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ON SOME LOGIC WITH A RELATION IMPOSED ON FORMULAE: TABLEAU SYSTEM $\mathcal{F}$

Abstract

In the paper we present tableau system $\mathcal{F}$ for logic with relation imposed on formulae. The language for that logic consists of formulae built of the classical connectives and two new intensional connectives: related conjunction $\vartriangle$ and related material conditional $\looparrowright$.

Models for $\mathcal{F}$ are of the form $\langle v, R\rangle$ where v is a Boolean valuation and R is a binary relation on the set of formulae. The interpretation of $A \vartriangle B$, $A \looparrowright B$ in $\mathcal{F}$ depends on the fact whether $\langle A, B\rangle \in R$. The system $\mathcal{F}$ is meant as a starting point for further development of logics with a relation on formulae, *e.g.* through imposing additional conditions on R.

We prove completeness, soundness and confluence of the system.

1. Introduction

Related connectives This paper presents a system of logic defined on extended Boolean language closed under two new intensional connectives: related conjunction $\vartriangle$ and related material conditional $\looparrowright$. A model for the new language is a pair $\mathfrak{M} = \langle v, R\rangle$, where v is a Boolean valuation and R is a binary relation on the set of formulae.

Relation R can be imposed in many different ways on the set of formulae and valid forms of argumentation expressed with $\looparrowright$ and $\vartriangle$ may vary accordingly. The related material conditional was studied for example in papers [1], [2] [3], where relation R was put on propositional letters and then extended to other formulae through additional, specific conditions (*e.g.* symmetry). Moreover, these authors use R also as a symbol of language, a connective allowing them to formulate axioms. In our exposition, on the other hand, we do not define any connective $\star$ such that $A \star B$ is true iff $R(A, B)$. Also, we add a new connective $\vartriangle$ whose expressive power relative to $\looparrowright$ varies depending on the system of logic.

Our task here is to give a general perspective on logics with a relation imposed on formulae. Thus, we present the system $\mathcal{F}$ in which only the relation between formulae, without any deeper analysis of their structure, is taken into account.

Extra-logical motivations The related connectives may be interpreted in many ways. Here we have some examples. Formulae $A \looparrowright B$ and $A \vartriangle B$ could be respectively read:

- causally: 'If A happens, it causes B', 'A happened and it caused B'
- temporally: 'If A happens, then B happens afterwards', 'A happened and B happened afterwards'
- analytically: 'If A is true, then by that fact it analytically follows that B is true', 'A is true and by that fact it analytically follows that B is true'.

Each such interpretation requires additional constraints on the relation R. The system presented in this paper can be extended by conditions such as reflexivity, symmetry, transitivity or by more specific conditions, to adjust them for various kinds of argumentation.

Tableau approach We adopt the abstract approach to tableau systems presented in [4], [5], but with some modifications concerning the notion of tableau. Under this approach a tableau consists of sequences of applications of rules and rules are applied to sets of formulae. Soundness, completeness and confluence theorems are provided for the presented system.

2. Language and syntax

Definition 2.0.1 (Alphabet). The *alphabet* of the language of our logics consists of the following:

(i) infinite denumerable set $\mathsf{Var} = \{p_0, p_1, p_2, \ldots\}$ of propositional letters (variables)

(ii) set $\mathsf{Con} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow \triangle, \nleftrightarrow\}$ of logical connectives
(iii) parentheses:), (, as auxiliary symbols.

We put $\mathsf{Con}^2 := \mathsf{Con} \smallsetminus \{\neg\}$.

Definition 2.0.2 (Formula). We say that A is a *formula* iff A is a finite string of elements of the alphabet and one of the following conditions holds:

(i) $A \in \mathsf{Var}$
(ii) A is of the form $\neg B$, where B is a formula
(iii) A is of the form $(B \star C)$, where B, C are formulae and $\star \in \mathsf{Con}^2$.

Sometimes we will use metavariables: x, y to represent propositional letters and metaformulae: A, B, C to represent formulae. In examples we will write letters p, q, r, s, rather than p_0, p_1, p_2, p_3.

We denote the set of all formulae by For. Often, where it is obvious how to understand a given formula, we omit parentheses.

To construct tableau proofs in the further parts we need expressions which will allow us to describe relations between formulae.

Definition 2.0.3 (Auxiliary expressions). To the set of *auxiliary expressions* (in short: Aux) belong all and only expressions of the form:

(i) ARB, where A, $B \in \mathsf{For}$
(ii) $A \not{R} B$, where A, $B \in \mathsf{For}$.

The set $\mathsf{Ex}\colon = \mathsf{For} \cup \mathsf{Aux}$ will be called the *set of expressions*.

3. System $\mathcal{F}$

3.1. Semantics

Definition 3.1.1 (Model). By a *model* $\mathfrak{M}$ we mean an ordered pair $\langle v, R\rangle$ where function $v\colon \mathsf{Var} \mapsto \{0,1\}$ is called a *valuation* of $\mathfrak{M}$ and $R \subseteq \mathsf{For} \times \mathsf{For}$ is a binary relation called an *underlying relation of* $\mathfrak{M}$.

When given a model $\mathfrak{M}$ we shall denote its valuation by $v^{\mathfrak{M}}$ and its underlying relation by $R^{\mathfrak{M}}$.

Definition 3.1.2 (Truth in a model). Let $\mathfrak{M}$ be a model and A a formula. The following list defines the conditions under which A *is true in* $\mathfrak{M}$ (in short: $\mathfrak{M} \vDash_{\mathcal{F}} A$), depending on the complexity of A:

(i) for $A \in \mathsf{Var}$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $v^{\mathfrak{M}}(A) = 1$

(ii) for A of the form $\neg B$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff it is not the case that $\mathfrak{M} \vDash_{\mathcal{F}} B$ (in short: $\mathfrak{M} \nvDash_{\mathcal{F}} B$)

(iii) for A of the form $(B \wedge C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $\mathfrak{M} \vDash_{\mathcal{F}} B$ and $\mathfrak{M} \vDash_{\mathcal{F}} C$

(iv) for A of the form $(B \vee C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $\mathfrak{M} \vDash_{\mathcal{F}} B$ or $\mathfrak{M} \vDash_{\mathcal{F}} C$

(v) for A of the form $(B \rightarrow C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $\mathfrak{M} \nvDash_{\mathcal{F}} B$ or $\mathfrak{M} \vDash_{\mathcal{F}} C$

(vi) for A of the form $(B \leftrightarrow C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $\mathfrak{M} \vDash_{\mathcal{F}} B$ is equivalent to $\mathfrak{M} \vDash_{\mathcal{F}} C$

(vii) for A of the form $(B \vartriangle C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff both following conditions hold:

- $\mathfrak{M} \vDash_{\mathcal{F}} B$ and $\mathfrak{M} \vDash_{\mathcal{F}} C$
- it is the case that $R^{\mathfrak{M}}(B, C)$

(viii) for A of the form $(B \looparrowright C)$: $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff both following conditions hold:

- either $\mathfrak{M} \nvDash_{\mathcal{F}} B$ or $\mathfrak{M} \vDash_{\mathcal{F}} C$
- it is the case that $R^{\mathfrak{M}}(B, C)$.

The semantics for standard connectives ((i) – (vi)) is the same as in the classical logic. The meaning of new connectives $\vartriangle$, $\looparrowright$ is similar to the meaning of conjuction and implication respectively, however a condition concerning the underlying relation of a model is added.

Definition 3.1.3 (Satisfaction, semantic consequence, tautology). Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$. Let $\mathfrak{M}$ be a model.

(i) We say that $\mathfrak{M}$ *satisfies* X (in short: $\mathfrak{M} \vDash_{\mathcal{F}} X$) iff for each formula $B \in X$: $\mathfrak{M} \vDash_{\mathcal{F}} B$.

(ii) We say that X is *satisfiable* iff there is a model $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{F}} X$.

(iii) We say that A *follows from* X (is a *semantic consequence* of X, in short: $X \vDash_{\mathcal{F}} A$) iff for any model $\mathfrak{M}$: $\mathfrak{M} \vDash_{\mathcal{F}} A$, whenever $\mathfrak{M} \vDash_{\mathcal{F}} X$.

(iv) We say that A is *tautology* (in short: $\vDash_{\mathcal{F}} A$) iff for any model $\mathfrak{M}$: $\mathfrak{M} \vDash_{\mathcal{F}} A$.

On the grounds of the semantic notions introduced above we see that $\vartriangle$ is definable in the terms of $\wedge$ and $\nrightarrow$:

FACT 3.1.4. For any $A, B \in \mathsf{For}$: $\models_{\mathcal{F}} (A \vartriangle B) \leftrightarrow (A \wedge (A \nrightarrow B))$.

As is known from Post-Sierpiński Theorem for Boolean logic, all Boolean connectives can be defined by the means of connectives $\{\neg, \vee\}$. Hence, by 3.1.4 we have:

FACT 3.1.5. Let $\overline{\mathsf{For}}$ be set of formulae build only of connectives: $\{\neg, \vee, \nrightarrow\}$ (so $\overline{\mathsf{For}} \subset \mathsf{For}$). For any $A \in \mathsf{For}$ there is a formula $B \in \overline{\mathsf{For}}$ such that $\models_{\mathcal{F}} A \leftrightarrow B$.

It would be sufficient to define our system for the language $\overline{\mathsf{For}}$ and next introduce metalanguage definitions for the remaining formulae. However, we define the system for the full language For and in proofs check only cases of formulae build of $\{\neg, \vee, \nrightarrow\}$ or $\{\neg, \rightarrow, \nrightarrow\}$, since by 3.1.5 the other cases can be reduced in a well-known way.

DEFINITION 3.1.6 (Faithful model). Let X be a set of expressions and $\mathfrak{M}$ be a model. We say that $\mathfrak{M}$ is *faithful to* X iff the following conditions hold:

(i) $\mathfrak{M} \models_{\mathcal{F}} X \cap \mathsf{For}$

(ii) for any A, $B \in \mathsf{For}$:

- if $ARB \in X$, then it is the case that $R^{\mathfrak{M}}(A, B)$
- if $A \not\!R\, B \in X$, then it is not the case that $R^{\mathfrak{M}}(A, B)$.

If a model is faithful to $\{E\}$, we shall say it is faithful to E, for $E \in \mathsf{Ex}$.

DEFINITION 3.1.7 (Contradictory set). A set $X \subseteq \mathsf{Ex}$ is called *contradictory* iff at least one of the following conditions holds:

(i) There is $A \in \mathsf{For}$ such that $A, \neg A \in X$;

(ii) There are $A, B \in \mathsf{For}$ such that $ARB, A \not\!R\, B \in X$.

If a set is not contradictory, we call it *non-contradictory*.

From the Definitions 3.1.6 and 3.1.7 we draw an obvious conclusion:

FACT 3.1.8. Let X be a contradictory set of expressions and $\mathfrak{M}$ be any model. Then $\mathfrak{M}$ is not faithful to X.

3.2. Tableau proofs

DEFINITION 3.2.1 (Negative set). A set $X \subseteq \mathsf{Ex}$ is called *negative* iff there is $A \in \mathsf{For}$ such that $\neg A \in X$.

DEFINITION 3.2.2 (Extension rule.). By a *rule* we mean any set of $n-$tuples defined on the set $\wp(\mathsf{Ex})$. The first element of each of these $n-$tuples is called the *initial set*, while the remaining $n-1$ ones are called *resulting sets*. Initial sets are required to be non-contradictory and negative. An *extension rule* R_i $(1 \leqslant i \leqslant 13)$ has one of the following forms:

$$R_1\colon \frac{X \cup \{(A{\wedge}B)\}}{X \cup \{(A{\wedge}B), A, B\}} \qquad R_2\colon \frac{X \cup \{(A{\vee}B)\}}{X \cup \{(A{\vee}B), A\}, X \cup \{(A{\vee}B), B\}}$$

$$R_3\colon \frac{X \cup \{(A{\rightarrow}B)\}}{X \cup \{(A{\rightarrow}B), \neg A\}, X \cup \{(A{\rightarrow}B), B\}}$$

$$R_4\colon \frac{X \cup \{(A{\leftrightarrow}B)\}}{X \cup \{(A{\leftrightarrow}B), A, B\}, X \cup \{(A{\leftrightarrow}B), \neg A, \neg B\}}$$

$$R_5\colon \frac{X \cup \{\neg\neg A\}}{X \cup \{\neg\neg A, A\}} \qquad R_6\colon \frac{X \cup \{\neg(A{\wedge}B)\}}{X \cup \{\neg(A{\wedge}B), \neg A\}, X \cup \{\neg(A{\wedge}B), \neg B\}}$$

$$R_7\colon \frac{X \cup \{\neg(A{\vee}B)\}}{X \cup \{\neg(A{\vee}B), \neg A, \neg B\}} \qquad R_8\colon \frac{X \cup \{\neg(A{\rightarrow}B)\}}{X \cup \{\neg(A{\rightarrow}B), A, \neg B\}}$$

$$R_9\colon \frac{X \cup \{\neg(A{\leftrightarrow}B)\}}{X \cup \{\neg(A{\leftrightarrow}B), \neg A, B\}, X \cup \{\neg(A{\leftrightarrow}B), A, \neg B\}}$$

$$R_{10}\colon \frac{X \cup \{(A{\vartriangle}B)\}}{X \cup \{(A{\vartriangle}B), (A{\wedge}B), ARB\}}$$

$$R_{11}\colon \frac{X\cup\{(A\looparrowright B)\}}{X\cup\{(A\looparrowright B),(A\to B),ARB\}}$$

$$R_{12}\colon \frac{X\cup\{\neg(A\triangle B)\}}{X\cup\{\neg(A\triangle B),\neg(A\wedge B)\},X\cup\{\neg(A\triangle B),A\not R\ B\}}$$

$$R_{13}\colon \frac{X\cup\{\neg(A\looparrowright B)\}}{X\cup\{\neg(A\looparrowright B),\neg(A\to B)\},X\cup\{\neg(A\looparrowright B)\ A\not R\ B\}\}}$$

Definition 3.2.3 (Extension, indirect extension). Let $X, Y \subseteq \mathsf{Ex}$. We say that Y is an *extension* of X iff there is a rule R_i ($1 \leqslant i \leqslant 13$) and a finite list of sets $Y_1, \ldots, Y_n$ such that: $\langle X, Y_1, \ldots, Y_n\rangle \in R_i$ and $Y = Y_j$ ($1 \leqslant j \leqslant n$). When $n > 1$, we call Y an *indirect extension* of X.

When talking about extensions we will also use the phrase Y *is an extension of* X *under the rule* R_i (where $1 \leqslant i \leqslant 13$).

Definition 3.2.4 (Sequence of extensions). Let $K \subseteq \mathbb{N}$ be a nonempty set such that for any natural numbers $k < l$: if $l \in K$, then $k \in K$. Let X be a negative set. Any function $f\colon K \mapsto \wp(\mathsf{Ex})$ satisfying conditions:

(i) $f(1) = X$
(ii) for every i, if $i + 1 \in K$, then $f(i + 1)$ is an extension of $f(i)$

is called a *sequence of extensions* of X.

Any sequence of extensions f is *monotonic*, since it satisfies the condition:

- If $i < j$ (i, j belonging to the domain of f), then $f(i) \subseteq f(j)$.

Definition 3.2.5 (Injective sequence of extensions). If f is a sequence of extensions and for any arguments $i \neq j$: $f(i) \neq f(j)$, then we say that f is an *injective sequence of extensions*.

Fact 3.2.6. Let X be a finite negative set. Then all injective sequences of extensions of X are finite.

Proof: The proof is not difficult, but laborious. The whole proof for sequences of extensions made only under rules R_1–R_9 can be found in

[5, pp. 90–93]. Here the proof differs in that our sequences are more diverse, but still finitely complex, since by the additional rules R_{10}–R_{13} at most two new elements can be added. □

DEFINITION 3.2.7 (Branch). A sequence of extensions $f\colon K \mapsto \wp(\mathsf{Ex})$ is called a *branch* iff:

(i) f is injective
(ii) for any $1 < j \in K$ and any $X \subseteq \mathsf{Ex}$: if X is an indirect extension of $f(j-1)$, then $X \neq f(j-1)$.

If a sequence of extensions of X is a branch, we shall call it a branch starting from X. Sometimes, given a branch $f\colon K \mapsto \wp(\mathsf{Ex})$ instead of using the function notation we shall represent f:

1. as a finite list of the form $X_1, \ldots, X_n$ where $n = \mathsf{max}(K)$ and $X_k = f(k)$ $(1 \leqslant k \leqslant n)$
2. by abbreviations like f_M, where M is the domain of f, *i.e.* $f\colon M \mapsto \wp(\mathsf{Ex})$
3. by small Greek letters: ϕ, ψ.

The following is an obvious consequence of the Fact 3.2.6 and the definition of a branch:

FACT 3.2.8. Let X be a finite negative set. Then any branch f being a sequence of extensions of X is finite.

DEFINITION 3.2.9 (Complete branch). A branch $X_1, \ldots, X_n$ is called a *complete branch* iff there is no branch of the form $X_1, \ldots, X_n, X_{n+1}$.

Let us observe that the Fact 3.2.6 and the definition of complete branch yield:

FACT 3.2.10. Let X be a finite negative set. Then there is a complete branch starting from X.

FACT 3.2.11. If X_1 is a finite, negative set, then for every branch of the form $X_1, \ldots, X_n$, there is a complete branch of the form $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$.

PROOF: Let X_1 be any finite, negative set and let $X_1, \ldots, X_n$ be a branch. Hence, X_n is a finite extension. Because by the Fact 3.2.10 there is a complete branch $X_n^1, \ldots, X_{n+m}^{1+m}$, so there is a complete branch of the form $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$. □

DEFINITION 3.2.12 (Closed branch, open branch). A branch $X_1, \ldots, X_n$ is *closed* iff X_n is a contradictory set. A branch is called *open* iff it is not closed.

FACT 3.2.13. If a branch is closed, then it is complete.

PROOF: Assume that $X_1, \ldots, X_n$ is a closed branch. Then the set X_n is contradictory and by the Definition 3.2.2 no rule can be applied to it, hence there is no extension of X_n, and by the definition of complete branch, $X_1, \ldots, X_n$ is complete. □

DEFINITION 3.2.14 (Maximal branch). Let Φ be a set of branches and let a branch $X_1, \ldots, X_n$ be its element. We say that $X_1, \ldots, X_n$ is a *maximal branch* of Φ iff there is no branch $X_1, \ldots, X_k$ in Φ such that $k > n$.

Given a negative $X \subseteq \mathsf{Ex}$, by $\mathsf{Br}(X)$ we denote the set of all branches starting from X and by $\mathsf{MaxBr}(X)$ the set of all maximal branches of $\mathsf{Br}(X)$.

It is clear (*cf.* [5, p. 95]) that:

FACT 3.2.15. For every finite, negative $X \subseteq \mathsf{Ex}$: $\mathsf{MaxBr}(X) \subseteq \mathsf{Br}(X)$ and $\mathsf{MaxBr}(X) \neq \varnothing$.

Now, we come to the central notion: tableau consequence.

DEFINITION 3.2.16 (Tableau consequence). We say that a formula A is a *tableau consequence* of a set $X \subseteq \mathsf{For}$ (in short: $X \vdash_{\mathcal{F}} A$) iff there is a finite $Y \subseteq X$ and every complete branch of the form $X_1 = Y \cup \{\neg A\}, X_2, \ldots, X_n$ is closed.

EXAMPLE 3.2.17. Here is a simple example of tableau inference: $\{p\} \vdash_{\mathcal{F}} (p \vee (q \looparrowright r)$ is a correct tableau inference, because every complete branch of the form $X_1 = \{p, \neg(p \vee (q \looparrowright r))\}, \ldots, X_n$ is closed. In fact, there is only one such complete branch $\{p, \neg(p \vee (q \looparrowright r))\}, \{p, \neg(p \vee (q \looparrowright r)), \neg p, \neg(q \looparrowright r)\}$ and it is indeed closed.

3.3. Metatheorems

Having presented the formal definitions, we can prove the basic metatheorems. Before we proceed to the soundness theorem, we need to start with some auxiliary lemmas.

Lemma 3.3.1. Let $\mathfrak{M}$ be any model and X be any negative set. Let $E(X)$ be the set of all extensions of X different of it. If $E(X)$ is not empty and $\mathfrak{M}$ is faithful to X, then there is at least one extension $Y \in E(X)$, such that $\mathfrak{M}$ is faithful to Y.

Proof: By inspection of the rules of extending. For example, if $(A \vee B) \in X$ and a model $\mathfrak{M}$ is faithful to X, then $\mathfrak{M} \vDash_{\mathcal{F}} A \vee B$. Notice that $X \cup \{A\}$, $X \cup \{B\} \in E(X)$ (by application of the rule R_2). Hence, $\mathfrak{M} \vDash_{\mathcal{F}} X \cup \{A\}$ or $\mathfrak{M} \vDash_{\mathcal{F}} X \cup \{B\}$, and therefore $\mathfrak{M}$ is faithful to at least one extension of X. For another example we consider a case with a non-classical formula. If $(A \looparrowright B) \in X$ and $\mathfrak{M}$ is faithful to X, then $\mathfrak{M} \vDash_{\mathcal{F}} (A \looparrowright B)$. Due to R_{11} $X \cup \{A \to B, ARB\} \in E(X)$ and $\mathfrak{M}$ is faithful to this extension. Reasoning for the remaining rules of extending is similar. □

Lemma 3.3.2. Let $\mathfrak{M}$ be a model and X_1 be a finite negative set. If $\mathfrak{M} \vDash_{\mathcal{F}} X_1$, then there is at least one complete and open branch $X_1, \ldots, X_n$, where $n \geqslant 1$.

Proof: We take a finite negative set X_1 and a model $\mathfrak{M} \vDash_{\mathcal{F}} X_1$. From the Fact 3.2.11 we know that for every branch $X_1, \ldots, X_n$, where X_1 is finite and $n \geqslant 1$, there is at least one complete branch: $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$. By the Fact 3.2.8 we also know that there are only finitely many branches starting from X_1, so in the proof we consider only finite and complete branches of the form $X_1, \ldots, X_j$, where $j \geqslant 1$. We assume they are all closed, hence X_j is a contradictory set.

By the Fact 3.2.15, some of them belong to the set of the maximal branches $\mathsf{MaxBr}(X_1)$ having the same length k. Hence, for every contradictory extension X_k, $\mathfrak{M}$ is not faithful to X_k, by the Fact 3.1.8.

Let us take any $n = k - m$, where $n \geqslant 2$ and assume that for any branch f_M under consideration, if $n \in M$, then $\mathfrak{M}$ is not faithful to X_n. Any X_{n-1} on any branch is either a contradictory extension, and then $\mathfrak{M}$ is not faithful to X_{n-1} (by the Fact 3.1.8), or for any extensions X_n of X_{n-1}, $\mathfrak{M}$ is not faithful to X_n, and then by the Lemma 3.3.1 $\mathfrak{M}$ is not faithful to X_{n-1}. Therefore, for any X_n, where $1 \leqslant n \leqslant k$, $\mathfrak{M}$ is not faithful to X_n. Specifically, when $X_n \subseteq \mathsf{For}$, we have $\mathfrak{M} \not\vDash X_n$. But this contradicts the assumption that $\mathfrak{M} \vDash_{\mathcal{F}} X_1$, for $n = 1$.

In consequence, there is at least one complete open branch $X_1, \ldots, X_n$, where $n \geqslant 1$. □

THEOREM 3.3.3 (Soundness). For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, if $X \vdash_{\mathcal{F}} A$, then $X \vDash_{\mathcal{F}} A$.

PROOF: We take any X and A such that $X \vdash_{\mathcal{F}} A$ and assume $X \nvDash_{\mathcal{F}} A$, *i.e.* there is a model $\mathfrak{M} \vDash_{\mathcal{F}} X \cup \{\neg A\}$. But there is a finite set $Y \subseteq X$, such that every complete branch of the form $X_1 = Y \cup \{\neg A\}, X_2, \ldots, X_n$, where $n \geqslant 1$, is closed. Because we know that $\mathfrak{M} \vDash_{\mathcal{F}} Y \cup \{\neg A\}$, there must be, by the Lemma 3.3.2, at least one complete open branch $X_1 = Y \cup \{\neg(A)\}, X_2, \ldots, X_n$, where $n \geqslant 1$, which provides a contradiction. Hence, $X \vDash_{\mathcal{F}} A$. □

LEMMA 3.3.4 (Compactness). For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$: $X \vDash_{\mathcal{F}} A$ iff there is a finite $Y \subseteq X$ such that $Y \vDash_{\mathcal{F}} A$.

PROOF:

1. We define the following language: $\mathsf{Var}' = \mathsf{Var} \cup \{r^{A,B} : A, B \in \mathsf{For}\}$. Next, we close Var' under Boolean connectives: $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$, obtaining the set For'. Now, we take classical consequence relation ($\vDash_{CL}$) defined semantically on For'. We know it is compact.
2. We define function $\dagger : \mathsf{For} \mapsto \mathsf{For}'$ by conditions:
   (i) for any $x \in \mathsf{Var}$, $\dagger(x) = x$
   (ii) for any $\neg A \in \mathsf{For}$, $\dagger(\neg A) = \neg\dagger(A)$
   (iii) for any $(A \star B) \in \mathsf{For}$, where $\star \in \mathsf{Con}^2 \smallsetminus \{\vartriangle, \looparrowright\}$, $\dagger((A \star B)) = (\dagger(A) \star \dagger(B))$
   (iv) for any $(A \vartriangle B) \in \mathsf{For}$, $\dagger(A \vartriangle B) = ((\dagger(A) \wedge \dagger(B)) \wedge r^{A,B})$
   (v) for any $(A \looparrowright B) \in \mathsf{For}$, $\dagger(A \looparrowright B) = ((\dagger(A) \rightarrow \dagger(B)) \wedge r^{A,B})$

Clearly, $\dagger$ is an injective function, since for any two structurally different formulae $A, B \in \mathsf{For}$, $\dagger(A) \neq \dagger(B)$.

3. We reduce classical consequence $\models_{CL}$ to the language $\dagger(\mathsf{For})$, writing for it $\models^{\dagger}_{CL}$ (so $\models^{\dagger}_{CL} \subseteq \wp(\dagger(\mathsf{For})) \times \dagger(\mathsf{For})$). Of course, $\models^{\dagger}_{CL} \subseteq \models_{CL}$, since for any $X \cup \{A\} \subseteq \dagger(\mathsf{For})$, if $X \models^{\dagger}_{CL} A$, then $X \models_{CL} A$. Moreover, the set of all valuations for the language of $\models^{\dagger}_{CL}$ is almost the same as for the language of $\models_{CL}$, since they have exactly the same propositional letters and valuations of propositional letters are extended exactly in the same way. The difference is that a valuation for $\models^{\dagger}_{CL}$ ascribes a logical value only to formulae $\dagger(\mathsf{For})$, but it can be extended in the unique way to some valuation for $\models_{CL}$; and *vice versa*: any valuation of $\models_{CL}$ can be reduced to some valuation for $\models^{\dagger}_{CL}$ only in a unique way, just by removing from the domain of that valuation formulae of the set $\mathsf{For} \smallsetminus \dagger(\mathsf{For})$. The set of all valuations for $\models^{\dagger}_{CL}$ shall be denoted by $\mathbf{V}$.
4. To show that consequence relation $\models^{\dagger}_{CL}$ is compact, we take some $X \cup \{A\} \subseteq \dagger(\mathsf{For})$ and assume $X \models^{\dagger}_{CL} A$. By 3 it holds that $X \models_{CL} A$, and since $\models_{CL}$ is compact, so there is a finite $Y \subseteq X$ that $Y \models_{CL} A$. If $Y \not\models^{\dagger}_{CL} A$, then there is a valuation $V \in \mathbf{V}$ such that $V(Y) = 1$ and $V(A) = 0$. Of course, V can be extended for the language of $\models_{CL}$ as V', which contradicts the conclusion $Y \models_{CL} A$, since V' agrees with V on $Y \cup \{A\}$. Hence consequence relation $\models^{\dagger}_{CL}$ is compact.
5. Let $\mathbf{M}$ be the set of all models for For and $\mathbf{V}$ be as in 3. We define function $\ddagger : \mathbf{M} \mapsto \mathbf{V}$ as follows. For any $\mathfrak{M}$, $\ddagger(\mathfrak{M}) = V$ iff both conditions hold:
   (i) for any $A \in \mathsf{For} \cap \mathsf{For}'$, $V(A) = v^{\mathfrak{M}}(A)$
   (ii) for any $r^{A,B} \in \mathsf{Var}'$, $V(r^{A,B}) = 1$ iff $R^{\mathfrak{M}}(A, B)$, where A, B, $\in \mathsf{For}$.

   We see that for the remaining formulae (any $A \in \mathsf{For}' \smallsetminus \mathsf{For}$) the logical value is determined by Boolean conditions for Boolean connectives (so $V(A)$ takes a value that follows from the truth-conditions for Boolean connectives).
   By induction and assumption that all models $\mathfrak{M}$ are well-defined it can be shown that the function $\ddagger$ is also well-defined, which means that for any $\mathfrak{M} \in \mathbf{M}$ and any $A \in \mathsf{For}'$:
   (i) $\ddagger(\mathfrak{M})(A) \in \{0, 1\}$

(ii) $\ddagger(\mathfrak{M})(A)$ satisfies all Boolean conditions for classical connectives (so for example, if $A = \neg B$ then $\ddagger(\mathfrak{M})(A) = 1$ iff $\ddagger(\mathfrak{M})(B) = 0$; if $A = B \vee C$ then $\ddagger(\mathfrak{M})(A) = 1$ iff $\ddagger(\mathfrak{M})(B) = 1$ or $\ddagger(\mathfrak{M})(C) = 1$; etc.).

Obviously, the function is surjective, since $\mathbf{V} \subseteq \ddagger(\mathfrak{M})$, we see that for any valuation V_1 we can find a model $\mathfrak{M}_1$ (by the equivalences in the condition of the definition of $\ddagger$) such that $\ddagger(\mathfrak{M}_1) = V_1$. Furthermore, the function is injective, since for two different models $\mathfrak{M}_1 \neq \mathfrak{M}_2$, $\ddagger(\mathfrak{M}_1) \neq \ddagger(\mathfrak{M}_2)$ (by equivalences in the condition of the definition of $\ddagger$). Hence, $\ddagger$ is a bijection.

6. Now we show that for any $\mathfrak{M} \in \mathbf{M}$ and any $A \in \mathsf{For}$:

   (✠) $\mathfrak{M} \vDash_{\mathcal{F}} A$ iff $\ddagger(\mathfrak{M})(\dagger(A)) = 1$.

   We assume $\mathfrak{M} \vDash_{\mathcal{F}} A$, for some $\mathfrak{M}$ and A. The proof will be an induction on the complexity of formula A. We will use the definitions of $\dagger$ and $\ddagger$.

   (a) The initial step. Let $A \in \mathsf{Var}$, so $\dagger(A) = A$. $\mathfrak{M} \vDash A$ iff $v^{\mathfrak{M}}(A) = 1$ iff $v^{\mathfrak{M}}(\dagger(A)) = 1$ iff there is V such that $V = \ddagger(\mathfrak{M})$ and $V(\dagger(A)) = 1$ iff $\ddagger(\mathfrak{M})(\dagger(A)) = 1$.

   (b) The inductive step. Let $B, C \in \mathsf{For}$ satisfy the thesis (✠). We examine the cases for more complex formulae:

      i. Let $A = \neg B$. $\mathfrak{M} \vDash A$ iff $\mathfrak{M} \vDash \neg B$ iff $\mathfrak{M} \nvDash B$ iff $\ddagger(\mathfrak{M})(\dagger(B)) = 0$ iff $\ddagger(\mathfrak{M})(\dagger\neg(B)) = 1$ iff $\ddagger(\mathfrak{M})(\dagger(A)) = 1$.

      ii. Let $A = B \vee C$. $\mathfrak{M} \vDash A$ iff $\mathfrak{M} \vDash B \vee C$ iff $\mathfrak{M} \vDash B$ or $\mathfrak{M} \vDash C$ iff $\ddagger(\mathfrak{M})(\dagger(B)) = 1$ or $\ddagger(\mathfrak{M})(\dagger(C)) = 1$ iff $\ddagger(\mathfrak{M})(\dagger(B) \vee \dagger(C)) = 1$ iff $\ddagger(\mathfrak{M})(\dagger(A)) = 1$.

      iii. Let $A = (B \looparrowright C)$. $\mathfrak{M} \vDash A$ iff $\mathfrak{M} \vDash (B \looparrowright C)$ iff $\mathfrak{M} \vDash B \rightarrow C$ and it is the case that $R^{\mathfrak{M}}(B, C)$ iff there is V such that $\ddagger(\mathfrak{M}) = V$ and $\ddagger(\mathfrak{M})(\dagger(B \rightarrow C)) = 1$ and $V(r^{B,C}) = 1$ iff there is a formula $((\dagger(B) \rightarrow \dagger(C)) \wedge r^{B,C})$ and $\dagger(B \looparrowright C) = ((\dagger(B) \rightarrow \dagger(C)) \wedge r^{B,C})$ hence $\ddagger(\mathfrak{M})(\dagger((B \looparrowright C)) = 1$ iff $\ddagger(\mathfrak{M})(\dagger(A)) = 1$.

7. For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$: $X \vDash_{\mathcal{F}} A$ iff $\dagger(X) \vDash^{\dagger}_{CL} \dagger(A)$.

   ($\Rightarrow$) We assume $X \vDash_{\mathcal{F}} A$ and for some V, $V(\dagger(X)) = 1$. Since $\ddagger$ is a bijection, there is exactly one $\mathfrak{M}$ such that $\ddagger(\mathfrak{M}) = V$ and by 6, $\mathfrak{M} \vDash X$. By assumption, $\mathfrak{M} \vDash_{\mathcal{F}} A$, and by 6, $\ddagger(\mathfrak{M})(A) = 1$, hence $V(\dagger(A)) = 1$.

($\Leftarrow$) We assume $\dagger(X) \vDash^{\dagger}_{CL} \dagger(A)$ and for some $\mathfrak{M}$, $\mathfrak{M} \vDash_{\mathcal{F}} X$. By 6, $\ddagger(\mathfrak{M})(\dagger(X)) = 1$, and by assumption $\ddagger(\mathfrak{M})(\dagger(A)) = 1$. Thus, by 6, $\mathfrak{M} \vDash_{\mathcal{F}} A$.

8. If $\vDash^{\dagger}_{CL}$ is compact, then $\vDash_{\mathcal{F}}$ is compact.
   We assume that $\vDash^{\dagger}_{CL}$ is compact and for some $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$: $X \vDash_{\mathcal{F}} A$. By 7, $\dagger(X) \vDash^{\dagger}_{CL} \dagger(A)$. By assumption, there is a finite $Z' \subseteq \dagger(X)$ such that $Z' \vDash^{\dagger}_{CL} \dagger(A)$. Since $\dagger$ is an injection, there is some finite $Z \subseteq X$, $\dagger(Z) = Z'$. So $\dagger(Z) \vDash^{\dagger}_{CL} \dagger(X)$, and by 7, $Z \vDash_{\mathcal{F}} A$.
9. The consequence relation $\vDash_{\mathcal{F}}$ is compact, by 4 and 8. □

LEMMA 3.3.5. Let $X_1, \ldots, X_n$ be a complete open branch. We put $L(X_n) := \{l \in X_n : l \in \mathsf{Var}$ or l is of the form $\neg x$, where $x \in \mathsf{Var}$, or $l \in \mathsf{Aux}\}$. For any model $\mathfrak{M}$: if $\mathfrak{M}$ is faithful to $L(X_n)$, $\mathfrak{M}$ is faithful to X_n.

PROOF: Assume that $X_1, \ldots, X_n$ is a complete open branch. We define a sequence of sets:

$$\begin{aligned} Y_1 &= X_n \setminus X_{n-1} \\ Y_2 &= X_{n-1} \setminus X_{n-2} \\ &\vdots \\ Y_{n-1} &= X_2 \setminus X_1 \\ Y_n &= X_1 \end{aligned}$$

Put $\mathbf{Y} := \bigcup\{Y_i : 1 \leqslant i \leqslant n\}$. First, we will show that $X_n \subseteq \mathbf{Y}$.

Take any expression $E \in X_n$. If $E \notin X_1 = Y_n$, then by the definition of a branch, there is such $1 \leqslant j \leqslant n$ that $E \notin X_j$ and $E \in X_{j+1}$. Hence $E \in X_{j+1} \setminus X_j = Y_{n-j}$. Thus, if a model is faithful to $\mathbf{Y}$, it is also faithful to X_n.

Now, assume that $\mathfrak{M}$ is faithful to $L(X_n)$. To prove that $\mathfrak{M}$ is faithful to $\mathbf{Y}$ it is sufficient to show that it is faithful to Y_i, where $1 \leqslant i \leqslant n$. We will proceed with an induction with respect to i.

Inductive basis. We have that $Y_1 = X_n \setminus X_{n-1}$ and, since the branch $X_1, \ldots, X_n$ is complete, $Y_1 \subseteq L(X_n)$. Hence $\mathfrak{M}$ is faithful to Y_1.

Inductive step. The inductive hypothesis is that the model $\mathfrak{M}$ is faithful to $\bigcup\{Y_1, \ldots, Y_i\}$, where $i < n$. It will be shown that $\mathfrak{M}$ is faithful to Y_{i+1}.

Take any expression E such that $E \notin \bigcup\{Y_1, \ldots, Y_i\}$ and $E \in Y_{i+1}$. We will examine possible forms of E. Recall that $X_n, \ldots, X_n$ is a complete branch.

(i) For E of one of the forms x, $\neg x$, ARB, $A \not R B$, where $x \in \mathsf{Var}$, $A, B \in \mathsf{For}$. $E \in L(X_n)$ and by assumption $\mathfrak{M}$ is faithful to E.

(ii) For E of the form $\neg\neg B$. By rule R_5 it is the case that $B \in \bigcup\{Y_1, \ldots, Y_i\}$ and by the inductive hypothesis $\mathfrak{M}$ is faithful to B, thus being faithful to E.

(iii) For E of the form $\neg(B \to C)$. By rule R_8 it is the case that $B, \neg C \in \bigcup\{Y_1, \ldots, Y_i\}$ and by the inductive hypothesis $\mathfrak{M}$ is faithful to B and $\neg C$, hence being faithful to $\neg(B \to C)$.

(iv) For E of the form $\neg(B \looparrowright C)$. By rule R_{13} either $\neg(B \to C) \in \bigcup\{Y_1, \ldots, Y_i\}$, or $A \not R B \in \bigcup\{Y_1, \ldots, Y_i\}$. In both cases the inductive hypothesis gives us that $\mathfrak{M}$ is faithful to E.

The cases of E being of the form $B \to C$ and $B \looparrowright C$ can be dealt with analogously.

We have established that the model $\mathfrak{M}$ is faithful to $\mathbf{Y}$. Hence, $\mathfrak{M}$ is faithful to X_n. □

LEMMA 3.3.6. Let X_1 be a set of formulae such that there is a complete open branch of the form $X_1, \ldots, X_n$. Then there is a model $\mathfrak{M}$ satisfying X_1.

PROOF: Since $X_1, \ldots, X_n$ is a complete and open branch, each X_i ($1 \leqslant i \leqslant n$) is non-contradictory and no extension rule can be further applied to X_n. We take $L(X_n)$ to be as in the Lemma 3.3.5. $L(X_n)$ is non-contradictory, since $L(X_n) \subseteq X_n$, and non-empty, since the considered branch is complete. We define the model $\mathfrak{M}$ faithful to $L(X_n)$ as follows. For all variables $x \in L(X_n)$: $v^{\mathfrak{M}}(x) := 1$, for all negated variables $\neg x \in L(X_n)$: $v^{\mathfrak{M}}(x) := 0$, the valuation of the remaining, irrelevant variables is set to, say, 1. $R^{\mathfrak{M}} := \{\langle A, B\rangle \in \mathsf{For} \times \mathsf{For} \colon ARB \in L(X_n)\}$. By the Lemma 3.3.5, the model $\mathfrak{M}$ is faithful to X_n, and hence to its subset X_1. But X_1 is a set of formulae, thus $\mathfrak{M} \models_{\mathcal{F}} X_1$. □

THEOREM 3.3.7 (Completeness). For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$: if $X \models_{\mathcal{F}} A$, then $X \vdash_{\mathcal{F}} A$.

Proof: Assume that $X \vDash_{\mathcal{F}} A$ and not $X \vdash_{\mathcal{F}} A$. By compactness we know that there is a finite subset of X having A as a semantic consequence, let us call this set Z. From the second hypothesis there is no finite $Y \subseteq X$ such that all complete branches of the form $Y \cup \{\neg A\}, X_2, \ldots, X_n$ are closed. Thus, for every finite $Y \subseteq X$ there is some complete and open branch $Y \cup \{\neg A\} = X_1, X_2, \ldots, X_n$, which together with the Lemma 3.3.6 yields existence of a model faithful to $Y \cup \{\neg A\}$. This holds for Z being a finite subset of X as well. But existence of a model for $Z \cup \{\neg A\}$ contradicts what we have earlier established that $Z \vDash_{\mathcal{F}} A$. □

3.4. Tableaux and the Confluence Theorem

Definition 3.4.1 (Tableau, complete tableau). Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ and Φ be a set of branches. The triple $\langle X, A, \Phi \rangle$ is a *tableau for* $\langle X, A \rangle$ iff the following conditions are satisfied:

(i) Φ is a subset of the set of all complete branches starting with $X \cup \{\neg A\}$;

(ii) for any two branches $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ in Φ, for any number i: if for all $k \leqslant i$ it is the case that $X_k = Y_k$ and $X_{i+1} \neq Y_{i+1}$, then there is an extension rule R such that X_{i+1} and Y_{i+1} are both extensions of $X_i = Y_i$ under R.

A tableau is a *complete tableau* if it satisfies one additional condition:

(iii) there is no Ψ satisfying conditions (i) and (ii) such that $\Phi \subset \Psi$.

The third condition warrants that no applications of an extension rule are omitted, *i.e.* if the resulting set of a rule R has n elements and there is a branch ϕ in the tableau whose i-th element is a result of application of R, then in the tableau there will also be the remaining $n-1$ branches identical with ϕ up to the $i-1$-th element, each having a different element of the resulting set of R as their i-th element.

Definition 3.4.2 (Closed tableau, open tableau). Let $\langle X, A, \Phi \rangle$ be a complete tableau. It is a *closed tableau* iff for any $\phi \in \Phi$: ϕ is a closed branch. Otherwise it is an *open tableau*.

In the appendix we present an example of a complete tableau.

FACT 3.4.3. Let X be a finite subset of For, $A \in \mathsf{For}$. Then there is a set of branches Φ, such that $\langle X, A, \Phi \rangle$ is a complete tableau for $\langle X, A \rangle$.

PROOF: Obvious, by the definition of a tableau and Fact 3.2.11. □

DEFINITION 3.4.4 (The same way of applying an extension rule). Let X, Y be negative sets of expressions, $A \in \mathsf{For}$ and R be an extending rule. Let $W \subseteq \mathsf{Ex}$ be such that:

(i) $X \cup W \cup \{A\}$ is an extension of $X \cup \{A\}$ under rule R;
(ii) $Y \cup W \cup \{A\}$ is an extension of $Y \cup \{A\}$ under rule R.

In respect to the above extensions we say that R is *applied in the same way* to $X \cup \{A\}$ and $Y \cup \{A\}$ iff for any negative set $Z \cup \{A\} \subseteq \mathsf{Ex}$, $Z \cup W \cup \{A\}$ is an extension of $Z \cup \{A\}$ under the rule R.

EXAMPLE 3.4.5. Let $X = \{p \wedge q, p \wedge r\}$, $X_1 = \{p \wedge q, p \wedge r, p, q\}$, $X_2 = \{p \wedge q, p \wedge r, p, r\}$, $Y = \{p \wedge q\}$, $Y_1 = \{p \wedge q, p, q\}$. X_1 and X_2 are not results of applying R_1 in the same way to X, while X_1 and Y_1 each are a result of applying R_1 in the same way to X and Y respectively.

LEMMA 3.4.6. Let X be a finite subset of For and $A \in \mathsf{For}$. Then, the following two conditions are equivalent:

(i) there is a closed tableau for $\langle X, A \rangle$;
(ii) every complete tableau of the form $\langle X, A, \Phi \rangle$ is closed.

PROOF: (i $\Rightarrow$ ii) Let $\langle X, A, \Psi \rangle$ be a closed tableau. Let Φ be any set of complete branches such that $\langle X, A, \Phi \rangle$ is a complete tableau. $\langle X, A, \Phi \rangle$ is either closed, or open. We shall assume the latter. Then there is a complete open branch $\phi \in \Phi$ of the form $X_1 = X \cup \{\neg A\}, \ldots, X_n$. Any $\psi \in \Psi$ is a closed branch of the form $Y_1 = X \cup \{\neg A\}, \ldots, Y_m$, and hence there is the greatest natural number k, $1 \leqslant k \leqslant \mathsf{min}(m, n)$, such that $X_i = Y_i$ for $i \leqslant k$.

By the definition of a branch we know that for i, $k \leqslant i < n$, to each set of expressions X_i corresponds an extension rule R_{X_i} such that X_{i+1} is obtained by application of R_{X_i} to X_i. Thus, we have a list of rules $R_{X_k}, \ldots, R_{X_{n-1}}$. Analogously, to $Y_k, \ldots, Y_{m-1}$ correspond rules from the list $R_{Y_k}, \ldots, R_{Y_{m-1}}$.

Let R_l be the first rule from the list $R_{Y_k}, \ldots, R_{Y_{m-1}}$ that does not occur on the list $R_{X_k}, \ldots, R_{X_{n-1}}$ or although it occurs, it is not applied

in the same way to X_l in ϕ and to Y_l in ψ. Then $Y_l \subseteq X_n$, since all rules of extending that preceded R_l on the list $R_{Y_k}, \ldots, R_{Y_{m-1}}$ are also among $R_{X_k}, \ldots, R_{X_{n-1}}$ and were applied in the same way to expand branches ϕ and ψ. Then $R_l, \ldots, R_{Y_{m-1}}$ can be applied in turn to $X_n, \ldots, X_{n+m-1}$ at the end giving $X_{n+m} = X_n \cup Y_m$. Because Y_m is closed and X_n is open, $X_{n+m} \smallsetminus X_n \neq \varnothing$. Hence, X_n can be further extended by some rules, but this contradicts the assumption that ϕ is complete.
(ii $\Rightarrow$ i) By the Fact 3.4.3 there is a complete tableau for $\langle X, A\rangle$, and by (ii) it is closed. □

LEMMA 3.4.7. Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$. Then, there is a finite $Y \subseteq X$, such that all complete tableaux of the form $\langle Y, A, \Phi\rangle$ are closed iff $X \vdash_{\mathcal{F}} A$.

PROOF: ($\Rightarrow$) Assume that all complete tableaux of the form $\langle Y, A, \Phi\rangle$ are closed. Then, for any $\phi \in \Phi$, ϕ is a closed branch. But this means that all complete branches starting with $Y \cup \{\neg A\}$ are closed, so $Y \vdash_{\mathcal{F}} A$ and, hence, $X \vdash_{\mathcal{F}} A$.
($\Leftarrow$) Assume that $X \vdash_{\mathcal{F}} A$. Then there is a finite $Y \subseteq X$ such that every complete branch starting with $Y \cup \{\neg A\}$ is closed. Hence, every complete tableau of the form $\langle Y, A, \Phi\rangle$ is closed, as by the definition of a complete tableau, Φ is a subset of the set of all complete branches starting with $Y \cup \{\neg A\}$. □

These two lemmas immediately result in the following theorem:

THEOREM 3.4.8 (Confluence). Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ and Y be a finite subset of X. Then, there is a closed tableau of the form $\langle Y, A, \Phi\rangle$ iff $X \vdash_{\mathcal{F}} A$.

Appendix

In this appendix we present an example of a complete tableau for system $\mathcal{F}$ (an $\mathcal{F}$-tableau). Since the set-theoretic notation of branches is too spacious and hardly readable, we adopt the following way of representing tableaux schemata.

A complete open $\mathcal{F}$-tableau for formula $((p \vartriangle q) \wedge (r \vartriangle q)) \to ((p \wedge r) \looparrowright q)$:

$$\underline{\neg(((p \vartriangle q) \wedge (r \vartriangle q)) \to ((p \wedge r) \looparrowright q))}$$

$$(R_8)$$

$$\underline{(p \vartriangle q) \wedge (r \vartriangle q)},\ \neg((p \wedge r) \looparrowright q))$$

$$(R_1)$$

$$\underline{p \vartriangle q},\ r \vartriangle q,\ \neg((p \wedge r) \looparrowright q))$$

$$(R_{10})$$

$$p \wedge q,\ pRq,\ \underline{r \vartriangle q},\ \neg((p \wedge r) \looparrowright q))$$

$$(R_{10})$$

$$\underline{p \wedge q},\ pRq,\ r \wedge q,\ rRq,\ \neg((p \wedge r) \looparrowright q))$$

$$(R_1)$$

$$p,\ q,\ pRq,\ \underline{r \wedge q},\ rRq,\ \neg((p \wedge r) \looparrowright q))$$

$$(R_1)$$

$$p,\ q,\ pRq,\ r,\ rRq,\ \underline{\neg((p \wedge r) \looparrowright q))}$$

$$(R_{13})$$

$p,\ q,\ pRq,\ r,\ rRq,\ \underline{\neg((p \wedge r) \to q)}$	$p,\ q,\ pRq,\ r,\ rRq,\ (p \wedge r) \not{R}\, q$
(R_8)	
$p,\ q,\ pRq,\ r,\ rRq,\ p \wedge r,\ \neg q$	
$\times$	

The first point is a negated formula, next points are lists of formulae separated by names of rules in brackets. To the set represented at a given point belong formulae listed at the given point and all underlined formulae at points above the given one. The branches consist in sets of formulae at a given point and at the points above it. The underlined formula is the one analysed by the rule immediately under it, to result in the next expansion below. When the branches fork we divide the tableau schema into columns separated by vertical lines, each column for a separate branch. Closed branches are marked by $\times$.

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