Bulletin of the **S**ection of **L**ogic
Volume 43:1/2 (2014), pp. 43–52

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KLEENE LOGIC AND INFERENCE

Abstract

In the paper a distinguished three-valued construction by Kleene [2] is analyzed. The main feature of the Kleene's logic is that the "third" logical value of indefiniteness or neutrality is not independent since it may turn into either of two classical values, i.e. truth or falsity. The second property of K_3 is that no formula is its tautology and, therefore, an inference, or entailment relation, is a matter of particular importance.

The problem of inference based on the original Kleene matrix K_3 having the true as the only distinguished value appears to be not at all simple. The first serious attempt to clarify the situation is due to Cleave [1] who departing from Körner's [4] conception of inexact classes worked out a model theoretic framework for a logic of inexact predicates. Notwithstanding, his non-standard conception of a degree of truth preserving consequence defined on K_3 is far from being satisfactory since the partial ordering of Kleene values was changed into linear one.

We aim at showing a radically different and sound solution to the problem. The inspiration comes from Körner's unsuccessful use of "modified classical logic" as an instrument of evaluation of sentences and deduction and his call for appropriate idealisation procedures. Accordingly, we focus on two operators W and S defined using Kleene values, converting appropriately neutral propositions into false or true. We define, following the scheme of q-consequence from [7] and [6], a relation of empirical inference between true or undefinite premises and the true conclusion. Thus, a conclusion α may be empirically inferred from a set of premises X, whenever, for any interpretation, it is the case that if all elements of X are not false, then α is true. The empirical inference thus constructed is non-reflexive and it interacts with the weak W and strong S unary idealization operators. Any classically consistent sentence α is deductively situated between its two idealizations, $W\alpha$ and $S\alpha$, since $W\alpha \vdash K_3\ \alpha \vdash K_3\ S\alpha$.

Keywords: three-valued logic, many-valuedness, matrix, tautology, consequence operation, structurality, logical two-valuedness, Suszko's Thesis, non-fregean logic, logical three-valuedness, inferential many-valuedness, inferential values .

1. Kleene logic

The original three-valued logic by Kleene [2] has an epistemological motivation. The principal Kleene's inspiration came from the studies of the foundations of mathematics, especially the problem of algorithms, cf. also Kleene [3]. The third logical value was designed to mark indeterminacy of some proposition at a certain stage of scientific investigation. Consequently, its assumed status is somewhat different from the classical values of truth and falsity. The starting point of Kleene's construction consists in considering the "third" category of propositions, such as whose logical value of truth or falsity is not essential for actual consideration, undefined or indetermined by means of accessible algorithms. Besides the classical values of truth (t) and falsity (f) he introduces the value of indefiniteness (u). The connectives: $\sim$ (negation), $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction), and $\equiv$ (equivalence) are described through the following truth tables:

α	$\sim\alpha$
f	t
u	u
t	f

$\rightarrow$	f	u	t
f	t	t	t
u	u	u	t
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	t
t	t	t	t

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

$\equiv$	f	u	t
f	t	u	f
u	u	u	u
t	f	u	t

Kleene distinguishes the value of truth (t). Therefore, the matrix

$$K_3 = (\{f, u, t\},\ \sim,\ \rightarrow,\ \vee,\ \wedge,\ \{t\})$$

with operations introduced by the above truth-tables will be called the (three-valued) *Kleene matrix*.

K_3 defines the non-tautological logic since any valuation which assigns the value u to each propositional variable sends any formula into u. Kleene is aware of discarding even such law of logic as the law of identity $p \rightarrow p$ and its "equivalential" version $p \equiv p$. "Here "unknown" is a category into which we can regard any proposition as falling, whose value we either do not know or choose for the moment to disregard; and it does not then exclude the other two possibilities 'true' and 'false'." ([3], p. 335). We may therefore conclude that the author of the construction treated the added logical value as apparent or as pseudo-value, distinct from the real truth-values, compare [8].

2. Inexact classes and consequence

Körner's [4] set-theoretical interpretation of Kleene's logic in terms of inexact classes gave impetus to the definition of consequence relation. An *inexact class* of a given non-empty domain A is identified with a three-valued 'characteristic function' $X_P : A \rightarrow \{-1, 0, +1\}$ corresponding to the partition of A generated by a partial definition $D(P)$ of some property P (of elements of A):

$$X_P(a) = \begin{cases} -1 & \text{when } P(a) \text{ according to } D(P) \text{ is false} \\ 0 & \text{when } P(a) \text{ is } D(P)\text{-undecidable} \\ +1 & \text{when } P(a) \text{ according to } D(P) \text{ is true.} \end{cases}$$

The operations of union ($\cup$), meet ($\cap$) and complement ($-$) are defined as follows:

$$(X \cup Y)(a) = max\{X(a), Y(a)\}$$
$$(X \cap Y)(a) = min\{X(a), Y(a)\}$$
$$(-X)(a) = -X(a).$$

The family $N\{A\}$ of inexact classes of a given set A is closed under the operations introduced, and the algebra ($N\{A\}$, $\cup$, $\cap$, $-$) is a distributive lattice with an additional unary operation that meets the following:

$$-(X \cup Y) = -X \cap -Y \qquad -(X \cap Y) = -X \cup -Y \quad - - X = X.$$

Accordingly, it is a *de Morgan lattice* (see e.g. Grätzer (1968)).

The link between inexact classes and Kleene logic is apparent: if -1, 0, $+1$ are interpreted as f, u, t, respectively, then the counterparts of algebraic operations $\cup$, $\cap$, $-$ are the connectives of disjunction, conjunction and negation of the logic of inexact predicates. Nevertheless, Cleave as showed in [1] that the 'translation' of the inexact predicate logic into the algebra of inexact classes dictated by Lindenbaum construction is feasible only after a change in the notion of a matrix consequence operation. Cleave adopts the idea that the Kleene values f, u, t are mutually comparable and linearly ordered: $f\ u\ t$. In his notation -1 replaces f, 0 stands for u, and $+1$ for t and an appropriate relation $\mapsto_C$ is defined as follows:

$X \mapsto_C \alpha$ if and only if $min\{v(\beta) : \beta \in X\} \leq v(\alpha)$ for any valuation v : *For* $\rightarrow \{-1, 0, +1\}$ interpreting $\sim, \vee$ and $\wedge$ as the Kleene connectives.

The concept of a special matrix consequence operation is thus determined by the Cleave's variant of K_3,

$$C_3 = (\{-1, 0, +1\}, \sim, \rightarrow, \vee, \wedge, \equiv, +1).$$

α is a C_3 consequence of the set of formulas X, $X \models_{C_3} \alpha$, whenever for every valuation v of the language in K_3, $min\{v(\beta) : \beta \in X\} \leq v(\alpha)$. In view of the suggested ordering, $\models_{C_3}$ is a degree of truth preserving semantic consequence operation. However, given the prime significance associated by Kleene to the values f, u, t, one may doubt whether the mutual comparability of u with the classical values f and t is acceptable. If so, the problem of the ordering and the consequence requires an essential revision.

3. Inference

Departing from the idea of "neutrality" of sentences of the third category and accepting inexactness Körner [4] tries to apply the classical logic to empirical discourse. To this aim, he proposes a special evaluation device leading to a kind of "modified two-valued logic" and, ultimately, to an instrument of evaluation validity of sentences and deduction. An interesting and natural consideration of the problem ends negatively: "The classical two-valued logic as an instrument of deduction, however, presupposes that neutral propositions are treated as if they were true, and inexact predicates as if they were exact. ..." This obviously means that the use of the classical logic for drawing empirical conclusions from empirical premises is unsound. Another possibility for passing from the empirical inexact and indefinite discourse to the "heaven" of theory is using idealisation procedures. In case of (inexact) classes this means replacement by their exact counterparts, in case of sentences turning indefinite sentences into definite, i.e. true or false. We may even imagine that every instance of empirical inference has to be preceded by its proper procedure of an idealisation.

Accordingly, we focus on two idealizations of indefinite empirical sentences: first, turning them into false and the other, into truth sentence. We define the operators corresponding to these idealisations – Weak (W) and Strong (S) – through the Kleene-like tables:

α	$W\alpha$
f	f
u	f
t	t

α	$S\alpha$
f	f
u	t
t	t

and consider the W, S - extension K_3^* of the Kleene matrix

$$K_3^* = (\{f, u, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, W, S, \{t\}).$$

For getting empirical formalization of Kleene inference, we apply the q-consequence scheme from [7] on the generic division of the set of values $\{f, u, t\}$ into three subsets $\{f\}$, $\{u\}$, $\{t\}$. Thus, f and t are the only rejected and accepted elements, respectively. The resulting q-matrix has the form

$$K_3^{*1} = (\{f, u, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, W, S, \{f\}, \{t\}),$$

and the q-matrix consequence

$$X \vdash_{K_3^{*1}} \alpha \text{ iff for every } h \in Hom(L, A)\ (h(\alpha) = t \text{ if } h(X) \subseteq \{u, t\})$$

will play a role empirical inference. An intuition behind its definition is that a conclusion may be inferred empirically form a set of premises X, when for any interpretation it is the case that if all elements of X are not false then α is true.

Let us note that the empirical inference just defined may also be interpreted as relation which holds between a set of premises X and a conclusion

α whenever is true independently from idealisation of empirical sentences in X. Here are examples of theorems establishing some essential properties of the empirical inference of Kleene sentential logic:

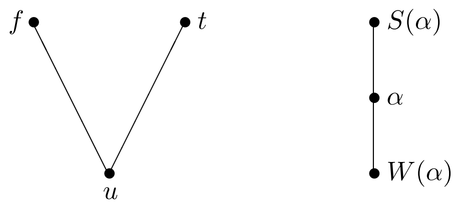
(e_1) If $X \vdash_{K_3^{*1}} \alpha$ and $X \vdash_{K_3^{*1}} \alpha \to \beta$, then $X \vdash_{K_3^{*1}} \beta$

(e_2) $X \vdash_{K_3^{*1}} \alpha \to \beta$ if and only if $X \cup \alpha \vdash_{K_3^{*1}} \beta$

(e_3) $\alpha \vdash_{K_3^{*1}} S(\alpha)$ and $W(\alpha) \vdash_{K_3^{*1}} \alpha$.

The first property is an inferential modus ponens , the second a conditional deduction theorem. The inferences in (e_3) state that from a sentence its strong idealisation follows while the sentence itself is inferred from its weak idealisation.

The diagram below shows how the Kleene values f, u, t are ordered and displays the inferential relations between a formula α and its two idealizations, $W(\alpha)$ and $S(\alpha)$:



Accordingly, a classically consistent sentence α is deductively situated between its two idealizations: $W(\alpha) \vdash_{K_3^{*1}} \alpha \vdash_{K_3^{*1}} S(\alpha)$. The logic defined by the *Kleene's matrix extension* K_3^* has tautologies. Among them, we find some formulas describing the basic properties of idealization operators:

- $\models^* W(\alpha) \to \alpha$, $\models^* \alpha \to S(\alpha)$, $\models^* W(\alpha) \to S(\alpha)$
- $\models^* W(\sim \alpha) \equiv \sim S(\alpha)$, $\models^* S(\sim \alpha) \equiv \sim W(\alpha)$,
- $\models^* W(S(\alpha)) \equiv S(\alpha)$, $\models^* (S(W(\alpha)) \equiv W(\alpha)$.

It is straightforward that a formula without any single appearance of an idealization operator is not a tautology of K_3^*.

4. Kleene four-valued logic and its inferences

A satisfying application of q-consequence to Kleene matrix extension tends to multiply indeterminacy values. The main problem is to organize a new generalized Kleene matrix structure appropriately. To start with, we take two u values instead of one, u_1, u_2 and put $u_1 \leq u_2 \leq f$, $u_1 \leq u_2 \leq t$. We also assume that f and t are incomparable, i.e. neither $f \leq t$, nor $t \leq f$. Now, we define the connectives of negation ($\sim$), disjunction ($\vee$) and conjunction ($\wedge$) adopting Kleene rules and accept the following definitions for the remaining two: $x \to y = \sim x \vee y$ and $x \equiv y = (x \to y) \wedge (y \to x)$. The truth tables of all connectives are following:

α	$\sim \alpha$
f	t
u_1	u_2
u_2	u_1
t	f

$\to$	f	u_1	u_2	t
f	t	t	t	t
u_1	u_2	u_2	u_2	t
u_2	u_1	u_1	u_2	t
t	f	u_1	u_2	t

$\vee$	f	u_1	u_2	t
f	f	u_1	u_2	t
u_1	u_1	u_1	u_2	t
u_2	u_2	u_2	u_2	t
t	t	t	t	t

$\wedge$	f	u_1	u_2	t
f	f	f	f	f
u_1	f	u_1	u_1	u_1
u_2	f	u_1	u_2	u_2
t	f	u_1	u_2	t

$\equiv$	f	u_1	u_2	t
f	t	u_2	u_1	f
u_1	u_2	u_2	u_1	u_1
u_2	u_1	u_1	u_2	u_2
t	f	u_1	u_2	t

and the basic Kleene four-valued matrix as

$$K_4 = (\{f, u_1, u_2, t\}, \quad \sim, \to, \vee, \wedge, \equiv, \{t\})$$

We also introduce the following three idealization operators W, M, S:

	$W(\alpha)$	$M(\alpha)$	$S(\alpha)$
f	f	f	f
u_1	f	f	t
u_2	f	t	t
t	t	t	t

and the W, M, S-matrix extension K_4^* of K_4:

$$K_4^* = (\{f, u_1, u_2, t\}, \sim, \to, \vee, \wedge, \equiv, W, M, S, \{t\})$$

Now, it is possible to define two non-trivial q-matrices, depending on the division of the set of values $\{f, u_1, u_2, t\}$ which is either $\{\{f, u_1\}, \{u_2\}, \{t\}\}$ or $\{\{f\}, \{u_1, u_2\}, \{t\}\}$. In the second case f is the only rejected element and in the first f, u_1 are rejected.

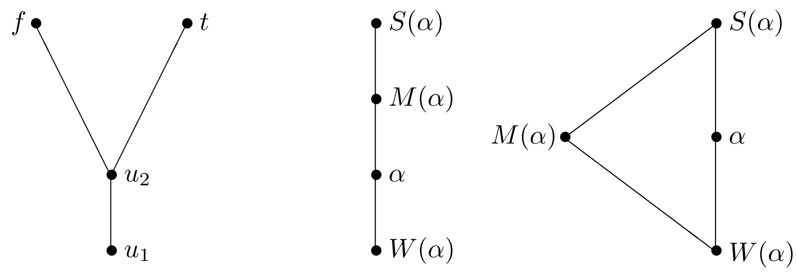
Accordingly, we get two q-matrices

$$K_4^{*1} = (\{f, u_1, u_2, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, W, M, S, \{f, u_1\}, \{t\})$$
$$K_4^{*2} = (\{f, u_1, u_2, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, W, M, S, \{f\}, \{t\})$$

Then, we have two Kleene 4-valued empirical inferences $\vdash_{K_4^{*1}}$, $\vdash_{K_4^{*2}}$ defined as follows:

$$X \vdash_{K_4^{*1}} \alpha \text{ iff for every } h \in Hom(L, A)\ (h(\alpha) = t \text{ if } h(X) \subseteq \{u_2, t\})$$
$$X \vdash_{K_4^{*2}} \alpha \text{ iff for every } h \in Hom(L, A)\ (h(\alpha) = t \text{ if } h(X) \subseteq \{u_1, u_2, t\})$$

The diagram below shows how Kleene values f, u_1, u_2, t are four ordered and displays the inferential relations between a formula α and its three idealizations $W(\alpha)$, $M(\alpha)$, $S(\alpha)$ in two cases – the middle diagram displays the situation for $\vdash_{K_4^{*1}}$ and the right for $\vdash_{K_4^{*2}}$.



Among the tautologies of K_4^* one will find all formulas listed as K_3^* valid (W and S have new meanings), and also tautologies involving M, such as $\models^* W(\alpha) \rightarrow M(\alpha)$, $\models^* M(\alpha) \rightarrow S(\alpha)$, $\models^* W(\alpha) \rightarrow M(\alpha)$, $\models^* S(M(\alpha) \equiv M(\alpha)$, $\models^* M(S(\alpha) \equiv S(\alpha)$, and $\models^* M(\sim \alpha) \equiv \sim M(\alpha)$.

5. Prospects

Further exploration of multi-valuedness within the Kleene inference scheme is possible. For any finite $n \geqslant 3$, we take the increasing chain of values $u_1, u_2, \ldots, u_{n-2}$ of "indeterminacy" values, assuming that its biggest element u_{n-2} is below two maximal elements f and t, i.e. that $u_{n-2} < f$ and $u_{n-2} < t$. Next, we define standard propositional connectives following the rules applied in the four-valued case, cf. Section 4, we introduce the $(n-1)$ idealization operations $I_1, I_2, \ldots, I_{n-1}$:

$$I_k(x) = \begin{cases} f & \text{if} \quad x = f \quad \text{or} \quad x < u_{n-k} \\ t & \text{if} \quad x = t \quad \text{or} \quad x \geqslant u_{n-k} \end{cases}$$

Finally, we set the $(n-2)$ possible q-matrices, as the result of the *monotonic* division of logical values, and $(n-1)$ possible inference relation.

The conceptual framework for n-valued Kleene inference relation deserves a separate discussion. Below, we briefly consider the 5-valued scheme, which compared to the 3 and to 4-valued schemes, will sufficiently display the way of generating inferences for more valued constructions of this type.

Assume then that $n = 5$. Accordingly, the set of values is $\{f, u_1, u_2, u_3, t\}$ and the Kleene matrix has the form:

$$K_5 = (\{f, u_1, u_2, u_3, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, \{t\}).$$

Since then we will get four idealization operators: I_1, I_2, I_3, I_4, the I-extension of K_5^*,

$$K_5^* = (\{f, u_1, u_2, u_3, t\}, \sim, \rightarrow, \vee, \wedge, \equiv, I_1, I_2, I_3, I_4, \{t\})$$

generates three non-trivial q-matrices, and, thus three Kleene empirical inference relations. These inferences correspond to the following three divisions of the sets of values of K_5^* into the subsets: of rejected, neither rejected nor accepted, and the accepted elements, $\{f, u_1, u_2\}$, $\{u_3\}$, $\{t\}$; $\{f, u_1\}$, $\{u_2, u_3\}$, $\{t\}$; $\{f\}, \{u_1, u_2, u_3\}$, $\{t\}$. The q-matrices corresponding to them, $K_5^{*_1}$, $K_5^{*_2}$ and $K_5^{*_3}$ define three matrix q-consequences, or empirical inference relations, $\vdash_{K_5^{*_1}}$, $\vdash_{K_5^{*_2}}$ and $\vdash_{K_5^{*_3}}$. The deductive relations between a formula α and its idealizations are the following:

$I_1(\alpha) \vdash_{K_5^{*_1}} \alpha \vdash_{K_5^{*_1}} I_2(\alpha) \vdash_{K_5^{*_1}} I_3(\alpha) \vdash_{K_5^{*_1}} I_4(\alpha)$;
$I_1(\alpha) \vdash_{K_5^{*_2}} \alpha \vdash_{K_5^{*_2}} I_3(\alpha) \vdash_{K_5^{*_2}} I_4(\alpha)$ and $I_1(\alpha) \vdash_{K_5^{*_2}} I_3(\alpha)$;
$I_1(\alpha) \vdash_{K_5^{*_3}} \alpha \vdash_{K_5^{*_3}} I_4(\alpha)$ and $I_1(\alpha) \vdash_{K_5^{*_3}} I_2(\alpha) \vdash_{K_5^{*_3}} I_3(\alpha) \vdash_{K_5^{*_3}} I_4(\alpha)$.

Obviously, K_5^* has tautologies, which describe the properties of idealization operators.

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