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NOTE ON D-COMPLETENESS AND PRELINEARITY

Abstract

We prove that any set of axioms containing B, B′, and the prelinearity axiom is **D**-complete, i.e. the very same formulae are provable with the rule of condensed detachment as are with modus ponens and substitution.

1. Introduction

The rule of condensed detachment, proposed by C.A. Meredith, see e.g. [4], is a combination of the rule of modus ponens with the minimal amount of substitution. Informally, the rule of condensed detachment derives from the two given formulae the most general formula deducible from these formulae by modus ponens and substitution. Therefore, it is easy to simulate this rule by modus ponens and substitution. The opposite direction, whether the very same formulae are provable solely by the rule of condensed detachment as are by modus ponens and substitution, so called **D**-completeness, generally does not hold. In other words some substitution instances of formulae which are provable using modus ponens and substitution are not provable using condensed detachment. In particular not all substitution instances of axioms are always derivable, because otherwise all the formulae would be derivable.

It is well known, see e.g. [2], that if we take any nonempty set of axioms A containing formulae only from the following list

(I) $p \to p$
(B) $(p \to q) \to ((r \to p) \to (r \to q))$

(B′) $(p \to q) \to ((q \to r) \to (p \to r))$
(C) $(p \to (q \to r)) \to (q \to (p \to r))$
(K) $p \to (q \to p)$

then not all substitution instances of formulae provable by the rule of condensed detachment from A are provable. Therefore such an A is not a **D**-complete set of axioms.

However, if we take axioms B, B′, I, and the contraction axiom $(p \to (p \to q)) \to (p \to q)$ then such a set is **D**-complete, see [6]. We prove the same result for the set of axioms containing B, B′, and the prelinearity axiom P.

(P) $((p \to q) \to r) \to (((q \to p) \to r) \to r)$

The prelinearity axiom is a key component of mathematical fuzzy logics, see e.g. [1]. It is interesting that this **D**-complete set of axioms consists only of commonly used formulae, does not contain contraction, is not a subsystem of intuitionistic logic, and is independent even given modus ponens and substitution. Namely all these axioms are in their most general form—no special substitution instance of another axiom is needed. Moreover, as the anonymous referee pointed out this set of axioms does not contain a formula with two positive[1] and one negative occurrence of at least one variable, cf. **D**-complete sets in [5]. The prelinearity axiom does have one positive and two negative occurrences of r.

2. Preliminaries

For our purposes we understand a *logic* L as a set of formulae. We fix a countably infinite set of variables $Var = \{p, q, r, \dots\}$. The set of formulae Fml is defined in the standard way: any variable from Var is an element of Fml, if $\varphi, \psi \in Fml$ then also $(\varphi \to \psi) \in Fml$ and nothing else is a member of Fml. Hence the only connective we are interested in is the implication. The outermost brackets in formulae are mostly omitted in this paper.

A *substitution* σ is a function $\sigma\colon Var \to Fml$. We say that a substitution σ is a *renaming* if $\sigma\colon Var \to Var$ is a bijection. The result of an

[1] We say that a variable p has a positive occurrence in p. If an occurrence of p is positive (negative) in φ then this occurrence of p is positive (negative) in $\psi \to \varphi$ and negative (positive) in $\varphi \to \psi$.

application of a substitution σ on a formula φ, denoted $\sigma(\varphi)$, is the formula obtained by replacing variables in φ according to σ simultaneously. A composition of substitutions $\sigma\colon \mathit{Var} \to \mathit{Fml}$ and $\delta\colon \mathit{Var} \to \mathit{Fml}$ is a substitution $\sigma \circ \delta = \{\, \langle p, \psi\rangle \mid (\exists \psi')(\langle p, \psi'\rangle \in \sigma \text{ and } \psi = \delta(\psi'))\,\}$.

A formula ψ is a *variant of a formula* φ, abbreviated by $\psi \sim \varphi$, if there is a renaming σ such that $\psi = \sigma(\varphi)$, i.e. $\varphi = \sigma^{-1}(\psi)$. Moreover, we say that a substitution σ is a *variant of a substitution* δ if there is a renaming θ such that $\sigma = \delta \circ \theta$, i.e. $\delta = \sigma \circ \theta^{-1}$.

A *unification* of a set of formulae $F = \{\varphi_1, \dots, \varphi_n\}$ is such a substitution σ that $\sigma(\varphi_1) = \cdots = \sigma(\varphi_n)$. If such a substitution exists we say that F is unifiable. Due to the Unification Theorem of Robinson [7], for any unifiable set of formulae F there exists a most general unifier of F. A *most general unifier* (m.g.u.) σ of F is such a unification that for any other unification δ of F, there is a substitution θ such that $\sigma \circ \theta = \delta$. All the most general unifiers, if they exist, are the same up to renaming, they are variants of each other. Since this difference will be unimportant for us we shall write the m.g.u. instead of a m.g.u.

2.1. Hilbert-style proof systems

A Hilbert-style proof system consists of a set of axioms A, which is just a set of formulae, and deduction rules. The following axioms are discussed in the paper: B, B′, I, and P. The names of axioms are based on corresponding combinators in combinatory logic, with the exception of P which stands for the prelinearity axiom. To simplify notation we identify the names of axioms with the set of these axioms, e.g. $\mathsf{BB'P}$ is the set of axioms consisting of B, B′, and P.

We shall use only three deduction rules: modus ponens, substitution, and condensed detachment. The rule of modus ponens (or detachment) derives ψ from $\varphi \to \psi$ and φ. The rule of substitution derives $\sigma(\varphi)$ from φ for any substitution σ.

Definition 2.1 (Condensed Detachment). Let us have two formulae $\varphi \to \psi$ and χ. We produce a variant of χ called χ', which does not have a common variable with $\varphi \to \psi$. If there is the m.g.u. σ of φ and χ', then produce a variant σ' of σ such that no new variable in $\sigma'(\varphi)$ occurs in ψ. The condensed detachment of $\varphi \to \psi$ and χ is $\sigma'(\psi)$. Otherwise, the condensed detachment of $\varphi \to \psi$ and χ is not defined.

It is evident that the condensed detachment of $\varphi \to \psi$ and χ is defined uniquely up to variants (renaming).

A *proof* of φ in A is a finite sequence of formulae $\psi_1, \dots, \psi_n$, where $\psi_n = \varphi$, with the following properties. Every element is a member of A or is derived from the preceding elements of the sequence by a deduction rule. In this paper we study **MP**-*proofs* which have modus ponens and substitution as their only deduction rules, and **D**-*proofs* which have condensed detachment as the only deduction rule.

If there is a **D**-proof (**MP**-proof) of φ in A we say that φ is **D**-*provable* (**MP**-*provable*) in A. Since we already pointed out that the result of an application of condensed detachment is unique up to variants we mostly do not mention that if φ is **D**-provable in A then also all the variants of φ are **D**-provable in A.

To simplify notation we sometimes mix **D**-proofs and the formulae they prove. Using this we can write **D**-proofs by the standard bracket notation—(XY) means the condensed detachment of X and Y, where X and Y are axioms or derivations, but in both cases X and Y can be easily identified with unique formulae (up to renaming). The outermost brackets are omitted.

The following well known theorem, first explicitly shown probably in [3], connects **D**-provability and **MP**-provability.

THEOREM 2.1. *Let A be a set of axioms and $\mathcal{P}$ be an* **MP**-*proof in A. Then there is a* **D**-*proof $\mathcal{P}'$ in A such that every step in $\mathcal{P}$ is a substitution instance of a step in $\mathcal{P}'$.*

DEFINITION 2.2. We say that a set of formulae A *is* **D**-*complete* if any formula **MP**-provable in A is also **D**-provable in A.

It is important to point out that **D**-completeness is a property of a set of formulae (axioms). There can be two sets of formulae which **MP**-prove the very same formulae, but one of them can be **D**-complete and the other is not, see [5]. Thus strictly speaking in our sense they form two different logics if they have only the rule of condensed detachment.

3. D-completeness of $\mathsf{BB'P}$

We provide a full proof of **D**-completeness of $\mathsf{BB'P}$, following [6] and [5]. Although a short proof could be based on the fact that (PB′)((B′B′)B) is $(p \to (p \to q)) \to (p \to (p \to q))$, which together with $\mathsf{BB'I}$ forms a **D**-complete set of axioms, see [5].

The standard way to prove **D**-completeness is to show that $\varphi \to \varphi$ is provable for any φ. Then **D**-completeness immediately follows from Theorem 2.1. For any φ provable by modus ponens and substitution we can prove some ψ by condensed detachment such that φ is a substitution instance of ψ. Thus the condensed detachment of $\varphi \to \varphi$ and ψ is φ.

Lemma 3.1. *The formula $p \to p$ is* **D***-provable in* BP *and* $\mathsf{B'P}$.

Proof: We obtain that ((PP)P)((PB)B) and ((PP)P)((PB′)B′) are both $p \to p$. □

Let us note that it is not difficult to show by standard semantic arguments that $\mathsf{BB'P}$, consequently also BP and $\mathsf{B'P}$, is an independent set of axioms (having modus ponens and substitution as the only rules) thus obviously none of them can be proved from the remaining ones using only the rule of condensed detachment.

Lemma 3.2. *For any formula φ with no repeated occurrence of variables $\varphi \to \varphi$ is* **D***-provable in* $\mathsf{BB'P}$.

Proof: We prove the lemma by induction on the length of φ. If φ is a variable then use Lemma 3.1. Otherwise $\varphi = \psi \to \chi$ such that $\psi \to \psi$ and $\chi \to \chi$ are provable by the induction hypothesis. We complete the proof by applying these formulae on

$$(p \to q) \to ((r \to s) \to ((q \to r) \to (p \to s)))$$

which is (B(B′B))((B′B′)B). □

Lemma 3.3. *For any formula φ containing p with no repeated occurrence of variables, $(p \to p) \to (\varphi \to \varphi)$ is* **D***-provable in* $\mathsf{BB'P}$.

Proof: We again proceed by induction. For $\varphi = p$ we get $(p \to p) \to (p \to p)$ as (PB′)B. For $\varphi = \psi \to \chi$ we know that p occurs only in either ψ or χ.

If p is in ψ then by the induction hypothesis and Lemma 3.2 we know that $(p \to p) \to (\psi \to \psi)$ and $\chi \to \chi$ are provable. Using

$$(p \to (q \to r)) \to ((s \to t) \to (p \to ((r \to s) \to (q \to t)))) \tag{1}$$

which is (B′((BB′)(BB′)))(B′((BB)B)) we obtain $(p \to p) \to ((\psi \to \chi) \to (\psi \to \chi))$.

If p is in χ then we know that $\psi \to \psi$ and $(p \to p) \to (\chi \to \chi)$ are provable. Using

$$(q \to r) \to ((p \to (s \to t)) \to (p \to ((r \to s) \to (q \to t)))) \tag{2}$$

which is (B′B′)((B′B′)((BB)(B′B))) we obtain $(p \to p) \to ((\psi \to \chi) \to (\psi \to \chi))$. □

Lemma 3.4. *For any formula φ containing p with no repeated occurrence of variables except p, $\varphi \to \varphi$ is* **D***-provable in* BB′P.

Proof: If p occurs only once in φ we use Lemma 3.2. Otherwise we use the following construction. We take some maximal subformula $\psi \to \chi$ of φ such that ψ and χ each contain only a single occurrence of p. We know that $(p \to p) \to (\psi \to \psi)$ and $(p \to p) \to (\chi \to \chi)$ are provable by Lemma 3.3. Using

$$(p \to (q \to r)) \to ((p \to (s \to t)) \to (p \to (p \to ((r \to s) \to (q \to t))))) \tag{3}$$

which is ((P(2))(((1)B′)B))(1) and Lemma 3.1 we obtain

$$(p \to p) \to ((\psi \to \chi) \to (\psi \to \chi)).$$

We construct $(p \to p) \to (\varphi \to \varphi)$ by repeatedly using the previous method, (3) and Lemma 3.1, (1), or (2). Finally, using Lemma 3.1 we obtain $\varphi \to \varphi$. □

Theorem 3.5. *For any formula φ, $\varphi \to \varphi$ is* **D***-provable in* BB′P.

Proof: Let $p_1, \ldots, p_n$ be all the variables occurring in φ and φ_i, $1 \le i \le n$, be the formulae which are created from φ by replacing all the occurrences of variables other than p_i by fresh variables. In φ_i the only variable which can occur repeatedly is p_i and all the other variables occur exactly once. From Lemma 3.4 we know that all $\varphi_i \to \varphi_i$, $1 \leq i \leq n$, are provable.

Using the formula

$$(p \to p) \to ((p \to p) \to (p \to p))$$

which is (P(BB′))(BB) and $\varphi_1 \to \varphi_1$ we obtain $(\varphi_1 \to \varphi_1) \to (\varphi_1 \to \varphi_1)$. Then using condensed detachment on this formula and $\varphi_2 \to \varphi_2$ we obtain a formula $\varphi_{1,2} \to \varphi_{1,2}$ which has the same occurrences of p_1 and p_2 as φ has. Repeating this construction for $\varphi_3 \to \varphi_3, \ldots, \varphi_n \to \varphi_n$ leads to $\varphi \to \varphi$. □

Corollary 3.6. *Any set of formulae containing* B*,* B′*, and* P *is* **D**-*complete.*

Corollary 3.7. *Any set of formulae in which* B*,* B′*, and* P *are* **D**-*provable is* **D**-*complete.*

Moreover, for sets of formulae in which B, B′, and P are not substitution instances of more general provable formulae **MP**-provability is sufficient, because **D**-provability follows from Theorem 2.1.

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