Based on this we describe the form of passive rules and provide an explicit basis for admissible rules in $\mathbf{S5^n}$. We also show that $\mathbf{S5^n}$ is almost structurally complete but not structurally complete.

Let us consider the standard *n-modal language*, for arbitrary but fixed $n \in \mathbb{N}$. $\mathcal{L}_n$ denotes a $n$-modal language built up by means of propositional variables $Var = \{x_1, x_2, \dots\}$, Boolean connectives $\wedge, \neg$ and the constant $\top$, for *truth*, and by means of modal operators $\Diamond_1, \dots, \Diamond_n$, representing 'possibility'. The remaining classical connectives $\to, \vee, \leftrightarrow, \bot$ and modal connectives $\Box_1, \dots, \Box_n$ (for 'necessity') are defined in the usual way; $Var(\varphi)$ denotes the set of variables occurring in a formula $\varphi$.

The fusion of $n$ copies of **S5** modal logic, $\mathbf{S5} \otimes \cdots \otimes \mathbf{S5}$, is defined by the set of **S5**-axioms, for each $\Diamond_i$, $i = 1, \dots, n$, separately, on the top of classical propositional logic (note that no interaction between $\Diamond_i$ and $\Diamond_j$, $i \neq j$, occurs):

$K_i : \Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$,
$T_i : \Box_i\varphi \to \varphi$,
$4_i : \Box_i\varphi \to \Box_i\Box_i\varphi$,
$B_i : \Diamond_i\Box_i\varphi \to \varphi$,

where, as usually, $\Box_i x \leftrightarrow \neg\Diamond_i\neg x$, with following rules:

$$RN_i : \frac{\varphi}{\Box_i\varphi}, \qquad MP : \frac{\varphi \to \psi, \varphi}{\psi}$$

We use basic definitions and results on $n$-frames, products of normal modal logics, in particular of **S5**, from the book [9]; in Chapter 3 and 8 the notion of the product of $n$-copies of normal modal logics is studied.

The $n$-dimensional *product of Kripke frames* $\mathfrak{F}_i = (W_i, R_i)$, for $i = 1, \dots, n$ is the $n$-frame $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n = (W_1 \times \cdots \times W_n, \overline{R_1}, \dots, \overline{R_n})$, where each $\overline{R_i}$, $i = 1, \dots, n$, is a binary relation on $W_1 \times \cdots \times W_n$ such that

$$(u_1, \dots, u_n)\overline{R_i}(v_1, \dots, v_n) \iff u_i R_i v_i \text{ and } u_k = v_k, \text{ for all } k \neq i, i \leq n.$$

For each $i = 1, \dots, n$, let $L_i$ be a Kripke complete modal logic determined by a class of all $L$-frames $\mathrm{Fr_i}$. The $n$-dimensional *product of modal logics* $L_i$, for $i = 1, \dots n$, is the $n$-modal logic determined by frames of the form $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, where $\mathfrak{F}_i \in \mathrm{Fr_i}$, for each $i = 1, \ldots, n$. Given the product of frames: $(W_1 \times \cdots \times W_n, \overline{R_1}, \ldots, \overline{R_n})$, a model based on it is defined in a standard way.

$\mathbf{S5}^n$ denotes the $n$-fold product $\mathbf{S5} \times \cdots \times \mathbf{S5}$. It is known that for $n$-times fusion we have: $\mathbf{S5} \otimes \cdots \otimes \mathbf{S5} \subset \mathbf{S5}^n$, and the inclusion is proper. The commutativity law, that states an interaction between $\Diamond_i$ and $\Diamond_j$:

$$\mathrm{comm_{ij}} : \Diamond_i \Diamond_j x \leftrightarrow \Diamond_j \Diamond_i x, \quad \text{for } i, j = 1, \ldots, n$$

is valid in every product of modal logics, in particular in $\mathbf{S5^n}$, but is not provable in the fusion $\mathbf{S5} \otimes \cdots \otimes \mathbf{S5}$. Note that $\mathbf{S5} \otimes \cdots \otimes \mathbf{S5} + \mathrm{comm_{ij}} \subset \mathbf{S5}^n$. For $n = 2$ the equality holds, $\mathbf{S5} \otimes \mathbf{S5} + \mathrm{comm_{ij}} = \mathbf{S5}^2$.
Uni-modal logic **S5** is determined by the universal frames: $(W, W \times W)$. $n$-modal logic $\mathbf{S5^n}$ is determined by products of $n$-copies of frames $(W_i, R_i)$, where $R_i = W_i \times W_i$, for $i = 1, \ldots, n$, see [9], p. 129.
A frame of the form $(W^n, \overline{R_1}, \ldots, \overline{R_n})$, where $(u_1, \ldots, u_n)\overline{R_i}(v_1, \ldots, v_n)$ iff $u_i, v_i \in W$ and $u_k = v_k$, for all $k \neq i, i \leq n$, is called the *cubic universal product frame*. In this case, having a string $\Diamond_1 \ldots \Diamond_n$ of all diamonds, any point $(w'_1, \ldots, w'_n)$ of $W^n$ can be accessed from any point $(w_1, \ldots, w_n)$ of $W^n$, i.e. $W^n$ is a '$\Diamond_1 \ldots \Diamond_n$-cluster'. We will use Prop. 3.12 of [9]:

PROPOSITION 3. $\mathbf{S5^n}$ *is determined by the cubic universal product frames.*

Due to the commutativity law $\Diamond_i \Diamond_j x \leftrightarrow \Diamond_j \Diamond_i x$, for $i, j \leq n$, the order of operators $\Diamond_i$ is not essential; hence, for fixed $n$, we use abbreviations:

$$\widehat{\Diamond}\varphi = \Diamond_1 \ldots \Diamond_n \varphi \text{ and } \widehat{\Box}\varphi = \Box_1 \ldots \Box_n \varphi.$$

Recall that $\Gamma \vdash_{S5^n} \varphi$ means that $\varphi$ can be derived from $\Gamma$ and $\mathbf{S5^n}$-theorems using the rules $MP$ and $RN_i : \psi / \Box_i \psi$, for every $i \leq n$; $\vdash_{S5^n}$ is a *global consequence relation.* Moreover, the Deduction Theorem holds.

THEOREM 4 (Deduction Theorem). *For every* $\Gamma, \varphi, \psi$ *in* $\mathcal{L}_n$,
$\Gamma, \varphi \vdash_{S5^n} \psi$ *iff* $\Gamma \vdash_{S5^n} \widehat{\Box}\varphi \to \psi$.

Using the following lemma on non-unifiable formulas we will find the basis for admissible passive rules. Some of the following lemmas are modifications of similar facts in monomodal logics over $S4.3$, see [7], [8].

Lemma 5. *If $\varphi$ is not unifiable in $\mathbf{S5}^n$ and $Var(\varphi) \subseteq \{x_1, \ldots, x_k\}$, then*

$$\varphi \vdash_{S5^n} (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k).$$

Proof: Let us proceed by induction on $k$. The formula is true for $k = 0$, as $\varphi$ must be $\bot$. Suppose the condition holds for each formula in $k$ variables and suppose that $\varphi(x_1, \ldots, x_{k+1})$ is not unifiable in $\mathbf{S5}^n$. So are $\varphi(x_1, \ldots, x_k, \top)$ and $\varphi(x_1, \ldots, x_k, \bot)$ (henceforth we omit '$\mathbf{S5}^n$'). We have

$$(x_{k+1} \leftrightarrow \top) \quad \vdash \varphi(x_1, \ldots, x_{k+1}) \leftrightarrow \varphi(x_1, \ldots, x_k, \top)$$
$$(x_{k+1} \leftrightarrow \bot) \quad \vdash \varphi(x_1, \ldots, x_{k+1}) \leftrightarrow \varphi(x_1, \ldots, x_k, \bot)$$

By induction hypothesis $\varphi(x_1, \ldots, x_k, \top) \vdash (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k)$, and $\varphi(x_1, \ldots, x_k, \bot) \vdash (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k)$. Hence, we get

$$\begin{array}{ll} x_{k+1}, \varphi(x_1, \ldots, x_{k+1}) & \vdash (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k) \\ \neg x_{k+1}, \varphi(x_1, \ldots, x_{k+1}) & \vdash (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k) \end{array}, \text{ and}$$

$$\varphi(x_1, \ldots, x_{k+1}) \quad \vdash \widehat{\Box} x_{k+1} \to (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k)$$
$$\varphi(x_1, \ldots, x_{k+1}) \quad \vdash \widehat{\Box} \neg x_{k+1} \to (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k)$$

from which it follows that $\varphi \vdash (\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_{k+1} \wedge \widehat{\Diamond} \neg x_{k+1})$. □

We use $ub(k)$ as an abbreviation of $(\widehat{\Diamond} x_1 \wedge \widehat{\Diamond} \neg x_1) \vee \cdots \vee (\widehat{\Diamond} x_k \wedge \widehat{\Diamond} \neg x_k)$ as this formula is an upper bound, in the ordering of the Lindenbaum-Tarski algebra, for non-unifiable formulas; so lemma 5 says: $\varphi \vdash_{S5^n} ub(k)$.

Let $\mathfrak{F}_0$ be an $n$-frame which consists of a single 1-element cluster $\{(u, u, \ldots, u)\}$, and $(u, u, \ldots, u)\overline{R_i}(u, u, \ldots, u)$ for all $i \leq n$, that is, $\mathfrak{F}_0$ is the product of $n$ copies of a 1-element unimodal reflexive frame. In $\mathfrak{F}_0$ modal operators $\Diamond_i$ are inessential, satisfiability of $\varphi$ in $\mathfrak{F}_0$ is equivalent to satisfiability of $\varphi$ (with all operators $\Diamond_i$ deleted) in classical logic. Note that $\mathfrak{F}_0$ is a model of $\mathbf{S5^n}$ and $\{\top, \bot\}$ is a subalgebra of the Lindenbaum-Tarski algebra for $\mathbf{S5^n}$.

Lemma 6. *In $\mathbf{S5^n}$ the following conditions are equivalent:*

1. *$\varphi$ is unifiable,*
2. *$\tau_0 \varphi \leftrightarrow \top$, for some substitution $\tau_0 : Var(\varphi) \to \{\top, \bot\}$,*
3. *$\varphi$ is satisfiable in $\mathfrak{F}_0$.*

Corollary 7. *In $\mathbf{S5^n}$ unifiability of formulas and recognizing passive rules is decidable.*

In $\mathfrak{F}_0$: $\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \leftrightarrow \bot$, hence $\tau(ub(k))$ is not satisfiable in $\mathfrak{F}_0$. Thus, if $\varphi \vdash_{S5^n} (\widehat{\Diamond}x_1 \wedge \widehat{\Diamond}\neg x_1) \vee \cdots \vee (\widehat{\Diamond}x_k \wedge \widehat{\Diamond}\neg x_k)$, then $\varphi$ is not unifiable in $\mathbf{S5^n}$.

COROLLARY 8. *$\varphi$ is not unifiable in $\mathbf{S5}^n$, with $Var(\varphi) \subseteq \{x_1, \ldots, x_k\}$, iff*

$$\varphi \vdash_{S5^n} (\widehat{\Diamond}x_1 \wedge \widehat{\Diamond}\neg x_1) \vee \cdots \vee (\widehat{\Diamond}x_k \wedge \widehat{\Diamond}\neg x_k).$$

LEMMA 9. *If $\varphi$ is not unifiable in $\mathbf{S5^n}$, then there is a formula $\psi$ such that*

$$\varphi \vdash_{S5^n} \widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi.$$

PROOF: Let $Var(\varphi) \subseteq \{x_1, \ldots, x_k\}$. We use Lemma 5. We define, by induction on $k$, a formula $\psi_k$ such that: $\psi_1 = x_1$ and
$\psi_{k+1} = (x_{k+1} \wedge \widehat{\Diamond}\neg x_{k+1}) \vee (\widehat{\Box}x_{k+1} \vee \widehat{\Box}\neg x_{k+1}) \wedge \psi_k$.
Its negation is: $\neg\psi_{k+1} = (\neg x_{k+1} \vee \widehat{\Box}\, x_{k+1}) \wedge \big((\widehat{\Diamond}\neg x_{k+1} \wedge \widehat{\Diamond}x_{k+1}) \vee \neg\psi_k\big)$.
Now we prove, by induction on $k$, that $\vdash_{S5^n} \widehat{\Box}ub(k) \to \widehat{\Diamond}\psi_k \wedge \widehat{\Diamond}\neg\psi_k$, i.e.:

$$(\circ) \qquad (\widehat{\Diamond}x_1 \wedge \widehat{\Diamond}\neg x_1) \vee \cdots \vee (\widehat{\Diamond}x_k \wedge \widehat{\Diamond}\neg x_k) \vdash_{S5^n} \widehat{\Diamond}\psi_k \wedge \widehat{\Diamond}\neg\psi_k.$$

By the definition, $(\circ)$ holds for $k = 1$.

For the induction step, suppose that $\psi_k$ satisfies $(\circ)$ and we show that: $w \Vdash \widehat{\Box}ub(k+1)$ implies $w \Vdash \widehat{\Diamond}\psi_{k+1} \wedge \widehat{\Diamond}\neg\psi_{k+1}$, for any $w \in W^n$. So, using Proposition 3, let us take a cubic universal product model for $\mathbf{S5^n}$, $(W^n, \overline{R_1}, \ldots, \overline{R_n}, \Vdash)$, and assume that $w \Vdash \widehat{\Box}ub(k+1)$, i.e. that

$$(AS) \ \ w \Vdash \widehat{\Box}\big((\widehat{\Diamond}x_1 \wedge \widehat{\Diamond}\neg x_1) \vee \cdots \vee (\widehat{\Diamond}x_{k+1} \wedge \widehat{\Diamond}\neg x_{k+1})\big) \quad \text{for any } w \in W^n.$$

There are two cases: (Case 1) either for each element $y$ in the set $W^n$.
(1) $y \Vdash \widehat{\Box}x_{k+1} \vee \widehat{\Box}\neg x_{k+1}$,
or (Case 2): the negation of (Case 1) holds.

(Case 1) Since $ub(k+1) = ub(k) \vee \neg(\widehat{\Box}x_{k+1} \vee \widehat{\Box}\neg x_{k+1})$ we get, by $(AS)$, $w \Vdash \big((\widehat{\Diamond}x_1 \wedge \widehat{\Diamond}\neg x_1) \vee \cdots \vee (\widehat{\Diamond}x_k \wedge \widehat{\Diamond}\neg x_k)\big)$; hence, by the induction hypothesis, there exists $\psi_k$ such that $w \Vdash \widehat{\Diamond}\psi_k \wedge \widehat{\Diamond}\neg\psi_k$, for each $w$ in $W^n$. Hence,

(1.1) $\exists_{y_1 \in W^n}\ y_1 \Vdash \psi_k$ and (1.2) $\exists_{y_2 \in W^n}\ y_2 \Vdash \neg\psi_k$.

Thus, by (1.1), $y_1 \Vdash (\widehat{\Box}x_{k+1} \vee \widehat{\Box}\neg x_{k+1}) \wedge \psi_k$, i.e. $w \Vdash \widehat{\Diamond}\psi_{k+1}$, for $w \in W^n$. Now, by (1), $y_2 \Vdash (\widehat{\Box}\neg x_{k+1} \vee \widehat{\Box}\, x_{k+1})$, in $\mathbf{S5^n}$: $y_2 \Vdash (\neg x_{k+1} \vee \widehat{\Box}\, x_{k+1})$, and by (1.2), we get $y_2 \Vdash \big((\widehat{\Diamond}\neg x_{k+1} \wedge \widehat{\Diamond}x_{k+1}) \vee \neg\psi_k\big)$, hence $y_2 \Vdash (\neg x_{k+1} \vee \widehat{\Box}\, x_{k+1}) \wedge \big((\widehat{\Diamond}\neg x_{k+1} \wedge \widehat{\Diamond}x_{k+1}) \vee \neg\psi_k\big)$, i.e. $w \Vdash \widehat{\Diamond}\neg\psi_{k+1}$, for any $w \in W^n$. Consequently, $w \Vdash \widehat{\Diamond}\psi_{k+1} \wedge \widehat{\Diamond}\neg\psi_{k+1}$, for any $w \in W^n$ in (Case 1).

(Case 2) - the negation of (Case 1); we have two conditions:

(2.1) $\exists_{z_1 \in W^n}\ z_1 \Vdash \neg x_{k+1}$ and (2.2) $\exists_{z_2 \in W^n}\ z_2 \Vdash x_{k+1}$.

Then, since $z_1, z_2 \in W^n$, $z_2 \Vdash x_{k+1} \wedge \widehat{\Diamond}\neg x_{k+1}$, hence $w \Vdash \widehat{\Diamond}\psi_{k+1}$.
Now we show that $z_1 \Vdash \neg\psi_{k+1}$. By (2.1), $z_1 \Vdash \neg x_{k+1} \vee \widehat{\Box}\ x_{k+1}$, the first part of $\neg\psi_{k+1}$. For the second part observe that, by (2.2), $z_1 \Vdash \widehat{\Diamond}\ x_{k+1}$. Now by (2.1), $z_1 \Vdash \widehat{\Diamond}\neg x_{k+1}$, hence $z_1 \Vdash \widehat{\Diamond}\ x_{k+1} \wedge \widehat{\Diamond}\neg x_{k+1}$, thus, $z_1 \Vdash \neg\psi_{k+1}$. Therefore $w \Vdash \widehat{\Diamond}\psi_{k+1} \wedge \widehat{\Diamond}\neg\psi_{k+1}$, for any $w \in W^n$, in (Case 2) too. □

From [5], 6.26, 6.29, (see also [2]) we have

LEMMA 10. *Unification in* $\mathbf{S5^n}$ *is projective. For every unifiable formula* $\varphi$ *with a ground unifier* $\tau_0 : \mathcal{L}_n \to \{\bot, \top\}$ *a unifier for* $\varphi$ *of the following form is projective:*

$$\sigma(x) = (\widehat{\Box}\varphi \to x) \wedge (\widehat{\Box}\varphi \vee \tau_0(x)), \qquad \textit{for } x \in Var(\varphi).$$

Let us consider the following rule, which can be seen as a generalization of the rule $P_2$ in monomodal logic, see e.g. [17], [7].

$$P_2^n\ :\ \frac{\Diamond_1 \ldots \Diamond_n \varphi\ \wedge\ \Diamond_1 \ldots \Diamond_n \neg\varphi}{\bot}, \quad \text{in an abbreviated form: } \frac{\widehat{\Diamond}\varphi \wedge \widehat{\Diamond}\neg\varphi}{\bot}.$$

Recall that a rule $\varphi/\psi$ is passive in a logic $L$ if $\varphi$ is not unifiable in $L$.
The rule $P_2^n$ is passive and hence, admissible, in $\mathbf{S5^n}$. But $\widehat{\Diamond}x \wedge \widehat{\Diamond}\neg x$ is satisfiable, hence $\widehat{\Diamond}x \wedge \widehat{\Diamond}\neg x \nvdash_{S5^n} \bot$, i.e. $P_2^n$ is not derivable in $\mathbf{S5^n}$.

COROLLARY 11. *n-modal logic* $\mathbf{S5^n}$ *is almost structurally complete but not structurally complete.*

From lemma 9 we get that $P_2^n$ is the strongest of all passive rules in $\mathbf{S5^n}$.

COROLLARY 12. *A modal consequence relation over* $\mathbf{S5^n}$ *obtained by extending an n-modal logic* $L \supseteq \mathbf{S5^n}$ *with the rule* $P_2^n$ *is structurally complete. The rule* $P_2^n$ *forms a basis for all passive (admissible) rules in* $\mathbf{S5^n}$.

For unimodal logics containing **S4** a similar description of non-unifiable formulas as in Lemma 9 and a similar basis for passive rules in unimodal logics was given in [18], [19]. Now we give a form of passive rules in $\mathbf{S5^n}$.

Theorem 13. *Each passive rule in* $\mathbf{S5^n}$ *is equivalent to a rule of the form*

$$\frac{\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi}{\theta} \qquad \textit{for some formulas } \psi, \theta.$$

Proof: Let $\varphi/\lambda$ be a passive rule in $\mathbf{S5^n}$ and assume that $\lambda = \widehat{\Box}\lambda$. By lemma 9 we have $\varphi \vdash_{S5^n} \widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi$, for some $\psi$, and hence $\varphi$ is deductively equivalent, in the sense of $\vdash_{S5^n}$, to $(\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi) \wedge (\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \to \varphi)$ (we will omit $S5^n$ from $\vdash_{S5^n}$ below).
Let us observe that $(\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \to \varphi)$ is unifiable and hence, by lemma 10, there is a *projective unifier* $\sigma$ for this formula. We will show that the following two rules are equivalent

$$\frac{\varphi}{\lambda} \qquad \text{and} \qquad \frac{\widehat{\Diamond}\sigma(\psi) \wedge \widehat{\Diamond}\neg\sigma(\psi)}{\sigma(\lambda)}$$

$(\to)$ Suppose that the rule $\varphi/\lambda$ holds, i.e. $\varphi \vdash \lambda$. Then, $\sigma(\varphi) \vdash \sigma(\lambda)$. Since $\sigma$ is a unifier for $\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \to \varphi$, this gives $\widehat{\Diamond}\sigma(\psi) \wedge \widehat{\Diamond}\neg\sigma(\psi) \vdash \sigma(\lambda)$.
$(\leftarrow)$ Assume that $\widehat{\Diamond}\sigma(\psi) \wedge \widehat{\Diamond}\neg\sigma(\psi) \vdash \sigma(\lambda)$. Since $\varphi \vdash \widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \to \varphi$ and $\sigma$ is projective, i.e. $(\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi \to \varphi) \vdash x \leftrightarrow \sigma(x)$, we get $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$, and hence, using $\varphi \vdash \widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi$ we get $\varphi \vdash \widehat{\Diamond}\varphi(\psi) \wedge \widehat{\Diamond}\neg\sigma(\psi)$. This gives $\varphi \vdash \sigma(\lambda)$, and hence, using again projectivity of $\sigma$, we get $\varphi \vdash \lambda$. $\Box$

We conclude that an arbitrary passive rule in $\mathbf{S5^n}$ is a subrule of the rule $P_2^n$. Since $\theta$ can be taken independently of $\psi$, infinitely many different rules of the form $\widehat{\Diamond}\psi \wedge \widehat{\Diamond}\neg\psi/\theta$ can be found.

Let us note that the variety $\mathrm{RDf_n}$ of $n$-dimensional diagonal-free representable cylindric algebras forms an algebraic semantics for $\mathbf{S5^n}$, see [9], 8.1, [13], [14]. A diagonal-free cylindric algebra of $n$-dimension is an algebra $\mathcal{C} = (C, 0, 1, \wedge, \vee, -, c_i)_{i\in\{1,\ldots,n\}}$, where $(C, 0, 1, \wedge, \vee, -)$ is a Boolean algebra and the operations of cylindrification $c_i$, for $i \leq n$, satisfy the following axioms, for every $x, y \in C$, $i, j \leq n$:
(1) $c_i 0 = 0$, (2) $x \leq c_i x$, (3) $c_i(x \wedge c_i y) = c_i x \wedge c_i y$, (4) $c_i c_j x = c_j c_i x$.
A representable (diagonal-free) cylindric algebra is a cylindric algebra that is isomorphic to a subdirect product of (diagonal-free) cylindric set algebras, see [14], [13].

If one substitutes $\Diamond_i$ for $c_i$ then the axioms (1) - (4) become provable in $\mathbf{S5^n}$, see [9]. The following quasi-identity:

$$\mathcal{P}_2^n : \qquad c_1 \ldots c_n x \ \wedge \ c_1 \ldots c_n - x = 1 \ \Rightarrow \ 1 = 0$$

holds in the $\omega$-generated free $\mathrm{RDf_n}$-algebra but does not hold in the variety $\mathrm{RDf_n}$. Similarly, expressions like $c_1 \ldots c_n x \ \wedge \ c_1 \ldots c_n - x = 1 \ \Rightarrow \ p(y) = q(z)$ hold in the free $\mathrm{RDf_n}$-algebra but may not hold in $\mathrm{RDf_n}$.

By [4] the variety $\mathrm{RDf_n}$ is a discriminator variety, hence it is almost structurally complete (see also [6]). Thus we have

Corollary 14. *The variety* $\mathrm{RDf_n}$ *is almost structurally complete but not structurally complete.*

There is a major difference between $\mathrm{RDf_n}$ (or $\mathbf{S5^n}$), for $n = 2$ and for $n \geq 3$ . For $n \geq 3$, $\mathrm{RDf_n}$ is undecidable (R. Maddux 1980), it is not finitely axiomatizable (J. Johnson 1969) and it does not have the f.m.p. (I. Nemeti 1984, A. Kurucz 2002). But $\mathbf{S5^2}$ (and $\mathrm{RDf_2}$) is finitely axiomatizable by Sahlqvist-formulas, it has the f.m.p. (N. Bezhanishvili, M. Marx 2003) and it is decidable by D. Scott, and satisfiability is NEXPTIME complete, (M. Marx 2003). Hence we have

Corollary 15. *Admissibility of rules is decidable in* $\mathbf{S5^2}$ *and in* $\mathrm{RDf_2}$.

## 3. Almost structural completeness in relation algebras

We will show that the theory of relation algebras, RA, is almost structurally complete but not structurally complete. A. Tarski presented the axioms for an equational theory of relation algebras in 1941, see [20], which consist of the axioms for Boolean algebras and axioms for relational operations: composition, conversion and identity.

Let $X$ be a set. An algebra $(S, \cup, ', X^2, \emptyset, \circ, ^{-1}, \imath\delta)$, where $S \subseteq \mathcal{P}(X^2)$, with operations $\circ, ^{-1}, \imath\delta$ (binary, unary and nullary, respectively) is called a *proper relation algebra*, (PRA), if:

1. $(S, \cup, ', X^2, \emptyset)$ is a field of sets,
2. $(S, \circ, ^{-1}, \imath\delta)$ is an involutive monoid, with the composition $\circ$, the converse $^{-1}$, and the identity $\imath\delta$ (which is $=$).
3. $\circ$ and $^{-1}$ are monotone operators,
4. $\circ$ and $^{-1}$ satisfy the so called *De Morgan theorem K*, that is

$$[(x \circ y) \leq z] \Rightarrow [(x^{-1} \circ -z) \leq -y] \text{ and } [(-z \circ y^{-1}) \leq -x].$$

A *relation algebra* (RA) is an algebra $(A, \vee, -, 1, 0, \circ, ^{\smile}, e)$ such that $(A, \vee, -, 1, 0)$ is a Boolean algebra and the operators: $\circ$ (binary), $^{\smile}$(unary) and $e$ (a constant) satisfy the following conditions:

1. $x \circ (y \vee z) = (x \vee y) \circ (x \vee z)$,
2. $x \circ (y \circ z) = (x \circ y) \circ z$,
3. $x \circ e = x = e \circ x$,
4. $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$,
5. $(x^{\smile})^{\smile} = x$,
6. $(-x)^{\smile} = -(x^{\smile})$,
7. $e^{\smile} = e$,
8. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$,
9. $(x^{\smile} \circ -(x \circ y)) \vee -y = -y$.

A relation algebra is called a *representable relation algebra* (RRA), if it is isomorphic to a subalgebra of a proper relation algebra. Not every relation algebra is representable (R. Lyndon 1950), see [14], [15].

The equational theory of relation algebras, RA, is undecidable (A. Tarski [20]). But unifiablility of terms in RA is decidable, see 3.4 in [4].

THEOREM 16 ([4]). *Terms $p$ and $q$ are unifiable in* RA*, iff the equation $p = q$ has a solution in the relation algebras with at most four elements.*

There are two four-element algebras on $\{1, 0, e, -e\}$, see [1],[14]; in [14] they are called the two-atom algebras. Two definitions of $\circ$ on $\{1, 0, e, -e\}$ are possible, since the result of $-e \circ -e$ can be $e$ or 1:

| $\circ$ | 1 | 0 | $e$ | $-e$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 |
| $e$ | 1 | 0 | $e$ | $-e$ |
| $-e$ | 1 | 0 | $-e$ | $e$ |

| $\circ$ | 1 | 0 | $e$ | $-e$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 |
| $e$ | 1 | 0 | $e$ | $-e$ |
| $-e$ | 1 | 0 | $-e$ | 1 |

Using these two-atom algebras we can effectively check unifiablility of terms in RA.

THEOREM 17. *The terms*

$$(x \circ y) \cap (-x \circ y) \cap (x \circ -y) \cap (-x \circ -y) \text{ and } 1$$

*are not unifiable in RA, but the equation*

$$(x \circ y) \cap (-x \circ y) \cap (x \circ -y) \cap (-x \circ -y) = 1$$

*is satisfiable in* RA.

Proof: Every calculation of the term in both four-element algebras give the result 0:

$$\begin{aligned}
(1,1)&: (1 \circ 1) \cap (0 \circ 1) \cap (1 \circ 0) \cap (0 \circ 0) = 1 \cap 0 \cap 0 \cap 0 = 0,\\
(1,0)&: (1 \circ 0) \cap (0 \circ 0) \cap (1 \circ 1) \cap (0 \circ 1) = 0 \cap 0 \cap 1 \cap 0 = 0,\\
(0,0)&: (0 \circ 0) \cap (1 \circ 0) \cap (0 \circ 1) \cap (1 \circ 1) = 0 \cap 0 \cap 0 \cap 1 = 0,\\
(0,e)&: (0 \circ e) \cap (1 \circ e) \cap (0 \circ -e) \cap (1 \circ -e) = 0 \cap 1 \cap 0 \cap 1 = 0,\\
(1,e)&: (1 \circ e) \cap (0 \circ e) \cap (1 \circ -e) \cap (0 \circ -e) = 1 \cap 0 \cap 1 \cap 0 = 0,\\
(0,-e)&: (0 \circ -e) \cap (1 \circ -e) \cap (0 \circ e) \cap (1 \circ e) = 0 \cap 1 \cap 0 \cap 1 = 0,\\
(1,-e)&: (1 \circ -e) \cap (0 \circ -e) \cap (1 \circ e) \cap (0 \circ e) = 1 \cap 0 \cap 1 \cap 0 = 0,\\
(-e,-e)&: (-e \circ -e) \cap (e \circ -e) \cap (-e \circ e) \cap (e \circ e) = \;?\; \cap -e \cap -e \cap e = 0,\\
(e,-e)&: (e \circ -e) \cap (-e \circ -e) \cap (e \circ e) \cap (-e \circ e) = -e \cap \;?\; \cap e \cap -e = 0,\\
(-e,e)&: (-e \circ e) \cap (e \circ e) \cap (-e \circ -e) \cap (e \circ -e) = -e \cap e \cap \;?\; \cap -e = 0,\\
(e,e)&: (e \circ e) \cap (-e \circ e) \cap (e \circ -e) \cap (-e \circ -e) = e \cap -e \cap -e \cap \;?\; = 0.
\end{aligned}$$

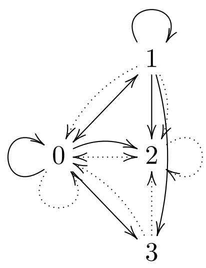
The results of $(-e \circ -e)$ are indicated by ?, as they have different values in the two four-element algebras, but the final value is 0. Hence the two terms are not unifiable in RA.

On the other hand, the equation

$$(x \circ y) \cap (-x \circ y) \cap (x \circ -y) \cap (-x \circ -y) = 1$$

is satisfiable in the following proper relation algebra with 16 atoms, $(\mathcal{P}(\{0,1,2,3\}^2), \cup, ', \{0,1,2,3\}^2, \emptyset, \circ, ^{-1}, \{(0,0),(1,1),(2,2),(3,3)\})$, with the valuation:
$x = \{(0,0),(1,0),(2,0),(3,0),(0,2),(1,2),(2,2),(3,2)\}$
$y = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$

The relations are shown on the following graph, with $x$ as a dotted line and $y$ as a solid line:




□

Hence, the following quasi-identity:

$$(x \circ y) \cap (-x \circ y) \cap (x \circ -y) \cap (-x \circ -y) = 1 \ \Rightarrow \ 1 = 0$$

holds in the $\omega$-generated free relation algebra but does not hold in RA.

By the result of A.Tarski, see [14], [4], [13], [15] it is known that

Theorem 18 (A.Tarski). *The variety* RA *of relation algebras is a discriminator variety.*

Corollary 19. *The variety* RA *of relation algebras is almost structurally complete but not structurally complete.*

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