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ON THE WEAKEST MODAL LOGICS DEFINING JAŚKOWSKI’S LOGIC D₂ AND THE D₂-CONSEQUENCE

Abstract

Jaśkowski’s logic $D_2$ (as a subset of the set $\text{For}^d$ of discursive formulae) was formulated with the help of the modal logic $S5$ as follows: $A \in D_2$ iff $\forall \exists A^* \in S5$, where $(-)^*$ is a translation from $\text{For}^d$ into the set $\text{Form}$ of modal formulae. We say that a modal logic $L$ defines $D_2$ iff $D_2 = \{A \in \text{For}^d : \forall \exists A^* \in L\}$. Let $S5_\forall$ be the set of all modal logics having the same theses beginning with ‘$\forall$’ as $S5$. All logics from $S5_\forall$ define $D_2$. In [8] we examined the logic $aS5^M$ which is the smallest in $S5_\forall$. In present paper we shall examine the logic $A$ which is the weakest modal logic defining $D_2$. As far as we know, it is the first example of a modal logic which defines $D_2$, but does not belong to $S5_\forall$. So $A \subseteq aS5^M$.

Studying Jaśkowski’s paper we can find the $D_2$-consequence relation $\vdash_{D_2}$ in $\varphi(\text{For}^d) \times \text{For}^d$ meant as follows: $II \vdash_{D_2} A$ iff $\forall \exists A^* \vdash_{S5} \forall \exists A^*$. We say that a modal logic $L$ defines the $D_2$-consequence iff $\vdash_{D_2}$ is the pure modus-ponens-style inference relations based on $L$. Thus, for all $\Phi \subseteq \text{Form}$ and $\psi \in \text{Form}$: $\Phi \vdash_L \psi$ iff there are $\varphi_1, \ldots, \varphi_n \in \Phi$ such that $\forall (\varphi_1 \land \cdots \land \varphi_n) \rightarrow \psi^\dagger \in L$. See p. 219.

Key words: Jaśkowski’s logic $D_2$, $D_2$-consequence, the weakest modal logic defining $D_2$, the weakest modal logic defining the $D_2$-consequence.
1. Basic notions

1.1. Modal logics

Modal formulae are formed in the standard way from propositional letters: ‘\(p\)’, ‘\(q\)’, ‘\(p_0\)’, ‘\(p_1\)’, ‘\(p_2\)’, . . .; truth-value operators: ‘\(\neg\)’, ‘\(\lor\)’, ‘\(\land\)’, ‘\(\rightarrow\)’, and ‘\(\leftrightarrow\)’ (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); modal operators: the necessity sign ‘\(\square\)’ and the possibility sign ‘\(\Diamond\)’; and brackets. By \(\text{For}_m\) we denote the set of all modal formulae. Of course, the set \(\text{For}_m\) includes the set of all classical formulae (without ‘\(\square\)’ and ‘\(\Diamond\)’); let \(\text{Taut}\) be the set of all classical tautologies. Besides, for any \(\phi, \psi, \chi \in \text{For}_m\), let \(\chi[^\phi/\psi]\) be any formula that results from \(\chi\) by replacing one or more occurrences of \(\phi\), in \(\chi\), by \(\psi\).

For any \(\phi \in \text{For}_m\) let \(\text{Sub}(\phi)\) be the set of modal formulae which are substitution instances of \(\phi\). Moreover, for any \(\Phi \subseteq \text{For}_m\) we put \(\square \Phi := \{\square \phi : \phi \in \Phi\}\) and \(\Diamond \Phi := \{\Diamond \phi : \phi \in \Phi\}\).

Modal logics are certain sets of formulae. We define a modal logic as a set \(L\) of modal formulae satisfying following conditions:

- \(\text{Taut} \subseteq L\),
- \(L\) includes the following set of formulae
  \[\{\varphi \rightarrow \square \varphi \rightarrow \Diamond \varphi / \phi : \varphi, \chi \in \text{For}_m\}\].

\(L\) is closed under the following two rules: modus ponens for ‘\(\rightarrow\)’:

<table>
<thead>
<tr>
<th>(\varphi)</th>
<th>(\varphi \rightarrow \psi)</th>
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<td>(\psi)</td>
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and uniform substitution of formulae for propositional letters in \(\varphi\).

Of course, by the uniform substitution, every modal logic includes the set \(\text{PL}\) of modal formulae which are substitution instances of classical tautologies from \(\text{Taut}\). In the present paper the term ‘modal logic’ is always understood as a set of modal formulae.

All members of a logic are called its theses. By (\(\text{rep}^{\text{\(\square\)}}\)), every modal logic has the following thesis:

\(\Diamond \varphi \leftrightarrow \square \neg \varphi\)  \(\text{(df } \Diamond)\)

Remark 1.1. In the present paper the symbol ‘\(\Diamond\)’ is a primary symbol; it is not an abbreviation of ‘\(\neg \square \neg\)’. Thus we have to add the set of axioms
The use of this set corresponds to the applying of the formula (df ♢) as a definition ruled by the definitional rule. Formulae from (rep□) allow to replace one or more occurrences of '¬ □ ¬' with '♢' and vice versa.

We say that a modal logic $L$ is an $rte$-logic iff $L$ is closed under replacement of tautological equivalents, i.e., for any $φ, ψ, χ ∈ For_m$

$$∀φ, ψ, χ ∈ For_m: [φ ⇔ ψ] ∈ PL \text{ and } χ ∈ L$$

A modal logic is an $rte$-logic iff it includes the following set

$$\{ [φ]_φ / ψ] ⇔ χ^γ: φ, ψ, χ ∈ For_m \text{ and } [φ ⇔ ψ^γ] ∈ PL \}.$$ (rep$\text{PL}$)

In any thesis of any $rte$-logic we can replace one or more occurrences of '¬ □ ¬' (resp. '□ ¬', '¬ □', '¬ □ ♢', '¬ □', '♢ ¬') by '♢' (resp. '♢', '♢', '□', '□', '♢', '♢') and vice versa. Thus, every $rte$-logic has the following thesis

$$□ p ⇔ ♢ ¬ p$$ (df □)

We say that a modal logic is congruential (or classical) iff it is closed under the congruence rule

$$φ ⇔ ψ \quad □ φ ⇔ □ ψ$$ (cgr)

Of course, every congruential logic is an $rte$-logic.

A modal logic is monotonic iff it is closed under the monotonicity rule

$$φ → ψ \quad □ φ → □ ψ$$ (mon)

Of course, every monotonic logic is congruential.

A modal logic is regular iff it is closed under the regularity rule

$$φ ∧ ψ → χ \quad □ φ ∧ □ ψ → □ χ$$ (reg)

iff it is monotonic and contains the following formula

$$□(p → q) → (□ p → □ q)$$ (K)

iff it is congruential and contains the following formula

$$□(p ∧ q) ⇔ (□ p ∧ □ q)$$ (R)
Every regular logic has the following thesis
\[ \Diamond (p \rightarrow q) \iff (\Box p \rightarrow \Diamond q) \] (R\^{\Box})

A modal logic is normal iff it contains (K) and is closed under the necessitation rule
\[ \varphi \]
\[ \Box \varphi \]
iff it is regular and contains at least one thesis of the form \( \Box \varphi \).

As usual, let K be the smallest normal modal logic. Using names of the above formulae, to simplify notation of normal logics we write the Lemmon code $KX_1 \ldots X_n$ to denote the smallest normal logic containing the formulae \((X_1), \ldots, (X_n)\). We standardly put $S_4 := KT_4$ and $S_5 := KT_5 = KT_4B = KD_4B = KD_5B$, where
\[ \Box p \rightarrow p \] (T)
\[ \Box p \rightarrow \Diamond p \] (D)
\[ \Box p \rightarrow \Box \Box p \] (4)
\[ p \rightarrow \Box \Diamond p \] (B)
\[ \Diamond p \rightarrow \Box \Diamond p \] (5)

As it is known, $KT \subseteq S_4 \subseteq S_5$, $KD_45 \subseteq S_5$, $KD_45 \not\subseteq S_4$ and $KT \not\subseteq KD_45$. We have also: \((D) \in K5_{c_1}, (5c) \in KD_4\), so $KD_4 = K5_{c_4}$ (see e.g. [8]), where
\[ \Box \Diamond p \rightarrow \Diamond p \] (5c)

Moreover, \((4) \in K5_{c_4}\), so $KD_45 = K5_{c_4}$.

Notice that the following formulae
\[ \Diamond (\Diamond p \rightarrow p) \] (1.1)
\[ \Diamond (\Diamond p \rightarrow (\Diamond q \rightarrow (p \land \Diamond q))) \] (1.2)
\[ \Diamond (p \land \Diamond q) \rightarrow (\Diamond p \land \Diamond q) \] (1.3)
\[ \Diamond (\Diamond p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q) \] (1.4)
\[ \Diamond (\Diamond p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q) \] (1.5)
belong to $KD_45$, so also to $S_5$.

For any modal logic $L$ and any formula $\varphi \in \text{For}_m$, let $L + \varphi$ be the smallest modal logic which includes $L$ and contains $\varphi$. We easily see:
Lemma 1.1. Let \( X \) be a set of modal logics which is closed under arbitrary intersections. If \( L \in X \), \( \varphi \in For_m \) and there is in \( X \) a logic including \( L \cup \{ \varphi \} \), then there is the smallest logic in \( X \) including \( L \cup \{ \varphi \} \). Let us denote this logic by \( L_+ \).

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Let \( \Phi \subseteq For_m \) and \( R \) be a set of rules on \( For_m \). We say that the pair \( \langle \Phi, R \rangle \) is an axiomatization of a modal logic \( L \) iff \( L \) is the smallest set including \( \Phi \), being closed under all rules from \( R \). Then for any \( \varphi \in For_m \):

For any modal logic \( L \) we define a consequence relation of \( \vdash_L \) with the help of modus ponens for \( \rightarrow \) as the only rule of inference, i.e., \( \vdash_L \) is the pure modus-ponens-style inference relation based on \( L \). Strictly speaking, for any \( \Phi \subseteq For_m \) and \( \psi \in For_m \):

\[
\Phi \vdash_L \psi \iff \text{there exists a sequence } \chi_1, \ldots, \chi_m \text{ in which } \chi_m = \psi \text{ and for any } i \leq m, \text{ either } \chi_i \in \Phi \text{ or there are } R \in R, \text{ } m < n, \text{ } j_1, \ldots, j_m < i \text{ such that } \langle \varphi_{j_1}, \ldots, \varphi_{j_m}, \varphi_i \rangle \in R.
\]

Lemma 1.2. For any \( \Phi \subseteq For_m \) and \( \psi \in For_m \):

\[
\Phi \vdash_L \psi \iff \text{for some } n \geq 0 \text{ and for some } \varphi_1, \ldots, \varphi_n \in \Phi \text{ we have that } \Gamma(\varphi_1 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \psi)\ldots)) \in L, \text{ or equivalently } \Gamma(\varphi_1 \land \ldots \land \varphi_n) \rightarrow \psi \in L.
\]

1.2. Discursive formulae and a translation

Discursive formulae are formed in the standard way from propositional letters: ‘\( p \)’, ‘\( q \)’, ‘\( p_0 \)’, ‘\( p_1 \)’, ‘\( p_2 \)’, …; truth-value operators: ‘\( \lnot \)’ and ‘\( \lor \)’ (negation and disjunction); discursive connectives: ‘\( \land^{d} \)’, ‘\( \rightarrow^{d} \)’, ‘\( \leftrightarrow^{d} \)’ (conjunction, implication and equivalence); and brackets. Let \( For^d \) be the set of all these formulae.

Let \( (-)^* \) be the translation from \( For^d \) into \( For_m \) such that:

1. \( (a)^* = a \), for any propositional letter \( a \),
2. for any \( A, B \in For^d \):
\[
(-A)^* = \Gamma \lnot \Gamma A^*,
(A \lor B)^* = \Gamma A^* \lor B^*,
(A \land B)^* = \Gamma A^* \land B^*.
\]
\[ (A \rightarrow^d B)^* = \Diamond \Diamond A^* \rightarrow B^* \land, \]
\[ (A \leftrightarrow^d B)^* = \Diamond (\Diamond A^* \rightarrow B^*) \land \Diamond (\Diamond B^* \rightarrow A^*) \land. \]

1.3. The discussive logic \( D_2 \) as a set of discussive formulae

Jaśkowski’s discussive logic \( D_2 \) can be treated either as some set of discussive formulae or as some consequence relation on the set of all discussive formulae (see Section 1.4).

Formulation. In the first case, \( D_2 \) is formulated with the help of the modal logic \( S5 \) as follows (see [4, 6]):

\[ D_2 := \{ A \in \text{For}^d : \Diamond \Diamond A^* \land \in S5 \} . \]

The set \( D_2 \) is closed under substitutions. Moreover, since \( (\Diamond) \in S5 \), the set \( D_2 \) is closed under \textit{modus ponens} for \( \rightarrow^d \) (i.e., for any \( A, B \in \text{For}^d \), if \( A, \Diamond A \rightarrow^d B \land \in D_2 \) then \( B \in D_2 \)). Besides, by (1.1) and (1.2), the following formulae

\[ p \rightarrow^d p \]  \hspace{1cm} \text{(1.1d)}
\[ p \rightarrow^d (q \rightarrow^d (p \land^d q)) \]  \hspace{1cm} \text{(1.2d)}

belong to \( D_2 \).

Notice that by the translation \((-)^*\), (1.4), (1.5), PL and by the induction on \( n \) we have

**Lemma 1.3.** For any \( n \geq 0, A_1, \ldots, A_n, B \in \text{For}^d :\)

\[ \Diamond A_1 \rightarrow^d (\ldots \rightarrow^d (A_n \rightarrow^d B) \ldots) \land \in D_2 \iff \]
\[ \Diamond (A_1 \land^d \ldots \land^d A_n) \rightarrow^d B \land \in D_2 . \]

**Modal logics defining \( D_2 \).** We say that a modal logic \( L \) \textit{defines} \( D_2 \) if \( D_2 = \{ A \in \text{For}^d : \Diamond \Diamond A^* \land \in L \} \). It is known that also other modal logics than \( S5 \) define \( D_2 \). By definitions we have:

**Fact 1.4.** If \( L \subseteq L' \subseteq S5 \) and \( L \) \textit{defines} \( D_2 \), then \( L' \) \textit{defines} \( D_2 \).

Since \( S5 \) defines \( D_2 \) and any intersection of modal logics defining \( D_2 \) is a modal logic defining \( D_2 \), so we have:
FACT 1.5. There exists the smallest modal logic defining \( D_2 \).

Let \( A \) be the smallest modal logic defining \( D_2 \) (“A” for “absolute”). Of course, \( A \subseteq S5 \). The logic \( A \) is examined in sections 2 and 4.

The family \( S5_\diamond \). While expressing the logic \( D_2 \) we refer to modal logics which have the same as \( S5 \) theses beginning with ‘\( \Diamond \)’. Let \( S5_\diamond \) be the family of all such modal logics. By definitions we have:

FACT 1.6. (i) Every logic from \( S5_\diamond \) defines \( D_2 \).
(ii) If \( L \subseteq L' \subseteq S5 \) and \( L \in S5_\diamond \), then \( L' \in S5_\diamond \).

Moreover, we obtain:

FACT 1.7 ([8]). For any \( rte \)-logic \( L \): \( L \) defines \( D_2 \) iff \( L \in S5_\diamond \).

In [8] we give a general method which, for some classes of modal logics determined by sets of joint axioms and rules, generates in the given class the weakest logic in \( S5_\diamond \). For example, by Fact 1.7, for \( rte \)-logics, congruential, monotonic, regular and normal we obtain the weakest — in the respective classes — logics defining \( D_2 \).\(^2\)

Let us recall (see [8]) that \( aS5^M \) is the smallest logic in \( S5_\diamond \) (the “absolute” one). By Fact 1.6i, \( A \subseteq aS5^M \). In Section 2 we prove that \( A \) has no member of the form ‘\( \Diamond \Diamond \chi \)’. Hence for any \( \tau \in PL \), ‘\( \Diamond \Diamond \tau \)’ \( \not\in A \), but ‘\( \Diamond \Diamond \tau \)’ \( \in S5 \). Consequently, \( A \not\subseteq S5_\diamond \) and \( A \subseteq aS5^M \). Hence, \( A \) is not an \( rte \)-logic (we use Fact 1.7 or the fact that \( aS5^M \) is not an \( rte \)-logic; see [8]).

Moreover, let \( rteS5^M \), \( eS5^M \), \( mS5^M \), \( rS5^M \), \( S5^M \) be respectively, the smallest \( rte \)-, congruential, monotonic, regular, normal logic in \( S5_\diamond \). By Fact 1.7, these logics are also respectively the smallest \( rte \)-, congruential, monotonic, regular, normal logic defining \( D_2 \). In [8] we proved that \( aS5^M \subseteq rteS5^M \subseteq eS5^M \subseteq mS5^M \subseteq rS5^M \subseteq S5^M \subseteq S5 \).

\(^2\)In [2], Furmanowski proved that \( S4 \in S5_\diamond \), so \( S4 \) defines \( D_2 \), by Fact 1.6i. In [11], Perzanowski defined the logic \( S5^M \) and proved that it is the smallest normal logic in \( S5_\diamond \); so also \( S5^M \) defines \( D_2 \). However, from this does not follow that \( S5^M \) is the smallest normal logic defining \( D_2 \). To achieve this result one needs a version of Fact 1.7 for normal logics; that was proved in Nasieniewski’s PhD thesis.
1.4. The discussive logic $D_2$ as a discussive consequence

Formulation. We may also consider the second way of understanding the discussive logic $D_2$, i.e., as a consequence relation between discussive formulae. In Jaskowski’s paper [3] the following definition of a discussive relation can be discovered: for any \( A \in For^d \) and \( B \in For^d \)

\[
\Pi \vdash_{D_2} B \iff \{ \Diamond A^* : A \in \Pi \} \vdash_{S5} \Diamond B^*.
\]

The relation \( \vdash_{D_2} \) (considered by Jąskowski) is called the $D_2$-consequence. It has been elaborated in [9, 10]. Notice that for any \( B \in For^d \),

\[
\emptyset \vdash_{D_2} B \iff B \in D_2 \text{ iff } \forall \Pi \subseteq For^d \Pi \vdash_{D_2} B.
\]

Discussive systems. Following Jąskowski, we say that a subset \( S \) of \( For^d \) is a discussive system iff \( S \) is closed under the $D_2$-consequence, i.e., for any \( \Pi \subseteq For^d \) and \( B \in For^d \), if \( \Pi \vdash_{D_2} B \) and \( \Pi \subseteq S \), then \( B \in S \). By (1), all discussive systems include the set \( D_2 \). The basic features of discussive systems are presented in [3, p. 61], [4, p. 37–38] (see also [9, 10]).

Properties of the $D_2$-consequence. Notice that, since (C) and (1.3) belong to \( S5 \), we obtain that for any \( A, B \in For^d \): \( A \rightarrow^d B, A \vdash_{D_2} B \) and \( A, B \vdash_{D_2} A \wedge^d B \).

Applying (1.5) and lemmas 1.2 and 1.3, on the basis of $D_2$ one can characterize the $D_2$-consequence in the following way:

FACT 1.8. For any \( \Pi \subseteq For^d \) and \( B \in For^d \):

\[
\Pi \vdash_{D_2} B \iff \text{for some } n \geq 0 \text{ and for some } A_1, \ldots, A_n \in \Pi \text{ we have }
\]

\[\forall A_1 \rightarrow^d (\ldots \rightarrow^d (A_n \rightarrow B \ldots))^\top \in D_2, \text{ or equivalently}
\]

\[\forall (A_1 \wedge^d \ldots \wedge^d A_n) \rightarrow^d A^\top \in D_2.
\]

Moreover, the relation \( \vdash_{D_2} \) can be characterized with the help of modus ponens for \( \rightarrow^d \) as the only rule of inference.

FACT 1.9 ([9]). For any \( \Pi \subseteq For^d \) and \( B \in For^d \):

\[
\Pi \vdash_{D_2} B \iff \text{there exists a sequence } C_1, \ldots, C_m \text{ in which } C_m = B
\]

and for any \( i \leq m \), either \( C_i \in \Pi \cup D_2 \) or there are \( j, k < i \) such that \( C_k = \forall C_j \rightarrow^d C_i^\top \).
Modal logics defining the $D_2$-consequence. We say that a modal logic $L$ defines the $D_2$-consequence iff $\vdash D_2 = \{ \langle I, A \rangle \in \wp(\text{For}^d) \times \text{For}^d : \Box I^* \vdash_L \Box A^* \}$. It is again known that there are other modal logics than $S5$ which define the $D_2$-consequence. By definitions and Lemma 1.2 we obtain:

**Fact 1.10.** If $L \subseteq L' \subseteq S5$ and $L$ defines $\vdash_{D_2}$, then also $L'$ defines $\vdash_{D_2}$.

**Fact 1.11 ([10]).** For any modal logic $L$:

(i) If $L$ defines $D_2$ and $(C) \in L \subseteq S5$, then $L$ defines $\vdash_{D_2}$.

(ii) If $L$ defines $\vdash_{D_2}$, then $L$ defines $D_2$ and $(C) \in L$.

From facts 1.4 and 1.11i, since $(C) \in S5$, we obtain

**Fact 1.12 ([10]).** If $L$ defines $D_2$ and $L \subseteq S5$, then $L + (C)$ defines $\vdash_{D_2}$.

Moreover, we have that:

**Fact 1.13 ([10]).** Let $X$ be a set of modal logics such that both $S5 \in X$ and there is the smallest logic in $X$ defining $D_2$. If $L$ is this logic, then:

(i) $L + (C)$ defines $\vdash_{D_2}$,

(ii) $L + (C)$ is included in all logics from $X$ that define $\vdash_{D_2}$.

We put $A_\circ := A + (C)$. By Fact 1.13 applied to the set of all modal logics we obtain that $A_\circ$ is the smallest modal logic defining $\vdash_{D_2}$. The logic $A_\circ$ will be examined in sections 3 and 4. It is shown there that $A_\circ \notin S5_0$. Similarly as in the case of $A$, for any $\tau \in \text{PL}$, $\Box \Box \Box \tau \notin A_\circ$.

**Fact 1.14 ([10]).** Consider a set of modal logics $X$ which contains $S5$ and is closed under arbitrary intersections. Let $L$ be the smallest logic in $X$ defining $D_2$. Then $L + X (C)$ is the smallest logic in $X$ defining $\vdash_{D_2}$.

The family $C_n S5$. Let $C_n S5$ be the set of modal logics which satisfy the following condition: for any modal logic $L$

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3 Notice that this logic exists, because the subset of $X$ consisting of logics defining $D_2$ includes $S5$ and of course is closed under arbitrary intersections.

4 By Lemma 1.1, the logic $L + X (C)$ exists, since $L, S5 \in X$ and $L \cup \{ (C) \} \subseteq S5$. 

\[ L \in \text{Cn}_s \text{S}_5 \] if for any \( I \subseteq F_{m} \) and \( B \in F_{m} \),
\[ \Diamond I \vdash \Diamond B \iff \Diamond I \vdash_{\text{S}_5} \Diamond B . \]

**Fact 1.15.**
(i) \( \text{Cn}_s \text{S}_5 \subseteq \text{S}_5 \).
(ii) Every logic from \( \text{Cn}_s \text{S}_5 \) defines the \( D_2 \)-consequence.
(iii) If \( L \subseteq L' \subseteq \text{S}_5 \) and \( L \in \text{Cn}_s \text{S}_5 \), then \( L' \in \text{Cn}_s \text{S}_5 \).

**Fact 1.16 ([10]).** For any modal logic \( L \):
(i) If \( L \in \text{S}_5 \), \( L \subseteq L' \subseteq \text{S}_5 \) and \( (C) \in L' \), then \( L' \in \text{Cn}_s \text{S}_5 \).
(ii) If \( L \in \text{Cn}_s \text{S}_5 \), then \( L \in \text{S}_5 \) and \( (C) \in L \).

**Fact 1.17 ([10]).** For any \( \text{rte} \)-logic \( L \) \( \vdash D_2 \) iff \( L \in \text{Cn}_s \text{S}_5 \).

In [10], a general method is given which, for some classes of modal logics determined by sets of joint axioms and rules, generates the weakest — in the given class — logic that belongs to \( \text{Cn}_s \text{S}_5 \). For example, by Fact 1.17, for \( \text{rte} \)-, congruential, monotonic, regular and normal logics we obtain the weakest — in the respective class — logic defining the \( D_2 \)-consequence.

Let us recall (see [10]) that \( \text{aS}_5^M \) is the smallest logic in \( \text{Cn}_s \text{S}_5 \). By Fact 1.15ii, \( A_r \subseteq \text{aS}_5^M \). In Section 3 we will prove that \( A_r \notin \text{Cn}_s \text{S}_5 \). Hence, \( A_r \) is not an \( \text{rte} \)-logic (we use Fact 1.17 or the fact that \( \text{aS}_5^M \) is not an \( \text{rte} \)-logic; see [10]).

Moreover, let \( \text{rteS}_5^M, \text{eS}_5^M, \text{mS}_5^M, \text{rS}_5^M \) be, respectively, the smallest \( \text{rte} \)-, congruential, monotonic and regular logic in \( \text{Cn}_s \text{S}_5 \). Besides, it is known (see [9]) that \( \text{KD45} \) is the smallest normal logic in \( \text{Cn}_s \text{S}_5 \). By Fact 1.17, these logics are also respectively the smallest \( \text{rte} \)-, congruential, monotonic, regular, normal logic defining \( D_2 \) (see [9, 10]). In [10] we proved that \( \text{aS}_r \subseteq \text{rteS}_5^M \subseteq \text{eS}_5^M \subseteq \text{mS}_5^M \subseteq \text{rS}_5^M \subseteq \text{KD45} \subseteq \text{S}_5 \).

2. The smallest modal defining \( D_2 \)

We define the following set of modal formulae:

\[ \text{Gen} := \{ \varphi \in F_{m} : \exists A \in D_2 \varphi = \Diamond A \} = \{ \Diamond A \in F_{m} : A \in D_2 \}. \]
**Lemma 2.1.** Every modal logic defining $D_2$ includes the set $Gen$.

**Proof:** Let $L$ define $D_2$ and $\varphi \in Gen$. Then for some $A \in D_2$ we have that $\varphi = \vdash \Diamond A \Diamond$ and $\vdash \Diamond A \Diamond \in L$.

Let $Ax_{PL}$ be a set of modal formulae such that the pair $(Ax_{PL}, \{mp\})$ is an axiomatization of a modal logic $PL$.

**The logic $A$.** Let us recall that $A$ is the smallest logic defining $D_2$.

**Fact 2.2.** $A$ is the smallest modal logic including the set $Gen$. Consequently, $A$ is axiomatized by the set $Ax_{PL} \cup (\text{rep}^{2}) \cup Gen$ and the rule $(mp)$.

**Proof:** Let $L$ be the smallest modal logic including the set $Gen$, i.e., $L$ is axiomatized by the set $Ax_{PL} \cup (\text{rep}^{2}) \cup Gen$ and the rule $(mp)$. So, by Lemma 2.1, $L \subseteq A$. Now we prove that also $A \subseteq L$.

Firstly, we show that $L \subseteq S5$. Indeed, if $\varphi \in L$, then $\varphi \in S5$, by the definition of $L$ and the inclusion $Ax_{PL} \cup (\text{rep}^{2}) \cup Gen \subseteq S5$.

Secondly, we show that $L$ defines $D_2$: hence $A \subseteq L$. Indeed, for any $A \in For^d$: if $A \in D_2$, then $\vdash \Diamond A \Diamond \in Gen \subseteq L$; if $\vdash \Diamond A \Diamond \in L$, then $\vdash \Diamond A \Diamond \in S5$ and $A \in D_2$.

Now notice that

**Lemma 2.3.** (i) No formula containing the operator ‘$\square$’ belongs to $Gen$.
(ii) No formula of the form $\vdash \Diamond \Diamond \Diamond$ or $\vdash (\Diamond (\e (\Diamond)))$ belongs to $Gen$.

**Proof:** (i) By definitions. (ii) Suppose that $\vdash \Diamond \Diamond \Diamond$ (resp. $\vdash (\Diamond (\e (\Diamond)))$) belongs to $Gen$. Then for some $A \in D_2$ we have that $A = \vdash \Diamond \Diamond \Diamond$ (resp. $A = \vdash (\Diamond (\e (\Diamond)))$). But this is impossible by the definition of the function $\Diamond$.

**Lemma 2.4.** $A \cap \Diamond For_m \subseteq \Diamond (\text{For}^d)^* \cup \{\varphi : \exists \psi \in (\text{For}^d)^* \cdot \psi = \varphi[\square -/\Diamond \Diamond] \}$. 

**Proof:** “$\supseteq$” — obvious. “$\subseteq$” Let $\psi$ be any valuation from $For_m$ into $\{0, 1\}$ such that it preserves classical truth conditions for classical constants and for any $\varphi \in For_m$:

- $\psi(\Diamond \varphi) = 1$ iff $\varphi \in (\text{For}^d)^*$ or $\exists \psi \in (\text{For}^d)^* \cdot \psi = \varphi[\square -/\Diamond \Diamond]$, 
- $\psi(\Box \varphi) = 0$ iff $\exists \psi \in (\text{For}^d)^* \cdot \varphi = \neg \psi \Diamond$ or $\exists \psi \in (\text{For}^d)^* \cdot (\varphi = \neg \psi \Diamond \land \chi = \psi[\square -/\Diamond \Diamond])$. 

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We show that for any \( \varphi \in A \): \( v(\varphi) = 1 \).

For any \( \varphi \) from \( PL \cup (\text{rep}^{[2]}) \cup \text{Gen} \) we have that \( v(\varphi) = 1 \). Thus, by the induction on the length of the proof, relative to the axiomatization considered in Fact 2.2, we obtain: if \( \varphi \in A \), then \( v(\varphi) = 1 \). \( \Box \)

From Lemma 2.4 we obtain the following corollaries.

**Fact 2.5.** \( A \) has neither member of the form \( \Box\Box \neg \chi \) nor \( \Box(\psi \rightarrow \Box \chi) \).

We can also give a more elementary proof of the above fact.

**Proof:** Let \( v \) be any valuation from \( \text{For}_m \) into \( \{0, 1\} \) such that it preserves classical truth conditions for classical constants and for any \( \varphi \in \text{For}_m \):

- \( v(\Box \varphi) = 1 \) iff \( \varphi \) has the form \( \Box \neg \chi \) or \( \neg \neg \Box \neg \chi \),
- \( v(\Diamond \varphi) = 0 \) iff \( \varphi \) has the form \( \Diamond \neg \chi \), or \( \neg \neg \Diamond \neg \chi \), or \( \Diamond \psi \rightarrow \neg \Diamond \neg \chi \), or
- \( \Diamond \psi \rightarrow \neg \Diamond \neg \chi \).

We show that for any \( \varphi \in A \): \( v(\varphi) = 1 \).

By Lemma 2.3, for any \( \varphi \) from \( PL \cup (\text{rep}^{[2]}) \cup \text{Gen} \) we have that \( v(\varphi) = 1 \). Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 2.2, we obtain: if \( \varphi \in A \), then \( v(\varphi) = 1 \).

**Corollary 2.6.** For any \( \tau \in PL \): \( \Box\Box\Box \neg \chi \notin A \) and \( \Box\Box\Box \neg \tau \notin S5 \). Consequently, \( A \notin S5_6 \).

By Fact 1.7 and Corollary 2.6, we obtain:

**Corollary 2.7.** \( A \) is not an rte-logic.

Moreover, the logic \( A \) has the following interesting property:

**Fact 2.8.** (i) The formula \( \Box(p \lor \neg p) \) belongs to \( A \).

(ii) The formula \( \Box(p \rightarrow p) \) does not belong to \( A \).

**Proof:** If \( \Box(p \rightarrow p) \) belonged to \( A \), then also \( \Box(p \rightarrow \Box p) \) would do. But this is excluded by Fact 2.5. \( \Box \)
Now, by Lemma 2.4, we obtain:

**FACT 2.9.** The logic $\mathbf{A}$ is closed under the following rule\(^5\)

\[
\frac{\Box \varphi \Box (\varphi \to \psi)}{\Box \psi} \quad (\text{RC})
\]

**Proof:** Let $\Box (\varphi \to \psi)$ and $\Box \varphi$ belong to $\mathbf{A}$. Then, by Lemma 2.4, these formulae belong to $\Box (\text{For}^d)^\dagger \cup \Box \{ \varphi : \exists \psi \in \text{For}^d, \psi = \varphi [\square \neg / \square \neg] \}$. We show that $\Box \psi$ also belongs to this set.

Firstly, let $\Box (\varphi \to \psi)$, $\varphi \in (\text{For}^d)^\dagger$. Then for some $A, B \in \text{For}^d$ we have that $\varphi = A^\dagger$, $\psi = B^\dagger$ and $\Box (\varphi \to \psi) = (A \to B)^\dagger$. Since $\mathbf{A}$ defines $\mathbf{D}_2$, so $\Box A \to B$, $A \in \mathbf{D}_2$. Hence $B \in \mathbf{D}_2$ and $\Box \psi$ belongs to $\mathbf{A}$.

Secondly, let for some $\chi_1, \chi_2 \in (\text{For}^d)^\dagger$ we have that $\chi_1 = \Box (\varphi \to \psi)[\neg \square \neg / \square \neg]$, and $\chi_2 = \varphi [\neg \square \neg / \square \neg]$. Then $\chi_1, \chi_2 \in \mathbf{A}$. We adopt the proof from the last paragraph.

Further, we apply a combination of proofs used in two foregoing paragraphs.

\[\Box\]

**The logic $\mathbf{A} + \Box (p \to p)$.** The logic $\mathbf{A} + \Box (p \to p)$ has also some interesting features. Since $\mathbf{A} \subseteq \mathbf{A} + \Box (p \to p) \subseteq \mathbf{S}5$, so $\mathbf{A} + \Box (p \to p)$ also defines $\mathbf{D}_2$.

**FACT 2.10.** $\mathbf{A} + \Box (p \to p)$ is the smallest modal logic including the set $\text{Gen} \cup \{ \Box (p \to p) \}$. Consequently, $\mathbf{A} + \Box (p \to p)$ is axiomatized by the set $\mathbf{AXPL} \cup (\text{rep}^\dagger) \cup \text{Gen} \cup \text{Sub}(\Box (p \to p))$ and the rule $(\text{mp})$.

Notice that also $\mathbf{A} + \Box (p \to p)$ does not belong to $\mathbf{S}5_\chi$, since we have:

**FACT 2.11.** The logic $\mathbf{A} + \Box (p \to p)$ has no member of the form $\Box \varphi$.

**Proof:** Let $v$ be any valuation from $\text{For}_m$ into $\{0, 1\}$ such that it preserves classical truth conditions for classical constants and for any $\varphi \in \text{For}_m$:

- $v(\varphi) = 0$ iff $\varphi$ has the form $\Box \chi$ or $\neg \square \neg \chi$,
- $v(\square \varphi) = 1$ iff $\varphi$ has the form $\neg \Box \chi$ or $\neg \neg \square \neg \chi$.

We show that for any $\varphi \in \mathbf{A} + \Box (p \to p)$: $v(\varphi) = 1$.

\(^5\)Cf. this rule with the rule $(\text{mp})^{\dagger \dagger}$ on p. 231.
By Lemma 2.3, for any \( \varphi \) from \( \text{PL} \cup (\text{rep}^2) \cup \text{Gen} \cup \text{Sub}(\Diamond(p \rightarrow p)) \) we have that \( v(\varphi) = 1 \). Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 2.10: if \( \varphi \in \text{A} + \Diamond(p \rightarrow p) \), then \( v(\varphi) = 1 \). Therefore \( \text{A} + \Diamond(p \rightarrow p) \) has no member of the form \( \Box \Diamond \Diamond \chi \). \( \vdash \)

We have another interesting characterization of the logic \( \text{A} + \Diamond(p \rightarrow p) \).

**Fact 2.12.** The logic \( \text{A} + \Diamond(p \rightarrow p) \) is not closed under the rule \( (\text{RC}) \).

**Proof:** Formulae ‘\( \Diamond(\Diamond(p \rightarrow p) \rightarrow \Diamond(p \rightarrow p)) \)’ and ‘\( \Diamond(p \rightarrow p) \)’ belong to \( \text{A} + \Diamond(p \rightarrow p) \), while by Fact 2.11, ‘\( \Diamond(\Diamond(p \rightarrow p)) \)’ does not. \( \vdash \)

### 3. The smallest logic defining the \( \text{D}_2 \)-consequence

Let us recall that we put \( \text{A}_\text{c} := \text{A} + (\text{C}) \) and by Fact 1.13 applied to the set of all modal logics we obtain that \( \text{A}_\text{c} \) is the smallest modal logic defining the \( \text{D}_2 \)-consequence.

From Fact 2.2, we obtain:

**Fact 3.1.** \( \text{A}_\text{c} \) is the smallest modal logic including the set \( \text{Gen} \cup \{ (\text{C}) \} \).

Consequently, we can consider \( \text{A}_\text{c} \) as being axiomatized by the set \( \text{AX}_{\text{PL}} \cup (\text{rep}^2) \cup \text{Gen} \cup \text{Sub}(\text{C}) \) and the rule \( (\text{mp}) \).

Of course, the rule \( (\text{RC}) \) is derivable in the logic \( \text{A}_\text{c} \). Hence \( \text{A}_\text{c} \) is closed under this rule.

By Lemma 2.3 we obtain:

**Lemma 3.2.** No formula of the form \( \Box \Diamond \Diamond \psi \) or \( \Box \Diamond(\psi \rightarrow \Diamond \chi) \) belongs to \( \text{Gen} \cup \text{Sub}(\text{C}) \).

**Fact 3.3.** \( \text{A}_\text{c} \) has neither member of the form \( \Box \Diamond \Diamond \psi \) nor \( \Box \Diamond(\psi \rightarrow \Diamond \chi) \).

**Proof:** Let \( v \) be the valuation given in the proof of Fact 2.5. By lemmas 2.3 and 3.2, for any \( \varphi \) from \( \text{PL} \cup (\text{rep}^2) \cup \text{Gen} \cup \text{Sub}(\text{C}) \) we have that \( v(\varphi) = 1 \). Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 3.1, we obtain: if \( \varphi \in \text{A}_\text{c} \), then \( v(\varphi) = 1 \). \( \vdash \)

By Fact 3.3 we obtain the following corollaries.
Corollary 3.4. For any \( \tau \in \text{PL} \): \( \vdash \Box \Box \Box \tau \) \( \not\in \text{A}_0 \). While \( \vdash \Box \Box \Box \tau \) \( \in S_5 \). Consequently, \( \text{A}_0 \) \( \not\in S_5 \). So also \( \text{A}_0 \) \( \not\in C_n, S_5 \).

By Fact 1.17 and Corollary 3.4, we obtain:

Corollary 3.5. \( \text{A}_0 \) is not an rt-logic.

Corollary 3.6. The formula \( \Box(p \to p) \) does not belong to \( \text{A}_0 \).\(^6\)

Of course, Fact 3.3 and corollaries 3.4–3.6 entail, respectively, Fact 2.5 and corollaries 2.6–2.8.

4. Axiomatizations of \( \text{A} \) and \( \text{A}_0 \) obtained from axiomatizations of the logic \( D_2 \)

Given Fact 2.2 (resp. Fact 3.1), the set \( \text{Ax}_{\text{PL}} \cup (\text{rep}^D) \cup \text{Gen} \) (resp. \( \text{Ax}_{\text{PL}} \cup (\text{rep}^D) \cup \text{Gen} \cup \text{Sub}(C) \)) and the rule (mp) can be treated as an axiomatization of the logic \( \text{A} \) (resp. \( \text{A}_0 \)). One can obtain more «economic» axiomatizations of these logics using an axiomatization of \( D_2 \). Refer to [7, 1] for specific axiomatizations of \( D_2 \).

Let \( \mathcal{A} \subseteq \text{For}^d \) and \( R \) be a set of rules on \( \text{For}^d \). We say that the pair \( (\mathcal{A}, R) \) is an axiomatization of the logic \( D_2 \) iff for any \( A \in \text{For}^d \): \( A \in D_2 \) iff there exists a sequence \( C_1, \ldots, C_n \) in which \( C_n = A \) and for any \( i \leq n \), either \( C_i \in \mathcal{A} \) or there are \( R \in R \), \( m < n \), \( j_1, \ldots, j_m < i \) such that \( \langle C_{j_1}, \ldots, C_{j_m}, C_i \rangle \in R \).

For any \( \Gamma \subseteq \text{For}^d \) we put

\[ \Gamma^{\ast} := \{ \Gamma \Box A^{\ast} \in \text{For}_m : A \in \Gamma \} . \]

Of course, \( \text{Gen} = D_2^{\ast} \). Moreover, for any rule \( R \) on \( \text{For}^d \) we define the following rule \( R^{\ast} \) on \( \text{For}_m \):

\[ R^{\ast} := \{ \langle \varphi_1, \ldots, \varphi_n, \psi \rangle : \exists A_1, \ldots, A_n, B \in \text{For}^a \; \varphi_1 = \Gamma \Box A_1^{\ast}, \ldots, \varphi_n = \Gamma \Box A_n^{\ast}, \psi = \Gamma \Box B^{\ast} \land \langle A_1, \ldots, A_n, B \rangle \in R \} . \]

\(^6\) Another proof of this fact: the formula \( \Box(\Box(p \to p) \to \Box(p \to p)) \to (\Box(p \to p) \to \Box(p \to p)) \) is a substitution instance of the axiom (C). If \( \Box(p \to p) \) were a thesis of the logic \( \text{A}_0 \), then \( \Box(p \to p) \) would be, but there is no thesis of the form \( \Box \Box \psi \).
In other words, for any \( A_1, \ldots, A_n, B \in \text{For}^{d} \):

\[
(A_1, \ldots, A_n, B) \in R \text{ iff } \langle \Diamond A_1^*, \ldots, \Diamond A_n^*, \Diamond B^* \rangle \in R^{\circ}.
\]

For any set of rules \( R \) on \( \text{For}^{d} \) we put: \( R^{\circ} := \{ R^{\circ} : R \in R \} \).

**LEMMA 4.1.** Let \( R \) be a rule such that \( D_2 \) is closed under \( R \). Then all modal logics defining \( D_2 \) are closed under the rule \( R^{\circ} \).

**PROOF:** Let \( L \) define \( D_2 \), \( \varphi_1, \ldots, \varphi_n \in L \) and \( \langle \varphi_1, \ldots, \varphi_n, \psi \rangle \in R^{\circ} \). Then for some \( A_1, \ldots, A_n \in \text{For}^{d} \) we have that \( \varphi_1 = \langle \Diamond A_1^* \rangle \), \( \ldots \), \( \varphi_n = \langle \Diamond A_n^* \rangle \), \( \psi = \langle \Diamond B^* \rangle \) and \( \langle A_1, \ldots, A_n, B \rangle \in R \). Since \( \langle \Diamond A_1^* \rangle, \ldots, \langle \Diamond A_n^* \rangle \) belong to \( L \), so \( A_1, \ldots, A_n \) belong to \( D_2 \). Hence also \( B \) belongs \( D_2 \). Thus \( \psi \in L \). \( \dashv \)

**FACT 4.2.** Let \( (A, R) \) be an axiomatization of \( D_2 \). Then \( \langle \text{AxPL} \cup \langle \text{rep}^2 \rangle \cup \langle A^\circ \rangle, \text{R}^\circ \cup \{(\text{mp})\} \rangle \) is an axiomatization of \( A \). Consequently, \( A \) is the smallest modal logic which includes the set \( A^\circ \) and is closed under the rules from \( R^\circ \).

**PROOF:** “\( \Rightarrow \)” Let \( \varphi \in A \). Then, by Fact 2.2, there is a sequence \( \chi_1, \ldots, \chi_n \) in which \( \chi_n = \varphi \) and for any \( i \leq n \), either \( \chi_i \in \text{AxPL} \cup \langle \text{rep}^2 \rangle \cup \text{Gen} \) or there are \( j, k \) such that \( \chi_k = \langle \chi_j \rightarrow \chi_i \rangle \).

Suppose that \( \gamma_1, \ldots, \gamma_l \) are all elements in the sequence \( \chi_1, \ldots, \chi_n \) that belong to \( \text{Gen} \). For every \( k \in \{1, \ldots, l\} \) there is \( A_{\gamma_k} \in D_2 \) such that \( \gamma_k = \langle \Diamond (A_{\gamma_k})^* \rangle \). Moreover, there is a sequence \( A_{\gamma_k}^1, \ldots, A_{\gamma_k}^{m_k} \) of formulae from \( D_2 \) in which \( A_{\gamma_k}^{m_k} = A_{\gamma_k}^1 \) and for any \( i \leq m_k \), either \( A_{\gamma_k}^i \in A \) or there are \( R \in \text{R}, q < m_k, t_1, \ldots, t_q < i \) such that \( \langle A_{\gamma_k}^{i-1}, \ldots, A_{\gamma_k}^1, A_{\gamma_k}^q \rangle \in R \). We transform the above sequence into the modal language. Thus, we obtain the sequence \( \langle \langle \Diamond (A_{\gamma_k}^1)^* \rangle, \ldots, \langle \Diamond (A_{\gamma_k}^{m_k})^* \rangle \rangle \) of formulae from \( \text{Form} \), where \( \langle \Diamond (A_{\gamma_k}^i)^* \rangle = \gamma_k \) and for any \( i \leq m_k \), either \( \langle \Diamond (A_{\gamma_k}^i)^* \rangle \in A^\circ \subseteq \text{Gen} \) or for some \( R \in \text{R}, q < m_k, t_1, \ldots, t_q < i \) such that \( \langle \Diamond (A_{\gamma_k}^{i-1})^*, \ldots, \Diamond (A_{\gamma_k}^1)^* \rangle \in R^\circ \).

Now we build the following sequence. We exchange the elements \( \gamma_1, \ldots, \gamma_l \) within the sequence \( \chi_1, \ldots, \chi_n \) respectively by sequences, \( \langle \Diamond (A_{\gamma_1}^1)^* \rangle, \ldots, \langle \Diamond (A_{\gamma_1}^{m_1})^* \rangle; \ldots; \langle \Diamond (A_{\gamma_l}^1)^* \rangle, \ldots, \langle \Diamond (A_{\gamma_l}^{m_l})^* \rangle \). Thus, there exists a sequence \( \psi_1, \ldots, \psi_s \) in which \( \psi_s = \varphi \) and for any \( i \leq s \), either \( \psi_i \in \text{AxPL} \cup \langle \text{rep}^2 \rangle \cup A^\circ \) or there are \( R \in \text{R}, q < s, j_1, \ldots, j_q < i \) such that \( \langle \psi_{j_1}, \ldots, \psi_{j_q}, \varphi_i \rangle \in R^\circ \), or there are \( j, k \) such that \( \psi_k = \langle \psi_j \rightarrow \chi_i \rangle \).
By Lemma 4.1, if there is a sequence presented in the last paragraph, then \( \phi \in A \).

One can consider any axiomatization of the logic \( D_2 \) including those defined by \textit{modus ponens} for \( ' \to \varphi^\ast \) as the only rule:

\[
\frac{A}{A \to \varphi^\ast B} \quad (\text{mp}_d)
\]

The following rule \( (\text{mp}_d)^{\circ\ast} \) corresponds to \( (\text{mp}_d) \):

\[
\frac{\Diamond A^\ast \quad \Diamond(\Diamond A^\ast \to B^\ast)}{\Diamond B^\ast} \quad (\text{mp}_d)^{\circ\ast}
\]

Of course, \( (\text{mp}_d)^{\circ\ast} \subseteq (\text{RC}) \). Thus, by facts 2.9 and 4.2, the logic \( A \) can be axiomatized by two rules: \( (\text{mp}) \) and either \( (\text{mp}_d)^{\circ\ast} \) or \( (\text{RC}) \).

**Corollary 4.3.** Let \( \langle A, (\text{mp}_d) \rangle \) be an axiomatization of the logic \( D_2 \). Then \( \langle \text{Ax}_{\text{PL}} \cup (\text{rep}^2) \cup A^{\circ\ast}, \{(\text{mp}_d)^{\circ\ast}, (\text{mp})\} \rangle \) and \( \langle \text{Ax}_{\text{PL}} \cup (\text{rep}^2) \cup A^{\circ\ast}, \{(\text{RC}), (\text{mp})\} \rangle \) are axiomatizations of \( A \). Consequently, \( A \) is the smallest modal logic which includes the set \( A^{\circ\ast} \) and is closed under the rule \( (\text{mp}_d)^{\circ\ast} \) (resp. \( (\text{RC}) \)).

**Proof:** For \( (\text{mp}_d)^{\circ\ast} \) consider Fact 4.2.

For \( (\text{RC}) \): We act similarly as in the proof of Fact 4.2. In the part \( \Rightarrow \) we use the inclusion \( (\text{mp}_d)^{\circ\ast} \subseteq (\text{RC}) \). In the part \( \Leftarrow \) we use Fact 2.9.

It is obvious, that for every logic that has the thesis \( (\text{C}) \), both rules \( (\text{mp}_d)^{\circ\ast} \) and \( (\text{RC}) \) are derivable. Therefore we obtain:

**Corollary 4.4.** Let \( \langle A, (\text{mp}_d) \rangle \) be an axiomatization of the logic \( D_2 \). Then \( \langle \text{Ax}_{\text{PL}} \cup (\text{rep}^2) \cup A^{\circ\ast} \cup \text{Sub}(\text{C}), \{(\text{mp})\} \rangle \) is an axiomatization of \( A \). Consequently, \( A \) is the smallest modal logic including the set \( A^{\circ\ast} \cup \{\text{C}\} \).

**References**


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