Bulletin of the **S**ection of **L**ogic
Volume 42:1/2 (2013), pp. 43–52

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A SEGERBERG-LIKE CONNECTION BETWEEN CERTAIN CLASSES OF PROPOSITIONAL LOGICS

Abstract

In [5, 6] and [7] two classes of logics called $\mathcal{K}$ and $\mathcal{R}$ were considered. The idea is to treat the negation as "it is possible that not". Equivalently, such a negation is understood as "it is not necessary" and comes from [2]. The use of the modal language force us to choose some modal logic. In [1] a logic called **Z** was formulated with the help of **S5**, the class $\mathcal{K}$ is obtained by the use of normal modal logics, while $\mathcal{R}$ is obtained by applying the idea to regular logics.

In the present paper we will indicate connections between classes $\mathcal{K}$ and $\mathcal{R}$, that mainly refer to a Segerberg theorem expressing a connection between normal and regular modal logics.

Introduction

In [1] the propositional logic **Z** is formulated. Negation is understood there as "it is possible that not". Other connectives are treated classically. The logic **Z** is obtained by using the respective translation of theses of the modal logic **S**5. In [5, 6] and [7] the same idea was applied to classes of normal and regular modal logics. In this way the classes $\mathcal{K}$ and $\mathcal{R}$ are respectively obtained. By the definition $\mathcal{K} \subseteq \mathcal{R}$ and it is easy to see that $\mathcal{K} \neq \mathcal{R}$.

In the case of logics obtained by translation of thesis of normal logics, i.e. in the case of logics from the set $\mathcal{K}$ we have (see [5, 6]) a general way of characterizing them syntactically and semantically. Up to our knowledge, for the case of the broader class $\mathcal{R}$ only some particular results of this kind are known (see [7]). Thus, a natural question arises, whether there exist some general connections between logics from the classes $\mathcal{K}$ and $\mathcal{R}$ — connections corresponding to those known to hold between classes of normal and regular modal logics; i.e. the question of some specification of the relation between sets $\mathcal{K}$ and $\mathcal{R}$ can be considered. In the present paper we give some results in this respect. Due to the use the modal logic we will refer to some standard notions taken from this field.

1. Classes $\mathcal{R}$ and $\mathcal{K}$

DEFINITION 1. Let For be the set of all propositional formulae in the language with connectives $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and the set of propositional variables Var.

DEFINITION 2 ([7]). Let $\mathcal{R}$ be the class of all logics being any non-trivial subset of For, containing the full positive classical logic $\mathbf{CL}^{+}$in the language $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$, additionally including (dM1$_{\rightarrow}$)

$$\sim(p \wedge q) \rightarrow (\sim p \vee \sim q), \qquad (\mathrm{dM1}_{\rightarrow})$$

and closed under Modus Ponens (MP),

$$\frac{\vdash A \rightarrow B, A}{\vdash B}. \qquad (\mathrm{MP})$$

any substitution (US), and (CONTR)

$$\frac{\vdash A \rightarrow B}{\vdash \sim B \rightarrow \sim A}. \qquad (\mathrm{CONTR})$$

DEFINITION 3 ([6]). Let $\mathcal{K}$ be the class of all logics being any non-trivial subset of For, containing $\mathbf{CL}^{+}$, including (dM1$_{\rightarrow}$)

$$\sim(p \rightarrow p) \rightarrow p. \qquad (\mathrm{EFQ})$$

and closed under (MP), (US), and (CONTR).

The smallest element in $\mathcal{R}$ (in $\mathcal{K}$) is denoted by $\mathbf{R_{C2}}$ ($\mathbf{P_K}$, respectively).

2. Semantics for $\mathcal{R}$ and $\mathcal{K}$.

DEFINITION 4 ([7]). A *relational frame* (a *frame* for short) is a triple $\langle W, R, N\rangle$ consisting of a nonempty set W, a binary relation R on W, and a subset N of W. Elements of W, N, and $W \backslash N$ are called *worlds*, *normal worlds*, and *non-normal worlds*[1], respectively, while R is an *accessibility relation*.

Definition 5. A *valuation* is any function $v : \mathrm{Var} \longrightarrow 2^W$.

Definition 6. A *model* is a quadruple $\langle W, R, N, v\rangle$, where $\langle W, R, N\rangle$ is a frame and v is a valuation. We say that $\langle W, R, N, v\rangle$ is *based* on the frame $\langle W, R, N\rangle$.

Definition 7 ([7]). A formula A is *true* in a world $w \in W$ under a valuation v (notation: $w \Vdash_v A$) iff

1. if A is a propositional variable,
   $w \Vdash_v A \Longleftrightarrow w \in v(A)$.
2. if A has a form $\sim B$, for some formula B, then
   for $w \in N$:
   $w \Vdash_v \sim B \Longleftrightarrow$ there is a world w' such that wRw' and it is not the case that $w' \Vdash_v B$ (in abbreviation $w' \nVdash_v B$);
   for $w \in W \backslash N$:
   $w \Vdash_v \sim B$,
3. if A is of the form $B \wedge C$, for some formulae B and C, then
   $w \Vdash_v B \wedge C \Longleftrightarrow w \Vdash_v B$ and $w \Vdash_v C$,
4. if A is of the form $B \vee C$, for some formulae B and C, then
   $w \Vdash_v B \vee C \Longleftrightarrow w \Vdash_v B$ or $w \Vdash_v C$,
5. if A is of the form $B \rightarrow C$, for some formulae B and C, then
   $w \Vdash_v B \rightarrow C \Longleftrightarrow w \nVdash_v B$ or $w \Vdash_v C$,
6. if A is of the form $B \leftrightarrow C$, for some formulae B and C, then
   $w \Vdash_v B \leftrightarrow C \Longleftrightarrow (w \Vdash_v B$ and $w \Vdash_v C)$ or $(w \nVdash_v B$ and $w \nVdash_v C)$.

Definition 8 ([7]). A formula A is $\mathcal{R}$-*true* (*true* for short) in a model $M = \langle W, R, N, v\rangle$ (notation $M \Vdash_{\mathcal{R}} A$) iff $w \Vdash_v A$, for each $w \in W$.

Definition 9 ([7]). A formula A is $\mathcal{R}$-*valid* (in short *valid*) in a frame $\langle W, R, N\rangle$ iff it is $\mathcal{R}$-true in all models based on $\langle W, R, N\rangle$.

[1]Since there are frames for which $N = W$, the considered class can be naturally treated as a superclass of the class of frames in the sense of [6].

One can easily observe that validity defined in [6] can be meant as $\mathcal{R}$-validity in frames where $W = N$. Thus frames with the empty set of non-normal worlds can be called $\mathcal{K}$-frames.

THEOREM 1 (Completeness for $\mathbf{R_{C2}}$ and $\mathbf{P_K}$, [6, 7]). *For any* $A \in \mathrm{For}$*,*

1. $A \in \mathbf{R_{C2}}$ *iff* A *is valid in every frame.*
2. $A \in \mathbf{P_K}$ *iff* A *is valid in every frame* $\langle W, R, N\rangle$*, where* $W = N$*.*

Since (EFQ) can be falsified in the frame $\langle \{1\}, \varnothing, \varnothing\rangle$, thus evidently

FACT 1.
1. $\mathbf{R_{C2}} \subsetneq \mathbf{P_K}$.
2. $\mathcal{K} \subsetneq \mathcal{R}$.

Directly from the definition of $\mathbf{P_K}$, by Theorem 1 we have:

COROLLARY 1 ([6, 7]). $p \to (\sim p \to q) \notin \mathbf{P_K}$*, thus* $\mathbf{P_K}$ *and* $\mathbf{R_{C2}}$ *are paraconsistent.*

We will need also an operation from For into the set of modal formulas $\mathrm{For}^{\mathrm{M}}$:

DEFINITION 10 ([6]). Let $-^m : \mathrm{For} \longrightarrow \mathrm{For}^{\mathrm{M}}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}^{\mathrm{M}}$ the following conditions:

1. $(a)^m = a$,
2. $(\sim A)^m = \Diamond\neg((A)^m)$,
3. $(A \wedge B)^m = (A^m \wedge B^m)$,
4. $(A \vee B)^m = (A^m \vee B^m)$,
5. $(A \to B)^m = (A^m \to B^m)$,
6. $(A \leftrightarrow B)^m = (A^m \leftrightarrow B^m)$.

Here we have an extension of the result from [6] that concerned the $\mathcal{R}$-validity. Below for $A \in \mathrm{For}^{\mathrm{M}}$, $w \vDash_v A$ means that A is true in the world w in the sense used for modal regular logics.

LEMMA 1. *For any* $A \in \mathrm{For}$*, any model* $\mathcal{M} = \langle W, N, R, v\rangle$*, and any* $w \in W$ *the following holds:* $w \Vdash_v A$ *iff* $w \vDash_v (A)^m$*.*

PROOF: Goes by induction. The case of propositional variable is obvious.

Assume that the equivalence under consideration holds for any formula of the complexity smaller than the complexity of a given formula A. Let us consider the main functor of A and any $w \in W$. The cases of $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are obvious. Let $A = \sim B$, for some $B \in \text{For}$. For $w \in N$ we have $w \Vdash_v A$ iff there is a world w' such that wRw' and $w' \nVdash_v B$. By the inductive hypothesis this holds iff there is a world w' such that wRw' and $w' \nvDash_v (B)^m$ iff $w \vDash_v \Diamond\neg((B)^m)$ iff $w \vDash_v (A)^m$.

For $w \in W \backslash N$ we have $w \Vdash_v \sim B$ and $w \vDash_v \Diamond\neg((B)^m)$ by definition of truth in non-normal worlds. □

And we have a reverse translation:

Definition 11 ([6]). Let $-^u : \text{For}^{\text{M}} \longrightarrow \text{For}$ be a function satisfying for any $a \in \text{Var}, A, B \in \text{For}$ following conditions:

1. $(a)^u = a$
2. $(\neg A)^u = ((A)^u \rightarrow \sim(p \rightarrow p))$
3. $(A \wedge B)^u = (A^u \wedge B^u)$
4. $(A \vee B)^u = (A^u \vee B^u)$
5. $(A \rightarrow B)^u = (A^u \rightarrow B^u)$
6. $(A \leftrightarrow B)^u = (A^u \leftrightarrow B^u)$
7. $(\Diamond A)^u = \sim((A)^u \rightarrow \sim(p \rightarrow p))$,
8. $(\Box A)^u = (\sim((A)^u) \rightarrow \sim(p \rightarrow p))$.

Let us recall that **K** (and **C2**) denotes the smallest element in the set of all normal (and respectively regular) modal logics.

Using Lemma 1 and respective completeness results for $\mathbf{R_{C2}}$ and **C2** we directly obtain:

Fact 2. *For any* $A \in \text{For}$*:* $\vdash_{\mathbf{R_{C2}}} A$ *iff* $A^m \in \mathbf{C2}$.

Definition 12.
1. For $X \subseteq \text{For}^{\text{M}}$, let $\mathbf{K}[X]$ be the smallest normal modal logic containing the logic **K** and the set X.
2. For $X \subseteq \text{For}$, let $\mathbf{R_{C2}}[X]$ ($\mathbf{P_K}[X]$) be the smallest logic in the class $\mathcal{R}$ (in the class $\mathcal{K}$) which contains $\mathbf{R_{C2}}$ ($\mathbf{P_K}$) and the set X. In the case that $X = \{A_1, \ldots, A_n\}$, we will write $\mathbf{R_{C2}A}_1 \ldots \mathbf{A}_n$ ($\mathbf{P_K A}_1 \ldots \mathbf{A}_n$).
3. For $X \subseteq \text{For}^{\text{M}}$, let $\mathbf{P}_{\mathbf{K}[X]} = \mathbf{P_K}[X^u]$, where $X^u = \{A^u : A \in X\}$.

Now we can refer to logics from $\mathcal{R}$. We have an easy:

FACT 3. 1. $\mathbf{R_{C2}}[X] \subseteq \mathbf{P_K}[X]$
2. *If* $\mathbf{P_K}[X]$ *is paraconsistent, so is* $\mathbf{R_{C2}}[X]$.

FACT 4 ([6]). *For* $X \subseteq \mathrm{For}^{\mathrm{M}}$, *such that* $p \to \Box p \notin \mathbf{K}[X]$ *the logic* $\mathbf{P}_{\mathbf{K}[X]}$ *is paraconsistent.*

Thus

COROLLARY 2. *If* $p \to \Box p \notin \mathbf{K}[X]$, *then the logic* $\mathbf{R_{C2}}[X^u]$ *is paraconsistent.*

Let us recall that $\mathbf{D} = \mathbf{K}[\Box p \to \Diamond p]$ and $\mathbf{Triv} = \mathbf{K}[\Box p \to \Diamond p, p \to \Box p]$. By the facts 3 and 4 we have:

FACT 5. *For any normal modal logic* $\mathbf{K}[X] \supseteq \mathbf{D}$, *such that* $\mathrm{For}^{\mathrm{M}} \neq \mathbf{K}[X] \neq \mathbf{Triv}$, *the logic* $\mathbf{R_{C2}}[X^u]$ *is paraconsistent.*

Let us recall that $\mathbf{R}_\bot$ is the smallest logic in $\mathcal{R}$ containing the formula

$$\sim(p \to p) \tag{$\bot$}$$

We have

LEMMA 2.

$$\vdash_{\mathbf{R_{C2}}} \sim p \to (\sim q \vee \sim(q \to p))$$

PROOF: Let us consider the following derivation

1. $\sim(q \wedge (q \to p)) \to \sim q \vee \sim(q \to p)$ (dM1$_\to$) and (US)
2. $\sim p \to \sim(q \wedge (q \to p))$ $\mathbf{CL}^+$ and (CONTR)
3. $\sim p \to (\sim q \vee \sim(q \to p))$ 1, 2, $\mathbf{CL}^+$ and (MP)

□

Let for any $A \in \mathrm{For}$

$$A^{(1)} = \sim(p \to p) \vee A$$

Consider the following[2].

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[2] Cf. [8, vol. II, Lemma 2.2].

LEMMA 3. *Let* $A_1, \ldots A_n \in$ For. *Assume that for any* $A \in$ For*:*

$$\vdash_{\mathbf{P_K A_1 \ldots A}_n} A \text{ iff } \vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} A^{(\mathbf{1})},$$

where

$$\sim(\sim(p \to p) \to q) \to \sim(p \to p) \qquad (\mathtt{N}_{\sim}^{(1)})$$

PROOF: ($\Leftarrow$) First observe that by (EFQ) and $\mathbf{CL}^+$ we see that $(p \to p) \to (\sim(p \to p) \to q) \in \mathbf{P_K}$, thus by (CONTR) we obtain that $\sim(\sim(p \to p) \to q) \to \sim(p \to p) \in \mathbf{P_K}$, i.e. $(\mathtt{N}_{\sim}^{(1)}) \in \mathbf{P_K}$. Of course, $\vdash_{\mathbf{P_K A_1 \ldots A_n}} A_i$ for every $1 \leqslant i \leqslant n$, so by $\mathbf{CL}^+$ we have $\vdash_{\mathbf{P_K A_1 \ldots A_n}} A_i^{(\mathbf{1})}$ for every $1 \leqslant i \leqslant n$. Now, since $\mathbf{P_K A_1} \ldots \mathbf{A}_n$ is closed on all rules indicated in Definition 2, thus $\vdash_{\mathbf{P_K A_1 \ldots A}_n} A^{(\mathbf{1})}$, but by (EFQ) and (US) we see that $\vdash_{\mathbf{P_K}} \sim(p \to p) \to A$, therefore using $\mathbf{CL}^+$ we conclude that $\vdash_{\mathbf{P_K A_1 \ldots A_n}} A$.

($\Rightarrow$) Consider the proof in the sense of $\vdash_{\mathbf{P_K A_1 \ldots A}_n}$ of a formula A. We will show by the induction, that for every member C of the considered proof it holds that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} C^{(\mathbf{1})}$.

($\mathbf{CL}^+$, dM1$_\to$). If $C \in \mathbf{CL}^+$ or C is an instance of (dM1$_\to$), then we obtain $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} C$ so also $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} C^{(\mathbf{1})}$.

(EFQ). If C is of the form $\sim(p \to p) \to p$, then by $\mathbf{CL}^+$ and (US) we have that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee C$, since $q \vee (q \to p) \in \mathbf{CL}^+$.

(MP). Assume that C was obtained in the initial proof by (MP). Thus by the inductive hypothesis $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee B$ and $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee (B \to C)$ for some $B \in$ For. By $\mathbf{CL}^+$ we obtain that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee C$.

(US). Assume that C was obtained by (US). By the inductive hypothesis $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee B$ for some $B \in$ For and a substitution s such that $C = s(B)$. By (US) we obtain $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(s(p) \to s(p)) \vee s(B)$, moreover since $\vdash_{\mathbf{CL}^+} (p \to p) \to (q \to q)$, therefore by (US) and (CONTR) we see that $\vdash_{\mathbf{R_{C2}}} \sim(s(p) \to s(p)) \to \sim(p \to p)$. Then, by $\mathbf{R_{C2}}$ we get $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee s(B)$

(CONTR). Assume that C was obtained by (CONTR). By the inductive hypothesis $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee (D \to E)$ for some $D, E \in$ For such that $C = \sim E \to \sim D$. By $\mathbf{CL}^+$ we have $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} (\sim(p \to p) \to (D \to E)) \to (D \to E)$ and by (CONTR) we see that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}}$

$\sim(D \to E) \to \sim(\sim(p \to p) \to (D \to E))$. Using $(\mathtt{N}^{(1)}_{\sim})$ and (US) we see that $\vdash_{\mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}} \sim(\sim(p \to p) \to (D \to E)) \to \sim(p \to p)$, therefore by the transitivity of $\to$ we obtain that $\vdash_{\mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1\ldots\mathbf{A}^{(1)}_n} \sim(D \to E) \to \sim(p \to p)$. Thus, by $(\mathbf{CL}^+)$ we deduce that $\vdash_{\mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1\ldots\mathbf{A}^{(1)}_n} \sim D \vee \sim(D \to E) \to \sim D \vee \sim(p \to p)$. By Lemma 2 and (US) we see that $\vdash_{\mathbf{R_{C2}}} \sim E \to (\sim D \vee \sim(D \to E))$, so using the transitivity of $\to$ we get: $\vdash_{\mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1\ldots\mathbf{A}^{(1)}_n} \sim E \to (\sim D \vee \sim(p \to p))$. Finally, by $(\mathbf{CL}^+)$ we conclude that $\vdash_{\mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1\ldots\mathbf{A}^{(1)}_n} \sim(p \to p) \vee (\sim E \to \sim D)$. □

We have the following fact:

FACT 6. *For any* $A \in \mathrm{For}$

$$\vdash_{\mathbf{R}_\perp} A \textit{ iff } \vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to A.$$

PROOF: Right-to-left implication is obvious.

($\Rightarrow$) We proceed as in the left-to-right part of the proof of Lemma 3 by showing that for every element C of the proof with respect to $\mathbf{R}_\perp$ of the formula A it holds that $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to C$. Consider the case of (CONTR). By $(\mathbf{CL}^+)$ and (CONTR) we obtain $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to \sim D$ for any formula D. Thus again by $(\mathbf{CL}^+)$ we have that $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to (\sim E \to \sim D)$.

For the case of (US) it is enough to recall that if $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to C$, then also $\vdash_{\mathbf{R_{C2}}} \sim(a \to a) \to C$ for any variable a that does not appear in C, so also $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to s(C)$ for any substitution s.

The other cases are obvious. □

Now we obtain (cf. [8, vol. II, Corollary 2.4])

THEOREM 2. $\mathbf{R}_\perp \cap \mathbf{P_K}\mathbf{A}_1 \ldots \mathbf{A}_n = \mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1 \ldots \mathbf{A}^{(1)}_n$.

PROOF: ($\supseteq$) As in the proof of Lemma 3, we see that $\vdash_{\mathbf{P_K}\mathbf{A}_1\ldots\mathbf{A}_n} \mathtt{N}^{(1)}_{\sim}$ and $\vdash_{\mathbf{P_K}\mathbf{A}_1\ldots\mathbf{A}_n} A^{(1)}_i$ for every $1 \leqslant i \leqslant n$.

Thanks to ($\perp$) and $(\mathbf{CL}^+)$ we obtain also that $\vdash_{\mathbf{R}_\perp} \mathtt{N}^{(1)}_{\sim}$ and $\vdash_{\mathbf{R}_\perp} A^{(1)}_i$ for every $1 \leqslant i \leqslant n$. Thus $\mathbf{R}_\perp \cap \mathbf{P_K}\mathbf{A}_1 \ldots \mathbf{A}_n \supseteq \mathbf{R_{C2}}\mathtt{N}^{(1)}_{\sim}\mathbf{A}^{(1)}_1 \ldots \mathbf{A}^{(1)}_n$.

($\subseteq$) Assume that $\vdash_{\mathbf{R}_\perp} A$ and $\vdash_{\mathbf{P_K}\mathbf{A}_1\ldots\mathbf{A}_n} A$. By the first assumption and Fact 6 we have that $\vdash_{\mathbf{R_{C2}}} \sim(p \to p) \to A$ while by the second assumption and Lemma 3 we conclude that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} \sim(p \to p) \vee A$. Therefore by $(\mathbf{CL}^{+})$ we see that $\vdash_{\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)}\ldots\mathbf{A}_n^{(1)}} A$. □

It easily follows that:

COROLLARY 3. *If $\mathbf{P_K A}_1 \ldots \mathbf{A}_n$ is paraconsistent, so is $\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)} \ldots \mathbf{A}_n^{(1)}$.*

Let us recall

COROLLARY 4 (Completeness for $\mathbf{R}_{\perp}$, [7]). *A formula A is $\mathcal{R}$-valid in every frame with empty set of normal worlds iff $A \in \mathbf{R}_{\perp}$.*

We can easily see:

LEMMA 4. *A formula A is valid in every frame with the empty set of normal worlds iff A is valid in the frame $\langle\{1\}, \varnothing, \varnothing\rangle$.*

Thus, we can deduce that[3]:

COROLLARY 5. *If $\mathbf{P_K A}_1 \ldots \mathbf{A}_n$ is sound and complete with respect to the class $\mathscr{C}$ of frames, then $\mathbf{R_{C2}}\mathtt{N}_{\sim}^{(1)}\mathbf{A}_1^{(1)} \ldots \mathbf{A}_n^{(1)}$ is sound and complete with respect to the class $\mathscr{C} \cup \langle\{1\}, \varnothing, \varnothing\rangle$.*

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[3] Cf. [8, vol. II, Theorem 2.1].

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