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Rafał Gruszczyński

MEREOLOGICAL FUSION AS AN UPPER BOUND

Abstract

Among characterizations of mereological set that can be found in the literature, the following two are the most important: these of *mereological sum* and *mereological fusion*. As it was proven in [5] the two relations coincide in the class of separative partial orders.

In this paper we show that the distinctive element which is sufficient for fusion to be sum is that fusion is an upper bound of its constituents. This motivates investigation of structures in which we axiomatically demand that fusion is an upper bound.

Moreover, we compare fusion relation with the standard relation of supremum and prove that in very weak structures interconnections between fusions and suprema coincide with those between suprema and mereological sums. For this reason the paper may be considered as the sequel to [2].

Keywords: mereology, logic, theory of relations

1. Basic axioms, definitions and facts about *part of* relation

We begin with a reminder of some standard definitions and basic facts about *part of* relation.

A relational structure $\mathfrak{M} = \langle M, \sqsubseteq\rangle$ is a partial order (*a poset*, for short) iff it satisfies the following three axioms of reflexivity, antisymmetry and transitivity:

$$\forall_{x\in M}\, x \sqsubseteq x\,, \tag{$\mathtt{r}_{\sqsubseteq}$}$$

$$\forall_{x,y \in M}(x \sqsubseteq y \wedge y \sqsubseteq x \to x = y)\,, \tag{$\mathtt{antis}_{\sqsubseteq}$}$$
$$\forall_{x,y,z \in M}(x \sqsubseteq y \wedge y \sqsubseteq z \to x \sqsubseteq z)\,. \tag{$\mathtt{t}_{\sqsubseteq}$}$$

In case $x \sqsubseteq y$ we say that x *is part of* y. We will call *reflexive* (resp. *antisymmetrical*, *transitive*) a structure which satisfies ($\mathtt{r}_{\sqsubseteq}$) (resp. ($\mathtt{antis}_{\sqsubseteq}$), ($\mathtt{t}_{\sqsubseteq}$)).

We define:

1. $\mathbf{R}$, $\mathbf{ANTIS}$ and $\mathbf{T}$ to be the classes of structures that are, respectively, reflexive, antisymmetrical and transitive;
2. $\mathbf{QPOS} := \mathbf{R} \cap \mathbf{T}$, i.e. $\mathbf{QPOS}$ is the class of all *quasi-posets*;
3. $\mathbf{POS} := \mathbf{R} \cap \mathbf{ANTIS} \cap \mathbf{T}$, i.e. $\mathbf{POS}$ is the class of all posets.

Remark 1.1. If $\mathbf{C}$ is a class of structures then any given sentence is said *to be true in this class* iff it is true in (satisfied by) every structure from this class. If φ is a formula expressing some property of the elements of $\mathbf{C}$, then $\mathbf{C} + \varphi$ is the class of all these structures from $\mathbf{C}$ that satisfy φ, that is:

$$\mathbf{C} + \varphi := \{\mathfrak{M} \in \mathbf{C} \mid \varphi \text{ is true in } \mathfrak{M}\}\,.$$

In $\langle M, \sqsubseteq \rangle$ we define the following auxiliary relations:

$$x \sqsubset y \overset{\mathrm{df}}{\longleftrightarrow} x \sqsubseteq y \wedge z \neq y\,, \tag{$\mathtt{df}\ \sqsubset$}$$
$$x \bigcirc y \overset{\mathrm{df}}{\longleftrightarrow} \exists_{z \in M}(z \sqsubseteq x \wedge z \sqsubseteq y)\,, \tag{$\mathtt{df}\ \bigcirc$}$$
$$x \wr y \overset{\mathrm{df}}{\longleftrightarrow} \neg\exists_{z \in M}(z \sqsubseteq x \wedge z \sqsubseteq y)\,, \tag{$\mathtt{df}\ \wr$}$$

In case $x \sqsubset y$ (resp. $x \bigcirc y$, $x \wr y$) we say that x *is a proper part of* y (resp. x *overlaps* y, x *is exterior to* y); '$x \not\sqsubseteq y$' is to mean that $\neg x \sqsubseteq y$. Moreover, we define the following operations from the domain M to its power set $\mathcal{P}(M)$[1]:

$$\mathsf{P}(x) := \{y \in M \mid y \sqsubseteq x\}\,, \tag{$\mathtt{df}\,\mathsf{P}(x)$}$$
$$\mathsf{PP}(x) := \{y \in M \mid y \sqsubset x\}\,, \tag{$\mathtt{df}\,\mathsf{PP}(x)$}$$
$$\mathsf{O}(x) := \{y \in M \mid y \bigcirc x\}\,. \tag{$\mathtt{df}\,\mathsf{O}(x)$}$$

It is easily verifiable that if $\langle M, \sqsubseteq \rangle$ satisfies ($\mathtt{r}_{\sqsubseteq}$), then:

$$z \sqsubseteq x \to z \bigcirc x \qquad \text{i.e. } \mathsf{P}(x) \subseteq \mathsf{O}(x)\,, \tag{1.1}$$
$$x \bigcirc x\,, \tag{$\mathtt{r}_{\bigcirc}$}$$

[1] $\mathcal{P}(M)$ is the power set of M, while $\mathcal{P}_+(M) := \mathcal{P}(M) \setminus \{\emptyset\}$.

and if $\langle M, \sqsubseteq\rangle$ satisfies $(\mathtt{t}_{\sqsubseteq})$, then:

$$z \sqsubseteq x \wedge z \bigcirc y \rightarrow x \bigcirc y\,. \tag{1.2}$$

Definition 1.1. An object x is called *the zero element* of a poset $\langle M, \sqsubseteq\rangle$ iff x is part of every object from M, i.e. $\forall_{y\in M}\ x \sqsubseteq y$. The uniqueness of the zero element follows from antisymmetry of $\sqsubseteq$.

2. Mereological sum and fusion

In $\langle M, \sqsubseteq\rangle$ we define two binary «hybrid» relations $\mathsf{Sum} \subseteq M \times \mathcal{P}(M)$ and $\mathsf{Fus} \subseteq M \times \mathcal{P}(M)$ of, respectively, *mereological sum* and *mereological fusion*[2]:

$$a \;\mathsf{Sum}\; X \overset{\text{df}}{\longleftrightarrow} \forall_{x\in X} x \sqsubseteq a \wedge \forall_{b\in M}(b \sqsubseteq a \rightarrow \exists_{x\in X} b \bigcirc x)\,, \tag{$\mathtt{df}\ \mathsf{Sum}$}$$

$$a \;\mathsf{Fus}\; X \overset{\text{df}}{\longleftrightarrow} \forall_{b\in M}(b \bigcirc a \longleftrightarrow \exists_{x\in X} b \bigcirc x)\,. \tag{$\mathtt{df}\ \mathsf{Fus}$}$$

Using the operations P and O we can give alternative versions of the definitions above:

$$a \;\mathsf{Sum}\; X \longleftrightarrow X \subseteq \mathsf{P}(a) \subseteq \bigcup \mathsf{O}[X]\,, \tag{$\mathtt{df}'\ \mathsf{Sum}$}$$

$$a \;\mathsf{Fus}\; X \longleftrightarrow \mathsf{O}(a) = \bigcup \mathsf{O}[X]\,. \tag{$\mathtt{df}'\ \mathsf{Fus}$}$$

where $\mathsf{O}[X]$ is the image of the set X under the operation O, i.e.:

$$\mathsf{O}[X] := \{\mathsf{O}(z) \mid z \in X\} \quad \text{and} \quad \bigcup \mathsf{O}[X] = \{y \in M \mid \exists_{z\in X}\ z \bigcirc y\}\,.$$

It follows from $(\mathtt{r}_{\sqsubseteq})$ that for every $X \subseteq M$:

$$\neg\exists_{a\in M} a \;\mathsf{Sum}\; \emptyset\,, \tag{2.1}$$

$$\neg\exists_{a\in M} a \;\mathsf{Fus}\; \emptyset\,. \tag{2.2}$$

Since $\forall_{x\in M}(x \bigcirc a \longleftrightarrow x \bigcirc a)$ is a logical truth, for an arbitrary $\mathfrak{M}$ it is the case that:

$$a \;\mathsf{Fus}\; \{a\}\,, \tag{2.3}$$

while for a reflexive structure $\mathfrak{M}$ it is routine to verify the following:

$$a \;\mathsf{Sum}\; \{a\} \tag{2.4}$$

[2] The *mereological fusion* relation was introduced by the creator of mereology, Stanisşaw Leśniewski, in [4]. Later, the relation was re-introduced in [3] by means of the language of set theory. Originally, in [4], the specific language of Leśniewski's systems had been applied. For more information see [5] and [7].

$$a \text{ Sum } \mathsf{P}(a) \tag{2.5}$$

$$\mathsf{PP}(a) \neq \emptyset \to a \text{ Sum } \mathsf{PP}(a)\,, \tag{2.6}$$

$$a \text{ Fus } \mathsf{P}(a)\,. \tag{2.7}$$

In order to prove the following we need $\mathfrak{M}$ to be both reflexive and transitive, i.e. $\mathfrak{M} \in \mathbf{QPOS}$:

$$\mathsf{PP}(a) \neq \emptyset \to a \text{ Fus } \mathsf{PP}(a)\,. \tag{2.8}$$

DEFINITION 2.1. A relational structure $\mathfrak{M} = \langle M, \sqsubseteq \rangle$ is *separative* iff it satisfies the so called *strong supplementation principle*:

$$x \not\sqsubseteq y \to \exists_{z \in M}(z \sqsubseteq x \wedge z \wr y)\,, \tag{SSP}$$

which we will also use in the following contraposed version:

$$\mathsf{P}(x) \subseteq \mathsf{O}(y) \to x \sqsubseteq y\,. \tag{$\mathtt{SSP}_{\bigcirc}$}$$

According to the conventions introduced, $\mathbf{QPOS} + (\mathtt{SSP})$ and $\mathbf{POS} + (\mathtt{SSP})$ are, respectively, the class of all separative quasi-posets and the one that consists of all separative posets.

In the sequel we present these features of mereological sums and fusions that that are important from point of view of main goals of the paper.

FACT 2.1 ([5, p. 75]). *The following extensionality principle*:

$$\mathsf{O}(x) \subseteq \mathsf{O}(y) \to x \sqsubseteq y\,.$$

is true in $\mathbf{R} + (\mathtt{SSP}_{\bigcirc})$.

PROOF: Assume $(\mathtt{SSP}_{\bigcirc})$ and $\mathsf{O}(x) \subseteq \mathsf{O}(y)$. If $z \in \mathsf{P}(x)$, then $z \in \mathsf{O}(x)$ by (1.1). So $z \in \mathsf{O}(y)$ and $x \sqsubseteq y$ by $(\mathtt{SSP}_{\bigcirc})$, as required. □

FACT 2.2 ([6, p. 217]). $\mathbf{T} + (\mathtt{SSP}) \subseteq \mathbf{R}$, *so*:

$$\mathbf{T} + (\mathtt{SSP}) = \mathbf{QPOS} + (\mathtt{SSP})\,.$$

$(\mathtt{SSP})$ *may of course be replaced with* $(\mathtt{SSP}_{\bigcirc})$.

PROOF: Suppose $x \not\sqsubseteq x$. By $(\mathtt{SSP})$ there is $y \in M$ such that (i) $y \sqsubseteq x$ and (ii) $y \wr x$. So by (ii) and $(\mathtt{df}\ \wr)$ we have that $\forall_{z \in M}(z \sqsubseteq y \to z \not\sqsubseteq x)$. From this, (i) and $(\mathtt{t}_{\sqsubseteq})$ it follows that $\neg\exists_{z \in M} z \sqsubseteq y$. In particular $y \not\sqsubseteq y$, and hence $\exists_{z \in M} z \sqsubseteq y$ by $(\mathtt{SSP})$, a contradiction. □

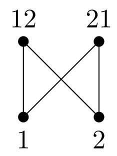

Fig. 1. Fusion does not have to be an upper bound.

3. Fusion as an upper bound

Let us start with the following standard definition.

DEFINITION 3.1. An element $a \in M$ is *an upper bound* of a set $X \subseteq M$ with respect to $\sqsubseteq$ iff $\forall_{x \in X} x \sqsubseteq a$ (i.e. $X \subseteq \mathsf{P}(a)$).

MODEL 1. Directly from ($\mathtt{df}\,\mathsf{Sum}$) it follows that if a Sum X, then a is an upper bound of X. On the other hand, mereological fusion, in stark contrast to mereological sum, does not have to be an upper bound of a set of which it is a fusion, even in the class **POS**. To see that consider the classical example in Figure 1, where we have[3]: $M := \{1, 2, 12, 21\}$, 12 Fus M, 21 Fus M, but (due to ($\mathtt{antis}_{\sqsubseteq}$)) neither $21 \not\sqsubseteq 12$ nor $12 \not\sqsubseteq 21$.

The lemma below demonstrates that in reflexive structures a sufficient condition for a fusion a of the set X to be a mereological sum of X is that a is an upper bound of X.

LEMMA 3.1. *Let* $\mathfrak{M} \in \mathbf{R}$. *Then*:

$$\forall_{a \in M} \forall_{X \in \mathcal{P}(M)} (X \subseteq \mathsf{P}(a) \wedge a\ \mathsf{Fus}\ X \rightarrow a\ \mathsf{Sum}\ X)\,.$$

PROOF: Suppose $X \subseteq \mathsf{P}(a)$ and a Fus X. Then $\mathsf{O}(a) = \bigcup \mathsf{O}[X]$. Since $\mathsf{P}(a) \subseteq \mathsf{O}(a)$ in reflexive structures, then $\mathsf{P}(a) \subseteq \bigcup \mathsf{O}[X]$. □

The aforementioned simple result is the main reason for turning our attention to the following two axioms put upon fusion relation:

$$\forall_{a \in M} \forall_{X \in \mathcal{P}(X)} \big(a\ \mathsf{Fus}\ X \rightarrow X \subseteq \mathsf{P}(a)\big)\,, \qquad (\mathtt{FUB})$$

$$\forall_{a,x,y \in M} \big(a\ \mathsf{Fus}\ \{x, y\} \rightarrow \{x, y\} \subseteq \mathsf{P}(a)\big)\,. \qquad (\mathtt{FUB}_2)$$

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[3]Diagrams are to be interpreted as follows. Filled dots (circles) mean that the object is its own part, thus for example in Figure 1 all circles are filled, which means that $\sqsubseteq$ is reflexive. Empty dots (like the one in Figure 2) mean that given element of the domain is not its own part. Lines are to be read «bottom-up», that is if two dots are linked by a line, then the one which is lower in a diagram is part of the one which is higher.

We will show that the two conditions are equivalent to each other in the class $\mathbf{T}$ and moreover, they are also equivalent to $(\mathtt{SSP})$ and to '$\mathsf{Fus} \subseteq \mathsf{Sum}$' (in the same class). For now, let us notice that:

$$\mathsf{Fus} \subseteq \mathsf{Sum} \rightarrow (\mathtt{FUB})\,, \tag{3.1}$$

$$(\mathtt{FUB}) \rightarrow (\mathtt{FUB}_2)\,. \tag{3.2}$$

4. Fusion vs. mereological sum

We will now shortly examine the following four conditions: '$\mathsf{Sum} \subseteq \mathsf{Fus}$', '$\mathsf{Fus} \subseteq \mathsf{Sum}$', '$\mathsf{Fus} = \mathsf{Sum}$' and

$$\forall_{a,x,y\in M}(a \;\mathsf{Fus}\; \{x,y\} \rightarrow a \;\mathsf{Sum}\; \{x,y\})\,. \tag{$\mathtt{Fus}_2 \subseteq \mathtt{Sum}_2$}$$

First a simple fact.

Fact 4.1. *If $\mathfrak{M} = \langle M, \sqsubseteq\rangle$ satisfies any one of the conditions* $(\mathtt{Fus}_2 \subseteq \mathtt{Sum}_2)$*, '$\mathsf{Fus} \subseteq \mathsf{Sum}$', $(\mathtt{FUB})$ or $(\mathtt{FUB}_2)$, then $\mathfrak{M} \in \mathbf{R}$.*

Proof: By (2.3) and either one of the conditions, we have that $a \sqsubseteq a$ for an arbitrary element of the domain. □

In light of the above fact we have:

$$\mathbf{T} + (\mathtt{FUB}) = \mathbf{QPOS} + (\mathtt{FUB})\,, \tag{4.1}$$

$$\mathbf{T} + (\mathtt{FUB}_2) = \mathbf{QPOS} + (\mathtt{FUB}_2)\,. \tag{4.2}$$

The following theorem points to crucial dependencies between sums, fusions and strong supplementation principle.

Theorem 4.2 ([6, p. 218]). *1. '$\mathsf{Sum} \subseteq \mathsf{Fus}$' is true in $\mathbf{T}$.*
2. $(\mathtt{SSP}_{\bigcirc})$ is true in $\mathbf{T} + (\mathtt{Fus}_2 \subseteq \mathtt{Sum}_2)$.
3. '$\mathsf{Fus} \subseteq \mathsf{Sum}$' is true in $\mathbf{T} + (\mathtt{SSP}_{\bigcirc})$.
4. The following equivalences hold in $\mathbf{T}$:

$$(\mathtt{Fus}_2 \subseteq \mathtt{Sum}_2) \longleftrightarrow (\mathtt{SSP}_{\bigcirc}) \longleftrightarrow \mathsf{Fus} \subseteq \mathsf{Sum}\,.$$

In consequence:

$$\mathbf{T}+(\mathtt{Fus}_2 \subseteq \mathtt{Sum}_2) = \mathbf{T}+ \text{'}\mathsf{Fus} \subseteq \mathsf{Sum}\text{'} = \mathbf{T}+ \text{'}\mathsf{Fus} = \mathsf{Sum}\text{'} = \mathbf{T}+(\mathtt{SSP}_{\bigcirc})\,.$$

Proof: 1. Let $x \;\mathsf{Sum}\; X$, i.e. (i) $\forall_{a\in X}\, a \sqsubseteq x$, (ii) $\forall_{b\in M}(b \sqsubseteq x \rightarrow \exists_{y\in X}\, y \bigcirc b)$. Take $a \in X$ and assume that $a \bigcirc x$, that is for some a_0, $a_0 \sqsubseteq a \wedge a_0 \sqsubseteq x$. So by (ii) there is $y \in X$ such that $y \bigcirc a_0$. Thus $a \bigcirc y$ by (1.2).

Let now $y_0 \in X$ and $y_0 \bigcirc a$. Since $y_0 \sqsubseteq x$ by (i), $a \bigcirc x$ by (1.2).

2. Suppose that $\mathsf{P}(x) \subseteq \mathsf{O}(y)$. By this and (1.2) we have that $\mathsf{O}(x) \subseteq \mathsf{O}(y)$. Therefore:

$$\mathsf{O}(y) = \bigcup \mathsf{O}[\{x, y\}]\,.$$

In consequence, by $(\mathtt{df}'\,\mathsf{Fus})$ we have that $y \mathrel{\mathsf{Fus}} \{y, x\}$. By the assumption $y \mathrel{\mathsf{Sum}} \{y, x\}$, and in consequence $x \sqsubseteq y$.

3. This time by Fact 2.2 we have $\mathbf{T} + (\mathtt{SSP}_{\bigcirc}) \subseteq \mathbf{R}$. Let $x \mathrel{\mathsf{Fus}} X$, i.e. $\forall_{y \in M}(y \bigcirc x \longleftrightarrow \exists_{z \in X} y \bigcirc z)$. From this we have:

$$\forall_{y \in M} \forall_{z \in X}(y \bigcirc z \rightarrow y \bigcirc x)\,,$$

that is $\forall_{z \in X} z \sqsubseteq x$, by Fact 2.1. On the other hand, if $y \sqsubseteq x$, then $y \bigcirc x$ by (1.1), so $\exists_{z \in X} y \bigcirc z$, as required. Thus $x \mathrel{\mathsf{Sum}} X$. □

Now we prove one of the main results of the paper.

Theorem 4.3. *1.* $(\mathtt{FUB_2})$ *is equivalent to* $(\mathtt{Fus_2} \subseteq \mathtt{Sum_2})$*.*

2. $(\mathtt{SSP}_{\bigcirc})$ *and '*$\mathsf{Fus} \subseteq \mathsf{Sum}$*' are true in* $\mathbf{T} + (\mathtt{FUB_2})$*.*

3. $(\mathtt{FUB})$ *is true in* $\mathbf{T} + (\mathtt{FUB_2})$*. So the following equivalence holds in* $\mathbf{T}$:

$$(\mathtt{FUB_2}) \longleftrightarrow (\mathtt{FUB})\,.$$

and thus:

$$\mathbf{T} + (\mathtt{FUB_2}) = \mathbf{T} + (\mathtt{FUB})\,.$$

Proof: 1. $(\rightarrow)$ Let $a \mathrel{\mathsf{Fus}} \{x, y\}$. By $(\mathtt{FUB_2})$, $\{x, y\} \subseteq \mathsf{P}(a)$. By Fact 4.1 the relation $\sqsubseteq$ is reflexive, so by (1.1), if $b \in \mathsf{P}(a)$, then $b \in \mathsf{O}(a)$ and $b \in \bigcup \mathsf{O}[\{x, y\}]$ by $(\mathtt{df}'\,\mathsf{Fus})$, as required.

$(\leftarrow)$ This implication follows from Sum being an upper bound.

2. By (4.2) and by the point 1 of this theorem it is enough to show that both conditions are true in the class $\mathbf{QPOS} + (\mathtt{Fus_2} \subseteq \mathtt{Sum_2})$. By Theorem 4.2.2 $(\mathtt{SSP}_{\bigcirc})$ is true in the class considered, so by point 3 of the same theorem the condition '$\mathsf{Fus} \subseteq \mathsf{Sum}$' must also be true.

3. The nontrivial inclusion is $\mathbf{T} + (\mathtt{FUB_2}) \subseteq \mathbf{T} + (\mathtt{FUB})$. But by 2 we have that '$\mathsf{Fus} \subseteq \mathsf{Sum}$' is true in $\mathbf{T} + (\mathtt{FUB_2})$, so $(\mathtt{FUB})$ must be true therein by (3.1). □

Let **FUB** be the class of all transitive structures in which fusion is an upper bound:

$$\mathbf{FUB} := \mathbf{T} + (\mathtt{FUB})\,(= \mathbf{T} + (\mathtt{FUB_2}))\,. \tag{$\mathtt{df}\,\mathbf{FUB}$}$$

By Fact 4.1 we have that $\mathbf{FUB} \subseteq \mathbf{R}$ and thus:

$$\mathbf{FUB} = \mathbf{QPOS} + (\mathtt{FUB})\,. \tag{4.3}$$

Theorem 4.4. *The following equivalence is true in* $\mathbf{T}$:

$$(\mathtt{FUB_2}) \longleftrightarrow (\mathtt{SSP_{\bigcirc}})\,,$$

Therefore:

$$\mathbf{FUB} = \mathbf{T} + (\mathtt{SSP_{\bigcirc}})\,.$$

Proof: $(\mathtt{FUB_2}) \to (\mathtt{SSP_{\bigcirc}})$ holds by Theorem 4.3.2.

$(\mathtt{SSP_{\bigcirc}})$ entails '$\mathsf{Fus} \subseteq \mathsf{Sum}$' by Theorem 4.2.3. So we obtain $(\mathtt{FUB_2})$ by (3.1) and (3.2). □

Corollary 4.5. *By Theorems 4.3 and 4.4 and Fact 4.1 we obtain the following chain of equalities*:

$$\begin{aligned}
\mathbf{FUB} &= \mathbf{T} + (\mathtt{FUB}) = \mathbf{T} + (\mathtt{FUB_2}) \\
&= \mathbf{T} + (\mathtt{SSP_{\bigcirc}}) = \mathbf{T} + (\mathtt{SSP}) \\
&= \mathbf{T} + \text{`}\mathsf{Fus} \subseteq \mathsf{Sum}\text{'} = \mathbf{T} + (\mathtt{Fus_2} \subseteq \mathtt{Sum_2}) \\
&= \mathbf{QPOS} + (\mathtt{FUB_2}) = \mathbf{QPOS} + (\mathtt{FUB}) \\
&= \mathbf{QPOS} + (\mathtt{SSP_{\bigcirc}}) = \mathbf{QPOS} + (\mathtt{SSP}) \\
&= \mathbf{QPOS} + \text{`}\mathsf{Fus} \subseteq \mathsf{Sum}\text{'} = \mathbf{QPOS} + (\mathtt{Fus_2} \subseteq \mathtt{Sum_2}) \\
&= \mathbf{QPOS} + \text{`}\mathsf{Fus} = \mathsf{Sum}\text{'}.
\end{aligned}$$

In consequence:

$$\mathbf{FUB} \cap \mathbf{ANTIS} = \mathbf{POS} + (\mathtt{SSP})\,.$$

5. Fusion as supremum

By a supremum of a given set $X \subseteq M$ of a relational structure $\langle M, \sqsubseteq\rangle$ we understand (standardly) the binary relation in $M \times \mathcal{P}(M)$ defined as below:

$$a \mathrel{\mathsf{Sup}} X \overset{\text{df}}{\longleftrightarrow} \forall_{x\in X}\, x \sqsubseteq a \wedge \forall_{b\in M}(\forall_{x\in X}\, x \sqsubseteq b \to a \sqsubseteq b)\,. \tag{df Sup}$$

We may also give an alternative definition of the relation in question:

$$a \mathrel{\mathsf{Sup}} X \overset{\text{df}}{\longleftrightarrow} X \subseteq \mathsf{P}(a) \wedge \forall_{b\in M}(X \subseteq \mathsf{P}(b) \to a \sqsubseteq b)\,. \tag{df$'$ Sup}$$

In [2] and [7] the following mutual interrelations between the relation mereological sum and supremum were examined[4]:

$$\mathsf{Sum} \subseteq \mathsf{Sup}\,, \tag{†}$$

[4]In $(\star)$ (and in the sequel) '$|M|$' denotes the cardinality of M.

$$\mathsf{Sup} \subseteq \mathsf{Sum}\,, \tag{‡}$$

$$\forall_{x \in M} \forall_{S \in \mathcal{P}_+(M)} (x \ \mathsf{Sup}\ S \rightarrow x \ \mathsf{Sum}\ S) \tag{$‡_+$}$$

$$\forall_{x \in M} \forall_{X \in \mathcal{P}(M)} (x \ \mathsf{Sum}\ X \longleftrightarrow X \neq \emptyset \wedge x \ \mathsf{Sup}\ X)\,, \tag{sum-sup}$$

$$|M| > 1 \longleftrightarrow \mathsf{Sup} \subseteq \mathsf{Sum}\,. \tag{$\star$}$$

Now we will be interested in the counterparts of the above conditions for the fusion relation:

$$\mathsf{Fus} \subseteq \mathsf{Sup}\,, \tag{$†^{\mathtt{f}}$}$$

$$\mathsf{Sup} \subseteq \mathsf{Fus}\,, \tag{$‡^{\mathtt{f}}$}$$

$$\forall_{x \in M} \forall_{S \in \mathcal{P}_+(M)} (x \ \mathsf{Sup}\ S \rightarrow x \ \mathsf{Fus}\ S) \tag{$‡^{\mathtt{f}}_+$}$$

$$\forall_{x \in M} \forall_{X \in \mathcal{P}(M)} (x \ \mathsf{Fus}\ X \longleftrightarrow X \neq \emptyset \wedge x \ \mathsf{Sup}\ X)\,, \tag{fus-sup}$$

$$|M| > 1 \longleftrightarrow \mathsf{Sup} \subseteq \mathsf{Fus}\,. \tag{$\star^{\mathtt{f}}$}$$

5.1. Structures in which fusions are suprema

We will make use of the following lemma, a proof of which can be found in either [5, p. 78] or [2, Theorem 5].

Lemma 5.1 ([5, p. 78]). *The following equivalence is true in* **QPOS**[5]:

$$(\mathtt{SSP}_{\bigcirc}) \longleftrightarrow (†)\,,$$

so:

$$\mathbf{QPOS} + (\mathtt{SSP}_{\bigcirc}) = \mathbf{QPOS} + (†)\,.$$

Theorem 5.2. **FUB** $= \mathbf{T} + (†^{\mathtt{f}})$, *which in light of Corollary 4.5 means that in* **T** *the following equivalences hold*:

$$(\mathtt{FUB}) \longleftrightarrow (\mathtt{FUB}_2) \longleftrightarrow (†^{\mathtt{f}}) \longleftrightarrow (\mathtt{SSP})\,.$$

Proof: ($\subseteq$) Suppose $\mathfrak{M} \in$ **FUB**. Thus by Corollary 4.5 and Lemma 5.1 we have that:

$$\mathfrak{M} \in \mathbf{QPOS} + (\mathtt{SSP}_{\bigcirc}) = \mathbf{QPOS} + \text{‘}\mathsf{Fus} \subseteq \mathsf{Sum}\text{’} = \mathbf{QPOS} + (†)\,.$$

Therefore $(†^{\mathtt{f}})$ must be true in $\mathfrak{M}$.

($\supseteq$) Directly from supremum being an upper bound. □

[5] See Appendix for an observation about the class **QPOS** + (†).

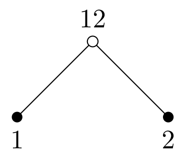

Fig. 2. A structure from $\mathbf{T} + (\dagger)$ which is not an element of $\mathbf{T} + (\dagger^{\mathtt{f}})$

LEMMA 5.3. $\mathbf{T} + (\dagger^{\mathtt{f}}) \subsetneq \mathbf{T} + (\dagger)$.

PROOF: Let $\mathfrak{M} \in \mathbf{T} + (\dagger^{\mathtt{f}})$. By Theorem 4.2.1 the condition '$\mathsf{Sum} \subseteq \mathsf{Fus}$' is true in the structure in question. Therefore $\mathsf{Sum} \subseteq \mathsf{Sup}$.

For the second part of this lemma we need to find $\mathfrak{M} \in \mathbf{T} + (\dagger)$ which does not satisfy $(\dagger^{\mathtt{f}})$. To this end take the structure from Figure 2, that is put $M := \{1, 2, 12\}$ with:

$$\sqsubseteq := \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 1,12\rangle, \langle 2,12\rangle\}\,.$$

The structure satisfies $(\mathtt{t}_{\sqsubseteq})$ in a trivial way.[6] Moreover:

$$\mathsf{Sum} = \{\langle 1, \{1\}\rangle, \langle 2, \{2\}\rangle, \langle 12, \{1,2\}\rangle\} = \mathsf{Sup}\,.$$

On the other hand, $\forall_{x\in M}\, x \bigcirc 12$, so $12 \ \mathsf{Fus}\ M$. Yet $12 \not\sqsubseteq 12$, so $(\mathtt{FUB})$ fails, and thus $(\dagger^{\mathtt{f}})$ fails either. □

THEOREM 5.4. *It is the case that*:

$$\mathbf{T} + (\dagger^{\mathtt{f}}) = \mathbf{QPOS} + (\dagger)\,.$$

In consequence, the conditions $(\dagger^{\mathtt{f}})$ *and* $(\dagger)$ *are equivalent in quasi-posets.*

PROOF: $(\subseteq)$ $\mathbf{T} + (\dagger^{\mathtt{f}}) \subseteq \mathbf{R}$, by Theorem 5.2 and Fact 4.1. So $\mathbf{T} + (\dagger^{\mathtt{f}}) \subseteq \mathbf{QPOS} + (\dagger)$, by Lemma 5.3.

$(\supseteq)$ Let $\mathfrak{M} \in \mathbf{QPOS} + (\dagger)$. By Lemma 5.1 we have that $\mathfrak{M} \in \mathbf{QPOS} + (\mathtt{SSP}_{\bigcirc})$. Thus $\mathfrak{M}$ satisfies $(\mathtt{FUB})$ by Corollary 4.5. Now we use Theorem 5.2 to conclude that $\mathfrak{M}$ must satisfy $(\dagger^{\mathtt{f}})$. □

Refereeing to the fact that in transitive structures mereological sum is fusion (see Theorem 4.2.1) we may conclude that in quasi-posets fusion is supremum iff fusion coincides with mereological sum (and iff fusion is an upper bound), which means that $\mathbf{QPOS} + (\dagger^{\mathtt{f}})$ is the class of all separative quasi-partial orders:

$$\mathbf{QPOS} + (\dagger^{\mathtt{f}}) = \mathbf{QPOS} + \text{`}\mathsf{Fus} = \mathsf{Sum}\text{'} = \mathbf{QPOS} + (\mathtt{SSP}) = \mathbf{FUB}\,.$$

[6]It may be worth noticing that the structure satisfies $(\mathtt{antis}_{\sqsubseteq})$ as well.

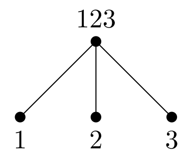

Fig. 3. A separative poset which does not satisfy $(\ddagger^{\mathbf{f}})$.

5.2. Structures in which suprema are fusions

Let us start with a simple model.

MODEL 2. The inclusion $(\ddagger^{\mathbf{f}})$ does not have to be true even in separative posets. The classical example from Figure 3 proves this. 123 is the supremum of the set $\{1, 2\}$ but $\neg 123\ \mathsf{Fus}\ \{1, 2\}$ since $3 \bigcirc 123$ and $3 \wr 1$ and $3 \wr 2$.

However, we now prove that $(\ddagger^{\mathbf{f}})$ and $(\ddagger)$ are equivalent in the class **QPOS**.

LEMMA 5.5. *The following equivalence is true in* **QPOS**:

$$\mathsf{Sup} \subseteq \mathsf{Fus} \longleftrightarrow \mathsf{Sup} \subseteq \mathsf{Sum}\,,$$

and thus:

$$\mathbf{QPOS} + (\ddagger^{\mathbf{f}}) = \mathbf{QPOS} + (\ddagger)\,.$$

PROOF: $(\rightarrow)$ By Lemma 3.1.

$(\leftarrow)$ By Theorem 4.2.1. □

In light of this, characterization of posets in which $(\ddagger^{\mathbf{f}})$ holds reduces to the problem of characterization of these that satisfy $(\ddagger)$. The similar fact holds for the remaining pairs of conditions: (i) $(\ddagger^{\mathbf{f}}_{+})$ and $(\ddagger_{+})$, (ii) $(\texttt{fus-sup})$ and $(\texttt{sum-sup})$ and (iii) $(\star^{\mathbf{f}})$ and $(\star)$, that is:

$$\mathbf{QPOS} + (\ddagger^{\mathbf{f}}_{+}) = \mathbf{QPOS} + (\ddagger_{+})\,, \tag{5.1}$$

$$\mathbf{QPOS} + (\texttt{fus-sup}) = \mathbf{QPOS} + (\texttt{sum-sup})\,, \tag{5.2}$$

$$\mathbf{QPOS} + (\star^{\mathbf{f}}) = \mathbf{QPOS} + (\star)\,. \tag{5.3}$$

PROOF: (5.1) $(\subseteq)$ Take $\mathfrak{M} \in \mathbf{QPOS} + (\ddagger^{\mathbf{f}}_{+})$ and let $x\ \mathsf{Sup}\ S \neq \emptyset$. Thus $x\ \mathsf{Fus}\ S$ and x is an upper bound of S. Therefore by Lemma 3.1 $x\ \mathsf{Sum}\ S$.

$(\supseteq)$ Let now $\mathfrak{M} \in \mathbf{QPOS} + (\ddagger_{+})$. If $x\ \mathsf{Sup}\ S \neq \emptyset$, then $x\ \mathsf{Sum}\ S$ by the assumption. So $x\ \mathsf{Fus}\ S$ by Theorem 4.2.

(5.2) ($\subseteq$) Let $\mathfrak{M} \in \mathbf{QPOS} + (\texttt{fus-sup})$.

($\rightarrow$) Let x Sum S. First, $S \neq \emptyset$ by (2.1). Second, x Fus S by Theorem 4.2. Thus x Sup $S \neq \emptyset$.

($\leftarrow$) Let now x Sup $S \neq \emptyset$. So x Fus S. But ($\texttt{FUB}$) must be true in the class considered, so '$\mathsf{Fus} \subseteq \mathsf{Sum}$' is true as well by Corollary 4.5. Thus x Sum S.

($\supseteq$) Let now $\mathfrak{M} \in \mathbf{QPOS} + (\texttt{sum-sup})$.

($\rightarrow$) $\mathfrak{M}$ satisfies ($\dagger$) by the assumption. So it must satisfy ($\dagger^{\mathtt{f}}$) by Theorem 5.4. Thus x Fus $S \rightarrow x$ Sup $S \wedge S \neq \emptyset$ by this and by (2.2).

($\leftarrow$) If x Sup $S \neq \emptyset$ then x Sum S by the assumption. Therefore x Fus S by Theorem 4.2.

(5.3) This equality follows immediately from Lemma 5.5. □

To conclude this part let us notice that:

$$\mathbf{QPOS} + (\texttt{fus-sup}) \subsetneq \mathbf{FUB}\,,$$

since the separative poset from Figure 3 shows that an element may be a supremum of a non-empty subset of the domain but fails to be a fusion.

6. Fusion as supremum and weak versions of the fusion existence axiom

In [2] the following conditions were introduced (later exhaustively examined in [7]):

$$\forall_{S\in\mathcal{P}_+(M)}(\exists_{u\in M} S \subseteq \mathsf{P}(u) \rightarrow \exists_{x\in M} x \ \mathsf{Sum}\ S), \tag{$\mathtt{W_1}\exists\mathsf{Sum}$}$$

$$\forall_{S\in\mathcal{P}_+(M)}\big(\forall_{y,z\in S}\exists_{u\in M}\ \{y,z\} \subseteq \mathsf{P}(u) \rightarrow \exists_{x\in M} x \ \mathsf{Sum}\ S\big). \tag{$\mathtt{W_2}\exists\mathsf{Sum}$}$$

and the following fact was proven about them.

FACT 6.1. *The sentence* ($\texttt{sum-sup}$) *is true in both* $\mathbf{POS} + (\mathtt{SSP}) + (\mathtt{W_1}\exists\mathsf{Sum})$ *and* $\mathbf{POS} + (\mathtt{SSP}) + (\mathtt{W_2}\exists\mathsf{Sum})$.

We will now briefly consider fusion versions of ($\mathtt{W_1}\exists\mathsf{Sum}$) and ($\mathtt{W_2}\exists\mathsf{Sum}$):

$$\forall_{S\in\mathcal{P}_+(M)}(\exists_{u\in M} S \subseteq \mathsf{P}(u) \rightarrow \exists_{x\in M} x \ \mathsf{Fus}\ S), \tag{$\mathtt{W_1}\exists\mathsf{Fus}$}$$

$$\forall_{S\in\mathcal{P}_+(M)}\big(\forall_{y,z\in S}\exists_{u\in M}\ \{y,z\}\subseteq \mathsf{P}(u)\to\exists_{x\in M} x \mathrel{\mathsf{Fus}} S\big). \tag{$\mathtt{W}_2\exists\mathsf{Fus}$}$$

and we will show that the counterpart of the above mentioned fact holds for fusion relation.

By Corollary 4.5 the sentence '$\mathsf{Fus} = \mathsf{Sum}$' is true in the class of all separative quasi-posets and in **FUB**, so we have:

$$\mathbf{QPOS} + (\mathtt{SSP}) + (\mathtt{W}_1\exists\mathsf{Fus}) = \mathbf{QPOS} + (\mathtt{SSP}) + (\mathtt{W}_1\exists\mathsf{Sum})\,, \tag{6.1}$$

$$\mathbf{QPOS} + (\mathtt{SSP}) + (\mathtt{W}_2\exists\mathsf{Fus}) = \mathbf{QPOS} + (\mathtt{SSP}) + (\mathtt{W}_2\exists\mathsf{Sum})\,, \tag{6.2}$$

$$\mathbf{FUB} + (\mathtt{W}_1\exists\mathsf{Fus}) = \mathbf{FUB} + (\mathtt{W}_1\exists\mathsf{Sum})\,, \tag{6.3}$$

$$\mathbf{FUB} + (\mathtt{W}_2\exists\mathsf{Fus}) = \mathbf{FUB} + (\mathtt{W}_2\exists\mathsf{Sum})\,. \tag{6.4}$$

In consequence we obtain the following fact, a counterpart of Fact 6.1.

FACT 6.2. *The sentence* (`fus-sup`) *is true in both* $\mathbf{POS} + (\mathtt{SSP}) + (\mathtt{W}_1\exists\mathsf{Fus})$ *and* $\mathbf{POS}+(\mathtt{SSP})+(\mathtt{W}_2\exists\mathsf{Fus})$. *Thus by Corollary 4.5 we have that* (`fus-sup`) *is true in* $(\mathbf{FUB}\cap\mathbf{ANTIS}) + (\mathtt{W}_1\exists\mathsf{Fus})$ *and* $(\mathbf{FUB}\cap\mathbf{ANTIS}) + (\mathtt{W}_2\exists\mathsf{Fus})$.

7. An application to mereoposets and mereological structures

7.1. Mereoposets

In light of (5.1) and Corollary 4.5 the class of mereological posets (mereoposets, for short) introduced in [2] and defined as follows:

$$\mathbf{MPOS} := \mathbf{POS} + (\mathtt{SSP}) + (\ddagger_+)$$

may be characterized as below:

$$\mathbf{MPOS} = \mathbf{FUB} + (\ddagger^{\mathsf{f}}_+)\,.$$

Similarly the class $\mathbf{MPOS}^+$ off all non-degenerate mereoposets, i.e.:

$$\mathbf{MPOS}^+ := \mathbf{POS} + (\mathtt{SSP}) + (\ddagger)$$

is equal to $\mathbf{FUB} + (\ddagger^{\mathsf{f}})$. The name 'non-degenerate' is a consequence of Lemma 11 from [2] according to which no poset from the class $\mathbf{POS} + (\ddagger)$ has a zero element. The counterpart of this lemma can as well be proven directly for $(\ddagger^{\mathsf{f}})$.

LEMMA 7.1. *No poset from* $\mathbf{POS} + (\ddagger^{\mathsf{f}})$ *has a zero element.*

PROOF: If $\mathfrak{M} \in \mathbf{POS}$ has a zero $\mathbf{0}$, then $\mathbf{0} \mathrel{\mathsf{Sup}} \emptyset$. But by (2.2) there is no fusion of the empty set and thus $(\ddagger^{\mathsf{f}})$ fails for $\mathfrak{M}$. □

7.2. Mereological structures

By *a mereological structure*[7] we mean any separative poset satisfying the following axiom of sum existence:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}x \mathsf{~Sum~} S\,. \tag{$\exists$Sum}$$

Let **MS** be the class of all mereological structures, i.e.:

$$\mathbf{MS} := \mathbf{POS} + (\mathtt{SSP}) + (\exists\mathsf{Sum})\,.$$

By Theorem 4.2 we have that:

$$\mathbf{MS} = \mathbf{POS} + (\mathtt{SSP}) + (\exists\,\mathsf{Fus})\,,$$

where:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}x \mathsf{~Fus~} S\,. \tag{$\exists$ Fus}$$

By Corollary 4.5 we may characterize the class of mereological structures in the following way:

$$\mathbf{MS} = (\mathbf{FUB}\cap\mathbf{ANTIS}) + (\exists\,\mathsf{Fus})\,.$$

Appendix. In the paper we have not been interested in conditions that are weaker that reflexivity, antisymmetry or transitivity. However, we would like to notice that in case of the class $\mathbf{QPOS} + (\dagger)$ we may alter the set of conditions to the following: $(\mathtt{t}_\sqsubseteq)$, $(\mathtt{wr}_\sqsubseteq)$ and $(\dagger)$, where:

$$\exists_{y\in M}y\sqsubseteq x \rightarrow x\sqsubseteq x \tag{$\mathtt{wr}_\sqsubseteq$}$$

is a weakened form of reflexivity. To be more precise, we may prove the following theorem.

Theorem A. *Any relational structure $\mathfrak{M}$ satisfies both $(\mathtt{r}_\sqsubseteq)$ and $(\dagger)$ iff it satisfies $(\mathtt{wr}_\sqsubseteq)$ and $(\dagger)$. In consequence:*

$$\mathbf{QPOS} + (\dagger) = \mathbf{T} + (\mathtt{wr}_\sqsubseteq) + (\dagger)\,.$$

Proof: $(\subseteq)$ Obvious.

[7] An exhaustive metamathematical analysis of mereological structures is carried out in [5] and [6].

($\supseteq$) Notice that:

$$\forall_{x\in M}(\mathsf{P}(x)=\emptyset\rightarrow x\ \mathsf{Sum}\ \emptyset\wedge\neg\, x\ \mathsf{Sup}\ \emptyset)\,,$$

since for $\mathsf{P}(x)=\emptyset$ we have that $\emptyset=\mathsf{P}(x)=\bigcup\mathsf{O}[\emptyset]$. Hence we obtain:

$$\exists_{x\in M}\ \mathsf{P}(x)=\emptyset\rightarrow\mathsf{Sum}\not\subseteq\mathsf{Sup}\,.$$

Thus,

$$\mathsf{Sum}\subseteq\mathsf{Sup}\rightarrow\forall_{x\in M}\exists_{y\in M}\ y\sqsubseteq x\,.$$

Therefore $\mathsf{Sum}\subseteq\mathsf{Sup}$ in conjunction with ($\mathtt{wr}_{\sqsubseteq}$) entails ($\mathtt{r}_{\sqsubseteq}$). □

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Department of Logic
Nicolaus Copernicus University
Toruń, Poland
e-mail: gruszka@umk.pl