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PSEUDO-BCI-LOGIC

Abstract
A non-commutative version of the BCI-logic, pseudo-BCI-logic, is introduced. Although it is not algebraizable, it is extended to logic which is so. The main result of the paper says that a pseudo-BCI-algebra is an algebraic counterpart of this extended logic (Theorem 3.2).

Keywords and phrases: pseudo-BCI-logic, pseudo-BCI-algebra, algebraizability of logic

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1. Introduction

The BCI-logic, mentioned by A. N. Prior in [11], is attributed to C. A. Meredith and dated in 1956. Its significance is due to a certain correspondence between combinators and implicational formulas (see [2] and [10]). The BCI-logic is the propositional logic with the axioms:

\[(B) \quad (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)),\]
\[(C) \quad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma)) ,\]
\[(I) \quad \alpha \rightarrow \alpha\]

and the only inference rule:

\[(MP): \quad \frac{\alpha, \alpha \rightarrow \beta}{\beta}.\]

In 1966 K. Iséki introduced the concept of BCI-algebras as an algebraic counterpart of the BCI-logic (see [5]). Unfortunately, BCI-algebras fails to
be the models of the BCI-logic. W. J. Blok and D. Pigozzi proved that the BCI-logic is not algebraizable (see Theorem 5.9 of [1]). A BCI-algebra is an algebraic counterpart of the BCI-logic extended on one additional inference rule (see [7]):

\[(Imp): \frac{\alpha, \beta}{\alpha \rightarrow \beta}.\]

In this paper we present a non-commutative version of the BCI-logic, pseudo-BCI-logic \(psBCI\). Although it is not algebraizable, we easily extend it to logic \(psBCI'\) which is so. Moreover, we show that pseudo-BCI-algebras are the models of logic \(psBCI'\), which is the main result of the paper. We do this similarly as it is done in [8] for pseudo-BCK-logic. The reader should also be familiar with [1].

2. Pseudo-BCI-algebras

A pseudo-BCI-algebra is a structure \(X = (X, \leq, \rightarrow, \Rightarrow, 1)\), where \(\leq\) is a binary relation on a set \(X\), \(\rightarrow\) and \(\Rightarrow\) are binary operations on \(X\) and 1 is an element of \(X\) such that for all \(x, y, z \in X\), we have

(a1) \(x \rightarrow y \leq (y \rightarrow z) \Rightarrow (x \rightarrow z), x \Rightarrow y \leq (y \Rightarrow z) \rightarrow (x \Rightarrow z),\)

(a2) \(x \leq (x \rightarrow y) \Rightarrow y, x \leq (x \Rightarrow y) \rightarrow y,\)

(a3) \(x \leq x,\)

(a4) if \(x \leq y\) and \(y \leq x\), then \(x = y,\)

(a5) \(x \leq y\) if \(x \Rightarrow y = 1\) if \(x \Rightarrow y = 1.\)

It is obvious that any pseudo-BCI-algebra \((X, \leq, \rightarrow, \Rightarrow, 1)\) can be regarded as a universal algebra \((X, \rightarrow, \Rightarrow, 1)\) of type \((2, 2, 0)\). Note that every pseudo-BCI-algebra satisfying \(x \Rightarrow y = x \Rightarrow y\) for all \(x, y \in X\) is a BCI-algebra. Notice also that every pseudo-BCI-algebra satisfying \(x \leq 1\) for all \(x \in X\) is a pseudo-BCK-algebra.

Now we list some basic properties of pseudo-BCI-algebras from [3], [6] and [9]. Let \(X\) be a pseudo-BCI-algebra. The following holds for all \(x, y, z \in X:\)

(b1) if \(1 \leq x\), then \(x = 1,\)

(b2) if \(x \leq y\), then \(y \rightarrow z \leq x \rightarrow z\) and \(y \Rightarrow z \leq x \Rightarrow z,\)

(b3) if \(x \leq y\) and \(y \leq z\), then \(x \leq z,\)

(b4) \(x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z),\)
(b5) \( x \leq y \rightarrow z \) iff \( y \leq x \rightsquigarrow z \),
(b6) \( x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightsquigarrow y) \),
(b7) if \( x \leq y \), then \( z \rightarrow x \leq y \) and \( z \rightsquigarrow x \leq z \rightsquigarrow y \),
(b8) \( 1 \rightarrow x = 1 \rightsquigarrow x = x \),
(b9) \( (x \rightarrow y) \rightsquigarrow y \rightarrow y = x \rightarrow y \),
(b10) \( x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1 \),
(b11) \( x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1 \),
(b12) \( x \rightarrow 1 = 1 \),
(b13) \( x \rightsquigarrow 1 = 1 \rightarrow y \rightarrow 1 \),
(b14) \( x \rightarrow 1 = x \rightsquigarrow 1 \).

Remark. If \( \mathcal{X} = (X, \leq, \rightarrow, \rightsquigarrow, 1) \) is a pseudo-BCI-algebra, then, by (a3),
(a4), (b3) and (b1), \( (X, \leq) \) is a poset with 1 as a maximal element.

The class of pseudo-BCI-algebras forms a quasivariety:

**Lemma 2.1.** An algebra \( \mathcal{X} = (X, \rightarrow, \rightsquigarrow, 1) \) of type \( (2, 2, 0) \) is a pseudo-BCI-
algebra if and only if it satisfies the following identities and quasi-identity:

(i) \( (x \rightarrow y) \rightsquigarrow ([y \rightarrow z] \rightsquigarrow (x \rightarrow z)) = 1 \),
(ii) \( (x \rightsquigarrow y) \rightarrow ([y \rightsquigarrow z] \rightarrow (x \rightsquigarrow z)) = 1 \),
(iii) \( 1 \rightarrow x = x \),
(iv) \( 1 \rightarrow x = x \),
(v) \( x \rightarrow y = 1 \) & \( y \rightarrow x = 1 \) \( \Rightarrow \) \( x = y \).

**Proof:** Every pseudo-BCI-algebra obviously satisfies (i)–(v). Conversely, assume that an algebra \( \mathcal{X} \) satisfies (i)–(v). Putting \( x = 1 \), \( y = 1 \) and \( z = x \) in (i) and (ii) and using (iii) and (iv), we have

\[
1 = (1 \rightsquigarrow 1) \rightarrow [(1 \rightsquigarrow x) \rightarrow (1 \rightsquigarrow x)] = x \rightarrow x
\]

and

\[
1 = (1 \rightarrow 1) \rightarrow [(1 \rightarrow x) \rightarrow (1 \rightarrow x)] = x \rightsquigarrow x.
\]

So, (a3) is satisfied. Now, putting \( x = 1 \), \( y = x \) and \( z = y \) in (i) and (ii) we get, by (iii) and (iv),

\[
1 = (1 \rightarrow x) \rightarrow [(x \rightarrow y) \rightsquigarrow (1 \rightarrow y)] = x \rightarrow [(x \rightarrow y) \rightsquigarrow y]
\]

and
\[1 = (1 \leadsto x) \rightarrow [(x \leadsto y) \rightarrow (1 \leadsto y)] = x \rightarrow [(x \leadsto y) \rightarrow y].\]

Hence, (a2) is also satisfied. Further, if \(x \rightarrow y = 1\), then, by (iv), \(x \leadsto y = x \leadsto (1 \leadsto y) = x \leadsto [(x \rightarrow y) \leadsto y] = 1\), and analogously, if \(x \leadsto y = 1\), then, by (iii), \(x \rightarrow y = x \rightarrow (1 \rightarrow y) = x \rightarrow [(x \leadsto y) \rightarrow y] = 1\). Thus, \(x \rightarrow y = 1\) iff \(x \leadsto y = 1\). It is therefore easily seen that the relation \(\leq\) is defined by
\[x \leq y \iff x \rightarrow y = 1 \iff x \leadsto y = 1\]
making the structure \((X, \leq, \rightarrow, \leadsto, 1)\) into a pseudo-BCI-algebra.

\[\Box\]

Remark. Since pseudo-BCI-algebras include BCI-algebras, which are not closed under homomorphic images (see [12]), it follows that the quasivariety of pseudo-BCI-algebras is not a variety.

3. Pseudo-BCI-logic

In this section we present pseudo-BCI-logic, a non-commutative version of BCI-logic. Following Hájek’s definition of his basic logic (see [4]), definition of pseudo-BCI-logic is as follows:

The formulas of pseudo-BCI-logic (psBCI, for short) are built from propositional variables and the primitive connectives \(\rightarrow\) and \(\leadsto\). The following formulas are the axioms of \(psBCI\) (where \(\alpha, \beta\) and \(\gamma\) are arbitrary formulas):

\[(B1) \ (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \leadsto (\alpha \rightarrow \gamma)),\]
\[(B2) \ (\alpha \leadsto \beta) \rightarrow ((\beta \leadsto \gamma) \rightarrow (\alpha \leadsto \gamma)),\]
\[(C1) \ (\alpha \rightarrow (\beta \leadsto \gamma)) \rightarrow (\beta \leadsto (\alpha \rightarrow \gamma)),\]
\[(C2) \ (\alpha \leadsto (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \leadsto \gamma)),\]
\[(I) \ \alpha \rightarrow \alpha.\]

The inference rules are:

\[(MP): \frac{\alpha \rightarrow \beta}{\beta},\]
\[(Imp1): \frac{\alpha \rightarrow \beta}{\alpha \leadsto \beta},\]
\[(Imp2): \frac{\alpha \leadsto \beta}{\alpha \rightarrow \beta}.\]

Remark. Using advanced methods and techniques of [1] it can be proved that the logic \(psBCI\) is not algebraizable (particularly see Theorem 5.9 of [1]).
In order to be algebraizable, we have to extend pseudo-BCI-logic on the inference rule:

\((\text{Imp}): \alpha, \beta \vdash \alpha \rightarrow \beta)\).

The extended logic, pseudo-BCI’-logic \((\text{psBCI}'\), for short) has the axioms: (B1), (B2), (C1), (C2) and (I), and the inference rules: (MP), (Imp1), (Imp2) and (Imp).

Next theorem shows the algebraizability of the logic \(\text{psBCI}'\) (in the sense of [1]).

**Theorem 3.1.** The logic \(\text{psBCI}'\) is algebraizable with the set of equivalence formulas \(\triangle = \{x \rightarrow y, y \rightarrow x\}\) and defining equation \(x = x \rightarrow x\).

**Proof:** Following the notation of [1], we write \(\alpha \triangle \beta\) as an abbreviation of \(\{\alpha \rightarrow \beta, \beta \rightarrow \alpha\}\) for any formulas \(\alpha, \beta\). In order to show that \(\text{psBCI}'\) is algebraizable, by Theorem 4.7 of [1], we have to prove the following properties, for all formulas \(\alpha, \beta, \gamma, \alpha_1, \beta_1\) (for the convenience we write \(\vdash\) instead of \(\vdash_{\text{psBCI}'}\)):

1. \(\vdash \alpha \triangle \alpha\),
2. \(\alpha \triangle \beta \vdash \beta \triangle \alpha\),
3. \(\alpha \triangle \beta, \beta \triangle \gamma \vdash \alpha \triangle \gamma\)
4. \(\alpha \triangle \beta, \alpha_1 \triangle \beta_1 \vdash (\alpha \rightarrow \alpha_1) \triangle (\beta \rightarrow \beta_1), (\alpha \leftrightarrow \alpha_1) \triangle (\beta \leftrightarrow \beta_1)\),
5. \(\alpha \vdash \alpha \triangle (\alpha \rightarrow \alpha)\).

(i): It is immediate consequence of (I).
(ii): It is trivial, because \(\alpha \triangle \beta = \beta \triangle \alpha\).
(iii): By (B1), \(\alpha \triangle \beta \vdash (\beta \rightarrow \gamma) \leftrightarrow (\alpha \rightarrow \gamma)\). Hence, \(\alpha \triangle \beta, \beta \triangle \gamma \vdash (\alpha \rightarrow \gamma)\). Now, replacing \(\alpha\) and \(\gamma\) we get \(\alpha \triangle \beta, \beta \triangle \gamma \vdash (\alpha \rightarrow \gamma)\). Thus (iii) holds.
(iv): From (B1) and (Imp2) it follows \(\alpha \triangle \beta \vdash (\alpha \rightarrow \alpha_1) \rightarrow (\beta \rightarrow \alpha_1)\) and \(\alpha \triangle \beta \vdash (\beta \rightarrow \alpha_1) \rightarrow (\alpha \rightarrow \alpha_1)\). So, \(\alpha \triangle \beta \vdash (\alpha \rightarrow \alpha_1) \triangle (\beta \rightarrow \alpha_1)\). \((1)\)

By (Imp1), \(\alpha \triangle \beta \vdash (\alpha \leftrightarrow \beta)\) and \(\alpha \triangle \beta \vdash (\beta \leftrightarrow \alpha)\). Hence, by (B2), \(\alpha \triangle \beta \vdash (\alpha \leftrightarrow \alpha_1) \rightarrow (\beta \leftrightarrow \alpha_1)\) and \(\alpha \triangle \beta \vdash (\beta \leftrightarrow \alpha_1) \rightarrow (\alpha \leftrightarrow \alpha_1)\). Thus, \(\alpha \triangle \beta \vdash (\alpha \leftrightarrow \alpha_1) \triangle (\beta \leftrightarrow \alpha_1)\). \((2)\)
Further, by (B1), \( \vdash (\beta \to \alpha_1) \to ((\alpha_1 \to \beta_1) \to (\beta \to \beta_1)) \) and \( \vdash (\beta \to \beta_1) \to ((\beta \to \beta_1) \to (\beta \to \alpha_1)) \). Hence, by (C1), \( \vdash (\alpha_1 \to \beta_1) \to ((\beta \to \alpha_1) \to (\beta \to \beta_1)) \) and \( \vdash (\beta_1 \to \alpha_1) \to ((\beta \to \beta_1) \to (\beta \to \alpha_1)) \). Thus,
\[
\alpha_1 \Delta \beta_1 \vdash (\beta \to \alpha_1) \Delta (\beta \to \beta_1). \tag{3}
\]

Similarly, by (B1) and (Imp1), \( \vdash (\beta \rightsquigarrow \alpha_1) \to ((\alpha_1 \rightsquigarrow \beta_1) \to (\beta \rightsquigarrow \beta_1)) \) and \( \vdash (\beta \to \beta_1) \to ((\beta \to \beta_1) \to (\beta \to \alpha_1)) \). Hence, by (C2), \( \vdash (\alpha_1 \to \beta_1) \to ((\beta \rightsquigarrow \alpha_1) \rightsquigarrow (\beta \rightsquigarrow \beta_1)) \) and \( \vdash (\beta_1 \to \alpha_1) \to ((\beta \to \beta_1) \rightsquigarrow (\beta \to \alpha_1)) \). Thus, \( \alpha_1 \Delta \beta_1 \vdash (\beta \to \alpha_1) \to (\beta \to \beta_1) \) and \( \alpha_1 \Delta \beta_1 \vdash (\beta \rightsquigarrow \beta_1) \to (\beta \rightsquigarrow \alpha_1) \) and so, by (Imp2), \( \alpha_1 \Delta \beta_1 \vdash (\beta \rightsquigarrow \alpha_1) \to (\beta \to \beta_1) \) and \( \alpha_1 \Delta \beta_1 \vdash (\beta \rightsquigarrow \beta_1) \to (\beta \to \alpha_1) \). Therefore,
\[
\alpha_1 \Delta \beta_1 \vdash (\beta \to \alpha_1) \Delta (\beta \rightsquigarrow \beta_1). \tag{4}
\]

Finally, by (iii), (1) and (3), we obtain
\[
\alpha \Delta \beta, \alpha_1 \Delta \beta_1 \vdash (\alpha \to \alpha_1) \Delta (\beta \to \beta_1)
\]
and similarly, by (iii), (2) and (4) we get
\[
\alpha \Delta \beta, \alpha_1 \Delta \beta_1 \vdash (\alpha \rightsquigarrow \alpha_1) \Delta (\beta \rightsquigarrow \beta_1)
\]
which end the proof of (iv).

(v): To prove (v) we must verify:

(a) \( \alpha \vdash \alpha \to (\alpha \to \alpha) \),
(b) \( \alpha \vdash (\alpha \to \alpha) \to \alpha \),
(c) \( \alpha \to (\alpha \to \alpha), (\alpha \to \alpha) \to \alpha \vdash \alpha \).

(a): We have it by (I) and (Imp).
(b): By (i) and (Imp1), \( \vdash (\alpha \to \alpha) \rightsquigarrow (\alpha \to \alpha) \), so by (C2), \( \vdash \alpha \to ((\alpha \to \alpha) \rightsquigarrow \alpha) \). Hence, \( \alpha \vdash (\alpha \to \alpha) \rightsquigarrow \alpha \) and, by (Imp2), \( \alpha \vdash (\alpha \to \alpha) \to \alpha \). Thus (b) holds.

(c): By (i) and (Imp1) we have \( \vdash ((\alpha \to \alpha) \to \alpha) \rightsquigarrow ((\alpha \to \alpha) \to \alpha) \), which implies, by (C2), \( \vdash (\alpha \to \alpha) \to ((\alpha \to \alpha) \to \alpha) \rightsquigarrow \alpha \). Since, by (i), \( \vdash \alpha \to \alpha \), it follows, by (MP), \( \vdash ((\alpha \to \alpha) \to \alpha) \rightsquigarrow \alpha \) and, by (Imp2), \( \vdash ((\alpha \to \alpha) \to \alpha) \to \alpha \). Thus, (c) also holds.

Therefore, the logic \( ps\text{BCI}' \) is algebraizable. \( \square \)

The equivalent quasivariety semantics (see [1]) for the logic \( ps\text{BCI}' \) is a quasivariety \( \mathcal{I} \) of algebras \( (X, \to, \rightsquigarrow) \) of type \( (2, 2) \) satisfying certain identities and quasi-identities, which are derived from the axioms and inference rules of \( ps\text{BCI}' \) using \( \Delta = \{ x \to y, y \to x \} \) and \( x = x \to x \), such that
(i) for every set of formulas $\Sigma$ and every formula $\alpha$,

$$
\Sigma \vdash_{\text{psBCT}} \alpha \iff \{ \beta = \beta : \beta \in \Sigma \} \vdash_I \alpha = \alpha \to \alpha,
$$

(ii) for every formulas $\alpha, \beta$,

$$
\alpha = \beta = || = I \{ \alpha \to \beta = (\alpha \to \beta) \to (\alpha \to \beta), \beta \to \alpha = (\beta \to \alpha) \to (\beta \to \alpha) \}.
$$

Notice that $\vdash_I \alpha \to \beta = (\alpha \to \beta) \to (\alpha \to \beta)$ iff $\vdash_{\text{psBCT}} \alpha \to \beta$, and similarly, $\vdash_I \beta \to \alpha = (\beta \to \alpha) \to (\beta \to \alpha)$ iff $\vdash_{\text{psBCT}} \beta \to \alpha$. Thus,

$$
\vdash_I \alpha = \beta \iff (\vdash_{\text{psBCT}} \alpha \to \beta \land \vdash_{\text{psBCT}} \beta \to \alpha) \iff \vdash_{\text{psBCT}} \alpha \triangle \beta.
$$

Next theorem is the main result of the paper and it says that the class of pseudo-BCI-algebras forms an algebraic semantics for the logic $\text{psBCT}$.

**Theorem 3.2.** The quasivariety of pseudo-BCI-algebras is definitionally equivalent to the equivalent quasivariety semantics for the logic $\text{psBCT}$.

**Proof:** First, note that by (I) and (Imp) we have $\vdash (\alpha \to \alpha) \to (\beta \to \beta)$ and $\vdash (\beta \to \beta) \to (\alpha \to \alpha)$. Thus, $\vdash (\alpha \to \alpha) \triangle (\beta \to \beta)$. Analogously, using additionally (Imp1), we obtain that $\vdash (\alpha \to \alpha) \triangle (\alpha \to \alpha)$ and $\vdash (\alpha \to \alpha) \triangle (\beta \to \beta)$. Hence, the equivalent algebraic semantics $I$ satisfies the identities $x \to x = y \to y = y \to y$. Thus, every algebra $(X, \to, \to)$ in $I$ possesses a constant 1 such that $1 = x \to x = x \to x$ for all $x \in X$.

Let $I^*$ be the class consisting of algebras $(X, \to, \to, 1)$ such that $(X, \to, \to)$ belongs to $I$. Using Theorem 2.17 of [1], we get that the quasivariety $I^*$ is axiomatized as follows:

1. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
2. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
3. $(x \to (y \to z)) \to (y \to (x \to z)) = 1$,
4. $(y \to (x \to z)) \to (x \to (y \to z)) = 1$,
5. $x \to x = 1$,
6. $x = 1 \land x \to y = 1 \Rightarrow y = 1$,
7. $x \to y = 1 \Rightarrow x \to y = 1$,
8. $x \to y = 1 \Rightarrow x \to y = 1$,
9. $x = 1 \land y = 1 \Rightarrow x \to y = 1$,
10. $x \to y = 1 \land y \to x = 1 \Rightarrow x = y$.

It is obvious that every pseudo-BCI-algebra satisfies (1)–(10). Hence, the quasivariety of pseudo-BCI-algebras is included in $I^*$. 


Conversely, let $(X, \to, \leadsto, 1)$ be an algebra belonging to $I^*$. From Lemma 2.1 it suffices to show the following equations

$$1 \to x = x \quad \text{and} \quad 1 \leadsto x = x.$$ 

From (3), (4) and (10) we get the following identity

$$x \to (y \leadsto z) = y \leadsto (x \to z).$$

Hence, by (5) and (7), $1 \to ((1 \to x) \leadsto x) = (1 \to x) \leadsto (1 \to x) = 1$ and $1 \leadsto ((1 \leadsto x) \to x) = (1 \leadsto x) \to (1 \leadsto x) = 1$. Thus, by (6) and (7), $(1 \to x) \leadsto x = 1$ and $(1 \leadsto x) \to x = 1$, and so, by (8), $(1 \to x) \to x = 1$ and $(1 \leadsto x) \to x = 1$. On the other hand, by (5), (7), (8), $x \to (1 \to x) = x \leadsto (1 \to x) = 1 \to (x \leadsto x) = 1 \to 1 = 1$ and $x \leadsto (1 \leadsto x) = 1 \leadsto (x \to x) = 1 \leadsto 1 = 1$. Thus, by (10), $1 \to x = x$ and $1 \leadsto x = x$. Therefore, $I^*$ is precisely the quasivariety of all pseudo-BCI-algebras.

4. Conclusion

The pseudo-BCI-logic is a non-commutative version of the BCI-logic — it has two different implications $\to$ and $\leadsto$. In order to be algebraizable we have to extend it on one inference rule (Imp). This leads us to formulate and prove the main result of the paper that pseudo-BCI-algebras are an algebraic counterpart of this extended logic (Theorem 3.2). We think this logic is so close to original one that it is worth studying its algebraic models — pseudo-BCI-algebras.

References


