Bulletin of the **S**ection of **L**ogic
Volume 41:3/4 (2012), pp. 215–232

Marek Nasieniewski and Andrzej Pietruszczak

ON THE WEAKEST MODAL LOGICS DEFINING JAŚKOWSKI'S LOGIC D_2 AND THE D_2-CONSEQUENCE

Abstract

Jaśkowski's logic $\mathbf{D_2}$ (as a subset of the set $\mathrm{For^d}$ of discussive formulae) was formulated with the help of the modal logic **S5** as follows: $A \in \mathbf{D_2}$ iff $\ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5}$, where $(-)^\bullet$ is a translation from $\mathrm{For^d}$ into the set $\mathrm{For_m}$ of modal formulae. We say that a modal logic $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{A \in \mathrm{For^d} : \ulcorner \Diamond A^\bullet \urcorner \in \boldsymbol{L}\}$. Let $\mathsf{S5}_\Diamond$ be the set of all modal logics having the same theses beginning with '$\Diamond$' as **S5**. All logics from $\mathsf{S5}_\Diamond$ define $\mathbf{D_2}$. In [8] we examined the logic $\mathbf{aS5^M}$ which is the smallest in $\mathsf{S5}_\Diamond$. In present paper we shall examine the logic **A** which is the weakest modal logic defining $\mathbf{D_2}$. As far as we know, it is the first example of a modal logic which defines $\mathbf{D_2}$, but does not belong to $\mathsf{S5}_\Diamond$. So $\mathbf{A} \subsetneq \mathbf{aS5^M}$.

Studying Jaśkowski's paper we can find the $\mathbf{D_2}$-consequence relation $\vdash_{\mathbf{D_2}}$ in $\wp(\mathrm{For^d}) \times \mathrm{For^d}$ meant as follows: $\Pi \vdash_{\mathbf{D_2}} A$ iff $\Diamond \Pi^\bullet \vdash_{\mathbf{S5}} \Diamond A^\bullet$.[1] We say that a modal logic $\boldsymbol{L}$ *defines* the $\mathbf{D_2}$-consequence iff $\vdash_{\mathbf{D_2}} = \{\langle \Pi, A\rangle \in \wp(\mathrm{For^d}) \times \mathrm{For^d} : \Diamond \Pi^\bullet \vdash_{\boldsymbol{L}} \Diamond A^\bullet\}$. Let $\mathsf{Cn}_\Diamond\mathsf{S5}$ be the set of all modal logics $\boldsymbol{L}$ such that for all $\Phi \subseteq \mathrm{For_m}$, $\psi \in \mathrm{For_m}$: $\Diamond\Phi \vdash_{\boldsymbol{L}} \Diamond\psi$ iff $\Diamond\Phi \vdash_{\mathbf{S5}} \Diamond\psi$. All logics from $\mathsf{Cn}_\Diamond\mathsf{S5}$ define $\vdash_{\mathbf{D_2}}$. In [10] we examined $\mathbf{aS5^M_\vdash}$ — the smallest logic in $\mathsf{Cn}_\Diamond\mathsf{S5}$. In this paper we shall study the logic $\mathbf{A}_\vdash$ which is the weakest modal logic defining the $\mathbf{D_2}$-consequence. We prove that $\mathbf{A}_\vdash \notin \mathsf{Cn}_\Diamond\mathsf{S5}$. So $\mathbf{A}_\vdash \subsetneq \mathbf{aS5^M_\vdash}$.

Key words: Jaśkowski's logic $\mathbf{D_2}$, $\mathbf{D_2}$-consequence, the weakest modal logic defining the logic $\mathbf{D_2}$, the weakest modal logic defining the $\mathbf{D_2}$-consequence.

---

[1] For any modal logic $\boldsymbol{L}$, let $\vdash_{\boldsymbol{L}}$ be the pure modus-ponens-style inference relations based on $\boldsymbol{L}$. Thus, for all $\Phi \subseteq \mathrm{For_m}$ and $\psi \in \mathrm{For_m}$: $\Phi \vdash_{\boldsymbol{L}} \psi$ iff there are $\varphi_1, \dots \varphi_n \in \Phi$ such that $\ulcorner(\varphi_1 \wedge \dots \wedge \varphi_n) \to \psi\urcorner \in \boldsymbol{L}$. See p. 219.

1. Basic notions

1.1. Modal logics

Modal formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); modal operators: the necessity sign '$\Box$' and the possibility sign '$\Diamond$'; and brackets. By $\mathrm{For_m}$ we denote the set of all modal formulae. Of course, the set $\mathrm{For_m}$ includes the set of all classical formulae (without '$\Box$' and '$\Diamond$'); let **Taut** be the set of all classical tautologies. Besides, for any $\varphi, \psi, \chi \in \mathrm{For_m}$, let $\chi[{}^{\varphi}/_{\psi}]$ be any formula that results from χ by replacing one or more occurrences of φ, in χ, by ψ.

For any $\varphi \in \mathrm{For_m}$ let $\mathrm{Sub}(\varphi)$ be the set of modal formulae which are substitution instances of φ. Moreover, for any $\Phi \subseteq \mathrm{For_m}$ we put $\Box\Phi := \{\ulcorner\Box\varphi\urcorner : \varphi \in \Phi\}$ and $\Diamond\Phi := \{\ulcorner\Diamond\varphi\urcorner : \varphi \in \Phi\}$.

Modal logics are certain sets of formulae. We define a *modal logic* as a set $\boldsymbol{L}$ of modal formulae satisfying following conditions:

- $\mathbf{Taut} \subseteq \boldsymbol{L}$,
- $\boldsymbol{L}$ includes the following set of formulae

$$\Big\{ \ulcorner\chi[{}^{\neg\Box\neg\varphi}/_{\Diamond\varphi}] \leftrightarrow \chi\urcorner \ : \ \varphi, \chi \in \mathrm{For_m} \Big\}. \qquad (\mathtt{rep}^{\Box})$$

- $\boldsymbol{L}$ is closed under the following two rules: *modus ponens* for '$\rightarrow$':

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad (\mathrm{mp})$$

  and *uniform substitution* of formulae for propositional letters in φ.

Of course, by the uniform substitution, every modal logic includes the set **PL** of modal formulae which are substitution instances of classical tautologies from **Taut**. In the present paper the term 'modal logic' is always understood as a set of modal formulae.

All members of a logic are called its *theses*. By $(\mathtt{rep}^{\Box})$, every modal logic has the following thesis:

$$\Diamond p \leftrightarrow \neg\Box\neg p \qquad (\mathtt{df}\,\Diamond)$$

REMARK 1.1. In the present paper the symbol '$\Diamond$' is a primary symbol; it is not an abbreviation of '$\neg\Box\neg$'. Thus we have to add the set of axioms

($\mathtt{rep}^{\Box}$). The use of this set corresponds to the applying of the formula ($\mathtt{df}\,\Diamond$) as a definition ruled by the definitional rule. Formulae from ($\mathtt{rep}^{\Box}$) allow to replace one or more occurrences of '$\neg\Box\neg$' with '$\Diamond$' and *vice versa.* ⊣

We say that a modal logic $\boldsymbol{L}$ is an *rte-logic* iff $\boldsymbol{L}$ is closed under replacement of tautological equivalents, i.e., for any $\varphi, \psi, \chi \in \mathrm{For_m}$

$$\text{if } \ulcorner\varphi \leftrightarrow \psi\urcorner \in \mathbf{PL} \text{ and } \chi \in \boldsymbol{L}, \text{ then } \chi[{}^{\varphi}/{}_{\psi}] \in \boldsymbol{L}. \tag{rte}$$

A modal logic is an rte-logic iff it includes the following set

$$\{\,\ulcorner\chi[{}^{\varphi}/{}_{\psi}] \leftrightarrow \chi\urcorner : \varphi, \psi, \chi \in \mathrm{For_m} \text{ and } \ulcorner\varphi \leftrightarrow \psi\urcorner \in \mathbf{PL}\,\}\,. \tag{$\mathtt{rep}_{\mathbf{PL}}$}$$

In any thesis of any rte-logic we can replace one or more occurrences of '$\neg\Box\neg$' (resp. '$\Box\neg$', '$\neg\Box$', '$\neg\Diamond\neg$', '$\neg\Diamond$', '$\Diamond\neg$') by '$\Diamond$' (resp. '$\neg\Diamond$', '$\Diamond\neg$','$\Box$', '$\Box\neg$', '$\neg\Box$') and *vice versa.* Thus, every rte-logic has the following thesis

$$\Box p \leftrightarrow \neg\Diamond\neg p \tag{$\mathtt{df}\,\Box$}$$

We say that a modal logic is *congruential* (or *classical*) iff it is closed under the congruence rule

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \tag{cgr}$$

Of course, every congruential logic is an rte-logic.

A modal logic is *monotonic* iff it is closed under the monotonicity rule

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \tag{mon}$$

Of course, every monotonic logic is congruential.

A modal logic is *regular* iff it is closed under the regularity rule

$$\frac{\varphi \wedge \psi \to \chi}{\Box\varphi \wedge \Box\psi \to \Box\chi} \tag{reg}$$

iff it is monotonic and contains the following formula

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

iff it is congruential and contains the following formula

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \tag{R}$$

Every regular logic has the following thesis

$$\Diamond(p \to q) \leftrightarrow (\Box p \to \Diamond q) \tag{$\mathtt{R}^{\Diamond\Box}$}$$

A modal logic is *normal* iff it contains (K) and is closed under the necessitation rule

$$\frac{\varphi}{\Box\varphi} \tag{nec}$$

iff it is regular and contains at least one thesis of the form $\ulcorner\Box\chi\urcorner$ iff it is congruential and contains (K) and at least one thesis of the form $\ulcorner\Box\chi\urcorner$.

As usual, let **K** be the smallest normal modal logic. Using names of the above formulae, to simplify notation of normal logics we write the *Lemmon code* $\mathbf{K}\mathbf{X}_1 \dots \mathbf{X}_n$ to denote the smallest normal logic containing the formulae $(\mathbf{X}_1)$, …, $(\mathbf{X}_n)$. We standardly put $\mathbf{S4} := \mathbf{KT4}$ and $\mathbf{S5} := \mathbf{KT5} = \mathbf{KT4B} = \mathbf{KD4B} = \mathbf{KD5B}$, where

$$\Box p \to p \tag{T}$$

$$\Box p \to \Diamond p \tag{D}$$

$$\Box p \to \Box\Box p \tag{4}$$

$$p \to \Box\Diamond p \tag{B}$$

$$\Diamond p \to \Box\Diamond p \tag{5}$$

As it is known, $\mathbf{KT} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$, $\mathbf{KD45} \subsetneq \mathbf{S5}$, $\mathbf{KD45} \nsubseteq \mathbf{S4}$ and $\mathbf{KT} \nsubseteq \mathbf{KD45}$. We have also: $(\mathtt{D}) \in \mathbf{K5_c}$, $(\mathtt{5_c}) \in \mathbf{KD4}$, so $\mathbf{KD4} = \mathbf{K5_c4}$ (see e.g. [8]), where

$$\Box\Diamond p \to \Diamond p \tag{$\mathtt{5_c}$}$$

Moreover, $(4) \in \mathbf{K55_c}$, so $\mathbf{KD45} = \mathbf{K55_c}$.

Notice that the following formulae

$$\Diamond(\Diamond p \to p) \tag{1.1}$$

$$\Diamond(\Diamond p \to (\Diamond q \to (p \wedge \Diamond q))) \tag{1.2}$$

$$\Diamond p \to (\Diamond q \to \Diamond(p \wedge \Diamond q)) \tag{1.3}$$

$$\Diamond(p \wedge \Diamond q) \leftrightarrow (\Diamond p \wedge \Diamond q) \tag{1.4}$$

$$\Diamond(\Diamond p \to q) \leftrightarrow (\Diamond p \to \Diamond q) \tag{1.5}$$

$$\Diamond(\Diamond p \to q) \to (\Diamond p \to \Diamond q) \tag{C}$$

belong to **KD45**, so also to **S5**.

For any modal logic $\boldsymbol{L}$ and any formula $\varphi \in \mathrm{For_m}$, let $\boldsymbol{L}{+}\varphi$ be the smallest modal logic which includes $\boldsymbol{L}$ and contains φ. We easily see:

LEMMA 1.1. *Let $\boldsymbol{X}$ be a set of modal logics which is closed under arbitrary intersections. If $\boldsymbol{L} \in \boldsymbol{X}$, $\varphi \in \mathrm{For_m}$ and there is in $\boldsymbol{X}$ a logic including $\boldsymbol{L} \cup \{\varphi\}$, then there is the smallest logic in $\boldsymbol{X}$ including $\boldsymbol{L} \cup \{\varphi\}$. Let us denote this logic by $\boldsymbol{L} +_{\boldsymbol{X}} \varphi$.*

Let $\Phi \subseteq \mathrm{For_m}$ and $\mathcal{R}$ be a set of rules on $\mathrm{For_m}$. We say that the pair $\langle \Phi, \mathcal{R} \rangle$ is an *axiomatization* of a modal logic $\boldsymbol{L}$ iff $\boldsymbol{L}$ is the smallest set including Φ, being closed under all rules from $\mathcal{R}$. Then for any $\varphi \in \mathrm{For_m}$: $\varphi \in \boldsymbol{L}$ iff there exists a sequence $\varphi_1, \ldots, \varphi_n = \varphi$ in which for any $i \leqslant n$, either $\varphi_i \in \Phi$ or there are $R \in \mathcal{R}$, $m < n$, $j_1, \ldots, j_m < i$ such that $\langle \varphi_{j_1}, \ldots, \varphi_{j_m}, \varphi_i \rangle \in R$.

For any modal logic $\boldsymbol{L}$ we define a consequence relation of $\vdash_{\boldsymbol{L}}$ with the help of modus ponens for '$\rightarrow$' as the only rule of inference, i.e., $\vdash_{\boldsymbol{L}}$ is the *pure modus-ponens-style inference relation based on* $\boldsymbol{L}$. Strictly speaking, for any $\Phi \subseteq \mathrm{For_m}$ and $\psi \in \mathrm{For_m}$:

$\Phi \vdash_{\boldsymbol{L}} \psi \overset{\mathrm{df}}{\iff}$ there exists a sequence $\chi_1, \ldots, \chi_m$ in which $\chi_m = \psi$ and for any $i \leqslant m$, either $\chi_i \in \Phi \cup \boldsymbol{L}$ or there are $j, k < i$ such that $\chi_k = \ulcorner \chi_j \rightarrow \chi_i \urcorner$.

LEMMA 1.2. *For any $\Phi \subseteq \mathrm{For_m}$ and $\psi \in \mathrm{For_m}$:*

$\Phi \vdash_{\boldsymbol{L}} \psi$ *iff for some $n \geqslant 0$ and for some $\varphi_1, \ldots, \varphi_n \in \Phi$ we have that $\ulcorner \varphi_1 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \psi) \ldots) \urcorner \in \boldsymbol{L}$, or equivalently $\ulcorner (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi \urcorner \in \boldsymbol{L}$.*

1.2. Discussive formulae and a translation

Discussive formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$' and '$\vee$' (negation and disjunction); discussive connectives: '$\wedge^{\mathrm{d}}$', '$\rightarrow^{\mathrm{d}}$', '$\leftrightarrow^{\mathrm{d}}$' (conjunction, implication and equivalence); and brackets. Let $\mathrm{For^d}$ be the set of all these formulae.

Let $(-)^{\bullet}$ be the translation from $\mathrm{For^d}$ into $\mathrm{For_m}$ such that:

1. $(a)^{\bullet} = a$, for any propositional letter a,
2. for any $A, B \in \mathrm{For^d}$:
   $(\neg A)^{\bullet} = \ulcorner \neg A^{\bullet} \urcorner$,
   $(A \vee B)^{\bullet} = \ulcorner A^{\bullet} \vee B^{\bullet} \urcorner$,
   $(A \wedge^{\mathrm{d}} B)^{\bullet} = \ulcorner A^{\bullet} \wedge \Diamond B^{\bullet} \urcorner$,

$(A \rightarrow^{\mathrm{d}} B)^\bullet = \ulcorner \Diamond A^\bullet \rightarrow B^{\bullet} \urcorner,$
$(A \leftrightarrow^{\mathrm{d}} B)^\bullet = \ulcorner (\Diamond A^\bullet \rightarrow B^\bullet) \wedge \Diamond(\Diamond B^\bullet \rightarrow A^\bullet) \urcorner.$

1.3. The discussive logic $\mathbf{D_2}$ as a set of discussive formulae

Jaśkowski's discussive logic $\mathbf{D_2}$ can be treated either as some set of discussive formulae or as some consequence relation on the set of all discussive formulae (see Section 1.4).

Formulation. In the first case, $\mathbf{D_2}$ is formulated with the help of the modal logic **S5** as follows (see [4, 6]):

$$\mathbf{D_2} := \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5} \}.$$

The set $\mathbf{D_2}$ is closed under substitutions. Moreover, since (C) $\in$ **S5**, the set $\mathbf{D_2}$ is closed under *modus ponens* for '$\rightarrow^{\mathrm{d}}$' (i.e., for any $A, B \in \mathrm{For}^{\mathrm{d}}$, if $A, \ulcorner A \rightarrow^{\mathrm{d}} B \urcorner \in \mathbf{D_2}$ then $B \in \mathbf{D_2}$). Besides, by (1.1) and (1.2), the following formulae

$$p \rightarrow^{\mathrm{d}} p \tag{1.1$^{\mathrm{d}}$}$$

$$p \rightarrow^{\mathrm{d}} (q \rightarrow^{\mathrm{d}} (p \wedge^{\mathrm{d}} q)) \tag{1.2$^{\mathrm{d}}$}$$

belong to $\mathbf{D_2}$.

Notice that by the translation $(-)^\bullet$, (1.4), (1.5), **PL** and by the induction on n we have

LEMMA 1.3. *For any $n \geqslant 0$, $A_1, \ldots, A_n$, $B \in \mathrm{For}^{\mathrm{d}}$:*

$$\ulcorner A_1 \rightarrow^{\mathrm{d}} (\ldots \rightarrow^{\mathrm{d}} (A_n \rightarrow^{\mathrm{d}} B) \ldots) \urcorner \in \mathbf{D_2} \iff \ulcorner (A_1 \wedge^{\mathrm{d}} \cdots \wedge^{\mathrm{d}} A_n) \rightarrow^{\mathrm{d}} B \urcorner \in \mathbf{D_2}.$$

Modal logics defining $\mathbf{D_2}$. We say that a modal logic $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^\bullet \urcorner \in \boldsymbol{L} \}$. It is known that also other modal logics than **S5** define $\mathbf{D_2}$. By definitions we have:

FACT 1.4. *If $\boldsymbol{L} \subseteq \boldsymbol{L}' \subseteq \mathbf{S5}$ and $\boldsymbol{L}$ defines $\mathbf{D_2}$, then $\boldsymbol{L}'$ defines $\mathbf{D_2}$.*

Since **S5** defines $\mathbf{D_2}$ and any intersection of modal logics defining $\mathbf{D_2}$ is a modal logic defining $\mathbf{D_2}$, so we have:

FACT 1.5. *There exists the smallest modal logic defining* $\mathbf{D_2}$.

Let $\mathbf{A}$ be the smallest modal logic defining $\mathbf{D_2}$ ("A" for "absolute"). Of course, $\mathbf{A} \subseteq \mathbf{S5}$. The logic $\mathbf{A}$ is examined in sections 2 and 4.

The family $\mathsf{S5}_\Diamond$. While expressing the logic $\mathbf{D_2}$ we refer to modal logics which have the same as $\mathbf{S5}$ theses beginning with '$\Diamond$'. Let $\mathsf{S5}_\Diamond$ be the family of all such modal logics. By definitions we have:

FACT 1.6. (i) *Every logic from* $\mathsf{S5}_\Diamond$ *defines* $\mathbf{D_2}$.
(ii) *If* $\boldsymbol{L} \subseteq \boldsymbol{L}' \subseteq \mathbf{S5}$ *and* $\boldsymbol{L} \in \mathsf{S5}_\Diamond$, *then* $\boldsymbol{L}' \in \mathsf{S5}_\Diamond$.

Moreover, we obtain:

FACT 1.7 ([8]). *For any rte-logic* $\boldsymbol{L}$: $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ *iff* $\boldsymbol{L} \in \mathsf{S5}_\Diamond$.

In [8] we give a general method which, for some classes of modal logics determined by sets of joint axioms and rules, generates in the given class the weakest logic in $\mathsf{S5}_\Diamond$. For example, by Fact 1.7, for rte-logics, congruential, monotonic, regular and normal we obtain the weakest — in the respective classes — logics defining $\mathbf{D_2}$.[2]

Let us recall (see [8]) that $\mathbf{aS5^M}$ is the smallest logic in $\mathsf{S5}_\Diamond$ (the "absolute" one). By Fact 1.6i, $\mathbf{A} \subseteq \mathbf{aS5^M}$. In Section 2 we prove that $\mathbf{A}$ has no member of the form $\ulcorner\Diamond\Diamond\chi\urcorner$. Hence for any $\tau \in \mathbf{PL}$, $\ulcorner\Diamond\Diamond\tau\urcorner \notin \mathbf{A}$, but $\ulcorner\Diamond\Diamond\tau\urcorner \in \mathbf{S5}$. Consequently, $\mathbf{A} \notin \mathsf{S5}_\Diamond$ and $\mathbf{A} \subsetneq \mathbf{aS5^M}$. Hence, $\mathbf{A}$ is not an rte-logic (we use Fact 1.7 or the fact that $\mathbf{aS5^M}$ is not an rte-logic; see [8]).

Moreover, let $\mathbf{rteS5^M}$, $\mathbf{eS5^M}$, $\mathbf{mS5^M}$, $\mathbf{rS5^M}$, $\mathbf{S5^M}$ be respectively, the smallest rte-, congruential, monotonic, regular, normal logic in $\mathsf{S5}_\Diamond$. By Fact 1.7, these logics are also respectively the smallest rte-, congruential, monotonic, regular, normal logic defining $\mathbf{D_2}$. In [8] we proved that $\mathbf{aS5^M} \subsetneq \mathbf{rteS5^M} \subsetneq \mathbf{eS5^M} \subsetneq \mathbf{mS5^M} \subsetneq \mathbf{rS5^M} \subsetneq \mathbf{S5^M} \subsetneq \mathbf{S5}$.

[2] In [2], Furmanowski proved that $\mathbf{S4} \in \mathsf{S5}_\Diamond$, so $\mathbf{S4}$ defines $\mathbf{D_2}$, by Fact 1.6i. In [11], Perzanowski defined the logic $\mathbf{S5^M}$ and proved that it is the smallest normal logic in $\mathsf{S5}_\Diamond$; so also $\mathbf{S5^M}$ defines $\mathbf{D_2}$. However, from this does not follow that $\mathbf{S5^M}$ is the smallest normal logic defining $\mathbf{D_2}$. To achieve this result one needs a version of Fact 1.7 for normal logics; that was proved in Nasieniewski's PhD thesis.

1.4. The discussive logic $\mathbf{D_2}$ as a discussive consequence

Formulation. We may also consider the second way of understanding the discussive logic $\mathbf{D_2}$, i.e., as a consequence relation between discussive formulae. In Jaśkowski's paper [3] the following definition of a discussive relation can be discovered: for any $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ and $B \in \mathrm{For}^{\mathrm{d}}$

$$\Pi \vdash_{\mathbf{D_2}} B \overset{\mathrm{df}}{\iff} \{\Diamond A^{\bullet} : A \in \Pi\} \vdash_{\mathbf{S5}} \Diamond B^{\bullet}.$$

The relation $\vdash_{\mathbf{D_2}}$ (considered by Jaśkowski) is called the $\mathbf{D_2}$-*consequence*. It has been elaborated in [9, 10]. Notice that for any $B \in \mathrm{For}^{\mathrm{d}}$,

$$\emptyset \vdash_{\mathbf{D_2}} B \;\text{ iff }\; B \in \mathbf{D_2} \;\text{ iff }\; \forall_{\Pi \subseteq \mathrm{For}^{\mathrm{d}}}\, \Pi \vdash_{\mathbf{D_2}} B\,. \tag{†}$$

Discussive systems. Following Jaśkowski, we say that a subset $\boldsymbol{S}$ of $\mathrm{For}^{\mathrm{d}}$ is a *discussive system* iff $\boldsymbol{S}$ is closed under the $\mathbf{D_2}$-consequence, i.e., for any $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ and $B \in \mathrm{For}^{\mathrm{d}}$, if $\Pi \vdash_{\mathbf{D_2}} B$ and $\Pi \subseteq \boldsymbol{S}$, then $B \in \boldsymbol{S}$. By (†), all discussive systems include the set $\mathbf{D_2}$. The basic features of discussive systems are presented in [3, p. 61], [4, p. 37–38] (see also [9, 10]).

Properties of the $\mathbf{D_2}$-consequence. Notice that, since (C) and (1.3) belong to $\mathbf{S5}$, we obtain that for any $A, B \in \mathrm{For}^{\mathrm{d}}$: $A \to^{\mathrm{d}} B, A \vdash_{\mathbf{D_2}} B$ and $A, B \vdash_{\mathbf{D_2}} A \wedge^{\mathrm{d}} B$.

Applying (1.5) and lemmas 1.2 and 1.3, on the basis of $\mathbf{D_2}$ one can characterize the $\mathbf{D_2}$-consequence in the following way:

Fact 1.8. *For any $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ and $B \in \mathrm{For}^{\mathrm{d}}$:*

$\Pi \vdash_{\mathbf{D_2}} B$ *iff* *for some $n \geqslant 0$ and for some $A_1, \ldots, A_n \in \Pi$ we have $\ulcorner A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to B)\ldots)\urcorner \in \mathbf{D_2}$, or equivalently $\ulcorner (A_1 \wedge^{\mathrm{d}} \cdots \wedge^{\mathrm{d}} A_n) \to^{\mathrm{d}} A\urcorner \in \mathbf{D_2}$.*

Moreover, the relation $\vdash_{\mathbf{D_2}}$ can be characterized with the help of modus ponens for '$\to^{\mathrm{d}}$' as the only rule of inference.

Fact 1.9 ([9]). *For any $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ and $B \in \mathrm{For}^{\mathrm{d}}$:*

$\Pi \vdash_{\mathbf{D_2}} B$ *iff* *there exists a sequence $C_1, \ldots, C_m$ in which $C_m = B$ and for any $i \leqslant m$, either $C_i \in \Pi \cup \mathbf{D_2}$ or there are $j, k < i$ such that $C_k = \ulcorner C_j \to^{\mathrm{d}} C_i\urcorner$.*

Modal logics defining the $\mathbf{D_2}$-consequence. We say that a modal logic $\boldsymbol{L}$ *defines* the $\mathbf{D_2}$-consequence iff $\vdash_{\mathbf{D_2}} = \{\langle \Pi, A\rangle \in \wp(\mathrm{For}^{\mathrm{d}}) \times \mathrm{For}^{\mathrm{d}} : \Diamond\Pi^{\bullet} \vdash_{\boldsymbol{L}} \Diamond A^{\bullet}\}$. It is again known that there are other modal logics than **S5** which define the $\mathbf{D_2}$-consequence. By definitions and Lemma 1.2 we obtain:

FACT 1.10. *If $\boldsymbol{L} \subseteq \boldsymbol{L}' \subseteq \mathbf{S5}$ and $\boldsymbol{L}$ defines $\vdash_{\mathbf{D_2}}$, then also $\boldsymbol{L}'$ defines $\vdash_{\mathbf{D_2}}$.*

FACT 1.11 ([10]). *For any modal logic $\boldsymbol{L}$:*

(i) *If $\boldsymbol{L}$ defines $\mathbf{D_2}$ and $(\mathtt{C}) \in \boldsymbol{L} \subseteq \mathbf{S5}$, then $\boldsymbol{L}$ defines $\vdash_{\mathbf{D_2}}$.*
(ii) *If $\boldsymbol{L}$ defines $\vdash_{\mathbf{D_2}}$, then $\boldsymbol{L}$ defines $\mathbf{D_2}$ and $(\mathtt{C}) \in \boldsymbol{L}$.*

From facts 1.4 and 1.11i, since $(\mathtt{C}) \in \mathbf{S5}$, we obtain

FACT 1.12 ([10]). *If $\boldsymbol{L}$ defines $\mathbf{D_2}$ and $\boldsymbol{L} \subseteq \mathbf{S5}$, then $\boldsymbol{L}+(\mathtt{C})$ defines $\vdash_{\mathbf{D_2}}$.*

Moreover, we have that:

FACT 1.13 ([10]). *Let $\boldsymbol{X}$ be a set of modal logics such that both $\mathbf{S5} \in \boldsymbol{X}$ and there is the smallest logic in $\boldsymbol{X}$ defining $\mathbf{D_2}$. If $\boldsymbol{L}$ is this logic, then:*

(i) *$\boldsymbol{L}+(\mathtt{C})$ defines $\vdash_{\mathbf{D_2}}$,*
(ii) *$\boldsymbol{L}+(\mathtt{C})$ is included in all logics from $\boldsymbol{X}$ that define $\vdash_{\mathbf{D_2}}$.*

We put $\mathbf{A}_{\vdash} := \mathbf{A}+(\mathtt{C})$. By Fact 1.13 applied to the set of all modal logics we obtain that $\mathbf{A}_{\vdash}$ is the smallest modal logic defining $\vdash_{\mathbf{D_2}}$. The logic $\mathbf{A}_{\vdash}$ will be examined in sections 3 and 4. It is shown there that $\mathbf{A}_{\vdash} \notin \mathbf{S5}_{\Diamond}$. Similarly as in the case of $\mathbf{A}$, for any $\tau \in \mathbf{PL}$, $\ulcorner \Diamond\Diamond\tau \urcorner \notin \mathbf{A}_{\vdash}$.

FACT 1.14 ([10]). *Consider a set of modal logics $\boldsymbol{X}$ which contains $\mathbf{S5}$ and is closed under arbitrary intersections. Let $\boldsymbol{L}$ be the smallest logic in $\boldsymbol{X}$ defining $\mathbf{D_2}$.*[3] *Then $\boldsymbol{L}+_{\boldsymbol{X}}(\mathtt{C})$ is the smallest logic in $\boldsymbol{X}$ defining $\vdash_{\mathbf{D_2}}$.*[4]

The family $\mathbf{Cn}_{\Diamond}\mathbf{S5}$. Let $\mathbf{Cn}_{\Diamond}\mathbf{S5}$ be the set of modal logics which satisfy the following condition: for any modal logic $\boldsymbol{L}$

[3] Notice that this logic exists, because the subset of $\boldsymbol{X}$ consisting of logics defining $\mathbf{D_2}$ includes **S5** and of course is closed under arbitrary intersections.

[4] By Lemma 1.1, the logic $\boldsymbol{L}+_{\boldsymbol{X}}(\mathtt{C})$ exists, since $\boldsymbol{L}, \mathbf{S5} \in \boldsymbol{X}$ and $\boldsymbol{L} \cup \{(\mathtt{C})\} \subseteq \mathbf{S5}$.

$$\boldsymbol{L} \in \mathsf{Cn}_{\Diamond}\mathsf{S5} \overset{\text{df}}{\iff} \text{for any } \varPi \subseteq \mathrm{For_m} \text{ and } B \in \mathrm{For_m}, \quad \Diamond\varPi \vdash_{\boldsymbol{L}} \Diamond B \quad \text{iff} \quad \Diamond\varPi \vdash_{\mathbf{S5}} \Diamond B\,.$$

FACT 1.15. (i) $\mathsf{Cn}_{\Diamond}\mathsf{S5} \subseteq \mathsf{S5}_{\Diamond}$.
(ii) *Every logic from* $\mathsf{Cn}_{\Diamond}\mathsf{S5}$ *defines the* $\mathbf{D_2}$*-consequence.*
(iii) *If* $\boldsymbol{L} \subseteq \boldsymbol{L}' \subseteq \mathbf{S5}$ *and* $\boldsymbol{L} \in \mathsf{Cn}_{\Diamond}\mathsf{S5}$*, then* $\boldsymbol{L}' \in \mathsf{Cn}_{\Diamond}\mathsf{S5}$.

FACT 1.16 ([10]). *For any modal logic* $\boldsymbol{L}$:
(i) *If* $\boldsymbol{L} \in \mathsf{S5}_{\Diamond}$*,* $\boldsymbol{L} \subseteq \boldsymbol{L}' \subseteq \mathbf{S5}$ *and* $(\mathtt{C}) \in \boldsymbol{L}'$*, then* $\boldsymbol{L}' \in \mathsf{Cn}_{\Diamond}\mathsf{S5}$.
(ii) *If* $\boldsymbol{L} \in \mathsf{Cn}_{\Diamond}\mathsf{S5}$*, then* $\boldsymbol{L} \in \mathsf{S5}_{\Diamond}$ *and* $(\mathtt{C}) \in \boldsymbol{L}$.

FACT 1.17 ([10]). *For any rte-logic* $\boldsymbol{L}$: $\boldsymbol{L}$ *define* $\vdash_{\mathbf{D_2}}$ *iff* $\boldsymbol{L} \in \mathsf{Cn}_{\Diamond}\mathsf{S5}$.

In [10], a general method is given which, for some classes of modal logics determined by sets of joint axioms and rules, generates the weakest — in the given class — logic that belongs to $\mathsf{Cn}_{\Diamond}\mathsf{S5}$. For example, by Fact 1.17, for rte-, congruential, monotonic, regular and normal logics we obtain the weakest — in the respective class — logic defining the $\mathbf{D_2}$-consequence.

Let us recall (see [10]) that $\mathbf{aS5}^{\mathbf{M}}_{\vdash}$ is the smallest logic in $\mathsf{Cn}_{\Diamond}\mathsf{S5}$. By Fact 1.15ii, $\mathbf{A}_{\vdash} \subseteq \mathbf{aS5}^{\mathbf{M}}_{\vdash}$. In Section 3 we will prove that $\mathbf{A}_{\vdash} \notin \mathsf{Cn}_{\Diamond}\mathsf{S5}$. Hence, $\mathbf{A}_{\vdash}$ is not an rte-logic (we use Fact 1.17 or the fact that $\mathbf{aS5}^{\mathbf{M}}_{\vdash}$ is not an rte-logic; see [10]).

Moreover, let $\mathbf{rteS5}^{\mathbf{M}}_{\vdash}$, $\mathbf{eS5}^{\mathbf{M}}_{\vdash}$, $\mathbf{mS5}^{\mathbf{M}}_{\vdash}$, $\mathbf{rS5}^{\mathbf{M}}_{\vdash}$ be, respectively, the smallest rte-, congruential, monotonic and regular logic in $\mathsf{Cn}_{\Diamond}\mathsf{S5}$. Besides, it is known (see [9]) that $\mathbf{KD45}$ is the smallest normal logic in $\mathsf{Cn}_{\Diamond}\mathsf{S5}$. By Fact 1.17, these logics are also respectively the smallest rte-, congruential, monotonic, regular, normal logic defining $\mathbf{D_2}$ (see [9, 10]). In [10] we proved that $\mathbf{aS5}^{\mathbf{M}}_{\vdash} \subsetneq \mathbf{rteS5}^{\mathbf{M}}_{\vdash} \subsetneq \mathbf{eS5}^{\mathbf{M}}_{\vdash} \subsetneq \mathbf{mS5}^{\mathbf{M}}_{\vdash} \subsetneq \mathbf{rS5}^{\mathbf{M}}_{\vdash} \subsetneq \mathbf{KD45} \subsetneq \mathbf{S5}$.

2. The smallest modal defining $\mathbf{D_2}$

We define the following set of modal formulae:

$$\mathrm{Gen} := \{\varphi \in \mathrm{For_m} : \exists_{A \in \mathbf{D_2}}\, \varphi = \ulcorner \Diamond A^{\bullet} \urcorner\} = \{\ulcorner \Diamond A^{\bullet} \urcorner \in \mathrm{For_m} : A \in \mathbf{D_2}\}.$$

Lemma 2.1. *Every modal logic defining* $\mathbf{D_2}$ *includes the set* Gen*.*

Proof: Let $\boldsymbol{L}$ define $\mathbf{D_2}$ and $\varphi \in \mathrm{Gen}$. Then for some $A \in \mathbf{D_2}$ we have that $\varphi = \ulcorner \Diamond A^{\bullet} \urcorner$ and $\ulcorner \Diamond A^{\bullet} \urcorner \in \boldsymbol{L}$. ⊣

Let $\mathrm{Ax}_{\mathbf{PL}}$ be a set of modal formulae such that the pair $\langle \mathrm{Ax}_{\mathbf{PL}}, \{(\mathrm{mp})\}\rangle$ is an axiomatization of a modal logic **PL**.

The logic A. Let us recall that **A** is the smallest logic defining $\mathbf{D_2}$.

Fact 2.2. **A** *is the smallest modal logic including the set* Gen*. Consequently,* **A** *is axiomatized by the set* $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen}$ *and the rule* (mp)*.*

Proof: Let $\boldsymbol{L}$ be the smallest modal logic including the set Gen, i.e., $\boldsymbol{L}$ is axiomatized by the set $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen}$ and the rule (mp). So, by Lemma 2.1, $\boldsymbol{L} \subseteq \mathbf{A}$. Now we prove that also $\mathbf{A} \subseteq \boldsymbol{L}$.

Firstly, we show that $\boldsymbol{L} \subseteq \mathbf{S5}$. Indeed, if $\varphi \in \boldsymbol{L}$, then $\varphi \in \mathbf{S5}$, by the definition of $\boldsymbol{L}$ and the inclusion $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen} \subseteq \mathbf{S5}$.

Secondly, we show that $\boldsymbol{L}$ defines $\mathbf{D_2}$; hence $\mathbf{A} \subseteq \boldsymbol{L}$. Indeed, for any $A \in \mathrm{For}^{\mathrm{d}}$: if $A \in \mathbf{D_2}$, then $\ulcorner \Diamond A^{\bullet} \urcorner \in \mathrm{Gen} \subseteq \boldsymbol{L}$; if $\ulcorner \Diamond A^{\bullet} \urcorner \in \boldsymbol{L}$, then $\ulcorner \Diamond A^{\bullet} \urcorner \in \mathbf{S5}$ and $A \in \mathbf{D_2}$. ⊣

Now notice that

Lemma 2.3. (i) *No formula containing the operator* '$\Box$' *belongs to* Gen*.*
(ii) *No formula of the form* $\ulcorner \Diamond\Diamond\chi \urcorner$ *or* $\ulcorner \Diamond(\psi \to \Diamond\chi) \urcorner$ *belongs to* Gen*.*

Proof: (i) By definitions. (ii) Suppose that $\ulcorner \Diamond\Diamond\chi \urcorner$ (resp. $\ulcorner \Diamond(\psi \to \Diamond\chi) \urcorner$) belongs to Gen. Then for some $A \in \mathbf{D_2}$ we have that $A^{\bullet} = \ulcorner \Diamond\chi \urcorner$ (resp. $A^{\bullet} = \ulcorner \psi \to \Diamond\chi \urcorner$). But this is impossible by the definition of the function $(\cdot)^{\bullet}$. ⊣

Lemma 2.4. $\mathbf{A} \cap \Diamond\mathrm{For_m} \subseteq \Diamond(\mathrm{For}^{\mathrm{d}})^{\bullet} \cup \Diamond\{\varphi : \exists_{\psi \in (\mathrm{For}^{\mathrm{d}})^{\bullet}}\ \psi = \varphi[\ulcorner \Box \urcorner/\Diamond]\}$.

Proof: "$\supseteq$" — obvious. "$\subseteq$" Let v be any valuation from $\mathrm{For_m}$ into $\{0,1\}$ such that it preserves classical truth conditions for classical constants and for any $\varphi \in \mathrm{For_m}$:

- $v(\Diamond\varphi) = 1$ iff $\varphi \in (\mathrm{For}^{\mathrm{d}})^{\bullet}$ or $\exists_{\psi \in (\mathrm{For}^{\mathrm{d}})^{\bullet}}\ \psi = \varphi[\ulcorner \Box \urcorner/\Diamond]$,
- $v(\Box\varphi) = 0$ iff $\exists_{\psi \in (\mathrm{For}^{\mathrm{d}})^{\bullet}}\ \varphi = \ulcorner \neg\psi \urcorner$ or $\exists_{\psi}\exists_{\chi \in (\mathrm{For}^{\mathrm{d}})^{\bullet}}\big(\varphi = \ulcorner \neg\psi \urcorner \ \&\ \chi = \psi[\ulcorner \Box \urcorner/\Diamond]\big)$.

We show that for any $\varphi \in \mathbf{A}$: $v(\varphi) = 1$.

For any φ from $\mathbf{PL} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen}$ we have that $v(\varphi) = 1$. Thus, by the induction on the length of the proof, relative to the axiomatization considered in Fact 2.2, we obtain: if $\varphi \in \mathbf{A}$, then $v(\varphi) = 1$. ⊣

From Lemma 2.4 we obtain the following corollaries.

FACT 2.5. $\mathbf{A}$ *has neither member of the form* $\ulcorner\Diamond\Diamond\chi\urcorner$ *nor* $\ulcorner\Diamond(\psi \to \Diamond\chi)\urcorner$.

We can also give a more elementary proof of the above fact.

PROOF: Let v be any valuation from $\mathrm{For_m}$ into $\{0, 1\}$ such that it preserves classical truth conditions for classical constants and for any $\varphi \in \mathrm{For_m}$:

- $v(\Box\varphi) = 1$ iff φ has the form $\ulcorner\neg\Diamond\chi\urcorner$ or $\ulcorner\neg\neg\Box\neg\chi\urcorner$,
- $v(\Diamond\varphi) = 0$ iff φ has the form $\ulcorner\Diamond\chi\urcorner$, or $\ulcorner\neg\Box\neg\chi\urcorner$, or $\ulcorner\psi \to \Diamond\chi\urcorner$, or $\ulcorner\psi \to \neg\Box\chi\neg\urcorner$.

We show that for any $\varphi \in \mathbf{A}$: $v(\varphi) = 1$.

By Lemma 2.3, for any φ from $\mathbf{PL}\cup(\mathtt{rep}^{\Box})\cup\mathrm{Gen}$ we have that $v(\varphi) = 1$. Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 2.2, we obtain: if $\varphi \in \mathbf{A}$, then $v(\varphi) = 1$. ⊣

COROLLARY 2.6. *For any* $\tau \in \mathbf{PL}$: $\ulcorner\Diamond\Diamond\tau\urcorner \notin \mathbf{A}$ *and* $\ulcorner\Diamond\Diamond\tau\urcorner \in \mathbf{S5}$. *Consequently,* $\mathbf{A} \notin \mathbf{S5}_{\Diamond}$.

By Fact 1.7 and Corollary 2.6, we obtain:

COROLLARY 2.7. $\mathbf{A}$ *is not an rte-logic.*

Moreover, the logic $\mathbf{A}$ has the following interesting property:

FACT 2.8. (i) *The formula* '$\Diamond(p \vee \neg p)$' *belongs to* $\mathbf{A}$.
(ii) *The formula* '$\Diamond(p \to p)$' *does not belong to* $\mathbf{A}$.

PROOF: If '$\Diamond(p \to p)$' belonged to $\mathbf{A}$, then also '$\Diamond(\Diamond p \to \Diamond p)$' would do. But this is excluded by Fact 2.5. ⊣

Now, by Lemma 2.4, we obtain:

FACT 2.9. *The logic* **A** *is closed under the following rule*[5]

$$\frac{\Diamond\varphi \qquad \Diamond(\Diamond\varphi \to \psi)}{\Diamond\psi} \tag{RC}$$

PROOF: Let $\ulcorner\Diamond(\Diamond\varphi \to \psi)\urcorner$ and $\ulcorner\Diamond\varphi\urcorner$ belong to **A**. Then, by Lemma 2.4, these formulae belong to $\Diamond(\mathrm{For^d})^\bullet \cup \Diamond\{\varphi : \exists_{\psi\in(\mathrm{For^d})^\bullet}\ \psi = \varphi[^{\neg\Box\neg}/_\Diamond]\}$. We show that $\ulcorner\Diamond\psi\urcorner$ also belongs to this set.

Firstly, let $\ulcorner\Diamond\varphi \to \psi\urcorner$, $\varphi \in (\mathrm{For^d})^\bullet$. Then for some $A, B \in \mathrm{For^d}$ we have that $\varphi = A^\bullet$, $\psi = B^\bullet$ and $\ulcorner\Diamond\varphi \to \psi\urcorner = (A \to^{\mathrm{d}} B)^\bullet$. Since **A** defines $\mathbf{D_2}$, so $\ulcorner A \to^{\mathrm{d}} B\urcorner$, $A \in \mathbf{D_2}$. Hence $B \in \mathbf{D_2}$ and $\ulcorner\Diamond\psi\urcorner \in \mathbf{A}$.

Secondly, let for some $\chi_1, \chi_2 \in (\mathrm{For^d})^\bullet$ we have that $\chi_1 = \ulcorner(\Diamond\varphi \to \psi)[^{\neg\Box\neg}/_\Diamond]\urcorner$ and $\chi_2 = \varphi[^{\neg\Box\neg}/_\Diamond]$. Then $\chi_1, \chi_2 \in \mathbf{A}$. We adopt the proof from the last paragraph.

Further, we apply a combination of proofs used in two foregoing paragraphs. ⊣

The logic A+$\Diamond(p \to p)$. The logic **A**+$\Diamond(p \to p)$ has also some interesting features. Since $\mathbf{A} \subseteq \mathbf{A}{+}\Diamond(p \to p) \subseteq \mathbf{S5}$, so **A**+$\Diamond(p \to p)$ also defines $\mathbf{D_2}$.

FACT 2.10. **A**+$\Diamond(p \to p)$ *is the smallest modal logic including the set* $\mathrm{Gen} \cup \{\Diamond(p \to p)\}$. *Consequently,* **A**+$\Diamond(p \to p)$ *is axiomatized by the set* $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^\Box) \cup \mathrm{Gen} \cup \mathrm{Sub}(\Diamond(p \to p))$ *and the rule* (mp).

Notice that also **A**+$\Diamond(p \to p)$ does not belong to $\mathbf{S5}_\Diamond$, since we have:

FACT 2.11. *The logic* **A**+$\Diamond(p \to p)$ *has no member of the form* $\ulcorner\Diamond\Diamond\chi\urcorner$.

PROOF: Let v be any valuation from $\mathrm{For_m}$ into $\{0, 1\}$ such that it preserves classical truth conditions for classical constants and for any $\varphi \in \mathrm{For_m}$:

- $v(\Diamond\varphi) = 0$ iff φ has the form $\ulcorner\Diamond\chi\urcorner$ or $\ulcorner\neg\Box\neg\chi\urcorner$,
- $v(\Box\varphi) = 1$ iff φ has the form $\ulcorner\neg\Diamond\chi\urcorner$ or $\ulcorner\neg\neg\Box\neg\chi\urcorner$.

We show that for any $\varphi \in \mathbf{A}{+}\Diamond(p \to p)$: $v(\varphi) = 1$.

[5] Cf. this rule with the rule $(\mathrm{mp_d})^{\Diamond\bullet}$ on p. 231.

By Lemma 2.3, for any φ from $\mathbf{PL} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen} \cup \mathrm{Sub}(\Diamond(p \to p))$ we have that $v(\varphi) = 1$. Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 2.10: if $\varphi \in \mathbf{A}{+}\Diamond(p \to p)$, then $v(\varphi) = 1$. Therefore $\mathbf{A}{+}\Diamond(p \to p)$ has no member of the form $\ulcorner\Diamond\Diamond\chi\urcorner$. ⊣

We have another interesting characterization of the logic $\mathbf{A}{+}\Diamond(p \to p)$.

Fact 2.12. *The logic* $\mathbf{A}{+}\Diamond(p \to p)$ *is not closed under the rule* (R$\mathtt{C}$).

Proof: Formulae '$\Diamond(\Diamond(p \to p) \to \Diamond(p \to p))$' and '$\Diamond(p \to p)$' belong to $\mathbf{A}{+}\Diamond(p \to p)$, while by Fact 2.11, '$\Diamond\Diamond(p \to p))$' does not. ⊣

3. The smallest logic defining the $\mathbf{D_2}$-consequence

Let us recall that we put $\mathbf{A}_{\vdash} := \mathbf{A}{+}(\mathtt{C})$ and by Fact 1.13 applied to the set of all modal logics we obtain that $\mathbf{A}_{\vdash}$ is the smallest modal logic defining the $\mathbf{D_2}$-consequence.

From Fact 2.2, we obtain:

Fact 3.1. $\mathbf{A}_{\vdash}$ *is the smallest modal logic including the set* $\mathrm{Gen} \cup \{(\mathtt{C})\}$. *Consequently, we can consider* $\mathbf{A}_{\vdash}$ *as being axiomatized by the set* $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen} \cup \mathrm{Sub}(\mathtt{C})$ *and the rule* (mp).

Of course, the rule (R$\mathtt{C}$) is derivable in the logic $\mathbf{A}_{\vdash}$. Hence $\mathbf{A}_{\vdash}$ is closed under this rule.

By Lemma 2.3 we obtain:

Lemma 3.2. *No formula of the form* $\ulcorner\Diamond\Diamond\psi\urcorner$ *or* $\ulcorner\Diamond(\psi \to \Diamond\chi)\urcorner$ *belongs to* $\mathrm{Gen} \cup \mathrm{Sub}(\mathtt{C})$.

Fact 3.3. $\mathbf{A}_{\vdash}$ *has neither member of the form* $\ulcorner\Diamond\Diamond\psi\urcorner$ *nor* $\ulcorner\Diamond(\psi \to \Diamond\chi)\urcorner$.

Proof: Let v be the valuation given in the proof of Fact 2.5. By lemmas 2.3 and 3.2, for any φ from $\mathbf{PL} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen} \cup \mathrm{Sub}(\mathtt{C})$ we have that $v(\varphi) = 1$. Thus, by the induction on the length of the proof, relative to the axiomatization from Fact 3.1, we obtain: if $\varphi \in \mathbf{A}_{\vdash}$, then $v(\varphi) = 1$. ⊣

By Fact 3.3 we obtain the following corollaries.

COROLLARY 3.4. *For any* $\tau \in \mathbf{PL}$: $\ulcorner\Diamond\Diamond\tau\urcorner \notin \mathbf{A}_{\vdash}$ *while* $\ulcorner\Diamond\Diamond\tau\urcorner \in \mathbf{S5}$. *Consequently,* $\mathbf{A}_{\vdash} \notin \mathsf{S5}_{\diamond}$; *so also* $\mathbf{A}_{\vdash} \notin \mathsf{Cn}_{\diamond}\mathsf{S5}$.

By Fact 1.17 and Corollary 3.4, we obtain:

COROLLARY 3.5. $\mathbf{A}_{\vdash}$ *is not an rte-logic.*

COROLLARY 3.6. *The formula* '$\Diamond(p \to p)$' *does not belong to* $\mathbf{A}_{\vdash}$.[6]

Of course, Fact 3.3 and corollaries 3.4–3.6 entail, respectively, Fact 2.5 and corollaries 2.6–2.8.

4. Axiomatizations of A and $\mathbf{A}_{\vdash}$ obtained from axiomatizations of the logic $\mathbf{D_2}$

Given Fact 2.2 (resp. Fact 3.1), the set $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen}$ (resp. $\mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathrm{Gen} \cup \mathrm{Sub}(\mathtt{C})$) and the rule (mp) can be treated as an axiomatization of the logic $\mathbf{A}$ (resp. $\mathbf{A}_{\vdash}$). One can obtain more «economic» axiomatizations of these logics using an axiomatization of $\mathbf{D_2}$. Refer to [7, 1] for specific axiomatizations of $\mathbf{D_2}$.

Let $\mathcal{A} \subseteq \mathrm{For}^{\mathrm{d}}$ and $\mathcal{R}$ be a set of rules on $\mathrm{For}^{\mathrm{d}}$. We say that the pair $\langle \mathcal{A}, \mathcal{R}\rangle$ is an *axiomatization* of the logic $\mathbf{D_2}$ iff for any $A \in \mathrm{For}^{\mathrm{d}}$: $A \in \mathbf{D_2}$ iff there exists a sequence $C_1, \ldots, C_n$ in which $C_n = A$ and for any $i \leqslant n$, either $C_i \in \mathcal{A}$ or there are $R \in \mathcal{R}$, $m < n$, $j_1, \ldots, j_m < i$ such that $\langle C_{j_1}, \ldots, C_{j_m}, C_i\rangle \in R$.

For any $\Gamma \subseteq \mathrm{For}^{\mathrm{d}}$ we put

$$\Gamma^{\Diamond\bullet} := \{\ulcorner\Diamond A^{\bullet}\urcorner \in \mathrm{For}_{\mathrm{m}} : A \in \Gamma\}.$$

Of course, $\mathrm{Gen} = \mathbf{D_2}^{\Diamond\bullet}$. Moreover, for any rule R on $\mathrm{For}^{\mathrm{d}}$ we define the following rule $R^{\Diamond\bullet}$ on $\mathrm{For}_{\mathrm{m}}$:

$$R^{\Diamond\bullet} := \{\langle \varphi_1, \ldots, \varphi_n, \psi\rangle : \exists_{A_1,\ldots,A_n,B\in\mathrm{For}^{\mathrm{d}}}\ \varphi_1 = \ulcorner\Diamond A_1^{\bullet}\urcorner, \ldots, \varphi_n = \ulcorner\Diamond A_n^{\bullet}\urcorner,\ \psi = \ulcorner\Diamond B^{\bullet}\urcorner \text{ and } \langle A_1, \ldots, A_n, B\rangle \in R\}.$$

[6] Another proof of this fact: the formula '$\Diamond(\Diamond(p \to p) \to \Diamond(p \to p)) \to (\Diamond(p \to p) \to \Diamond\Diamond(p \to p))$' is a substitution instance of the axiom ($\mathtt{C}$). If '$\Diamond(p \to p)$' were a thesis of the logic $\mathbf{A}_{\vdash}$, then '$\Diamond\Diamond(p \to p)$' would be, but there is no thesis of the form $\ulcorner\Diamond\Diamond\psi\urcorner$.

In other words, for any $A_1, \ldots, A_n, B \in \mathrm{For}^{\mathrm{d}}$:

$$\langle A_1, \ldots, A_n, B\rangle \in R \text{ iff } \langle \Diamond A_1^\bullet, \ldots, \Diamond A_n^\bullet, \Diamond B^\bullet\rangle \in R^{\Diamond\bullet}.$$

For any set of rules $\mathcal{R}$ on $\mathrm{For}^{\mathrm{d}}$ we put: $\mathcal{R}^{\Diamond\bullet} := \{ R^{\Diamond\bullet} : R \in \mathcal{R} \}$.

LEMMA 4.1. *Let R be a rule such that $\mathbf{D_2}$ is closed under R. Then all modal logics defining $\mathbf{D_2}$ are closed under the rule $R^{\Diamond\bullet}$.*

PROOF: Let $\boldsymbol{L}$ define $\mathbf{D_2}$, $\varphi_1, \ldots, \varphi_n \in \boldsymbol{L}$ and $\langle \varphi_1, \ldots, \varphi_n, \psi\rangle \in R^{\Diamond\bullet}$. Then for some $A_1, \ldots, A_n \in \mathrm{For}^{\mathrm{d}}$ we have that $\varphi_1 = \ulcorner\Diamond A_1^\bullet\urcorner$, …, $\varphi_n = \ulcorner\Diamond A_n^\bullet\urcorner$, $\psi = \ulcorner\Diamond B^\bullet\urcorner$ and $\langle A_1, \ldots, A_n, B\rangle \in R$. Since $\ulcorner\Diamond A_1^\bullet\urcorner$, …, $\ulcorner\Diamond A_n^\bullet\urcorner$ belong to $\boldsymbol{L}$, so A_1, …, A_n belong to $\mathbf{D_2}$. Hence also B belongs $\mathbf{D_2}$. Thus $\psi \in \boldsymbol{L}$. $\dashv$

FACT 4.2. *Let $\langle \mathcal{A}, \mathcal{R}\rangle$ be an axiomatization of $\mathbf{D_2}$. Then $\langle \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^\Box) \cup \mathcal{A}^{\Diamond\bullet}, \mathcal{R}^{\Diamond\bullet} \cup \{(\mathrm{mp})\}\rangle$ is an axiomatization of $\mathbf{A}$. Consequently, $\mathbf{A}$ is the smallest modal logic which includes the set $\mathcal{A}^{\Diamond\bullet}$ and is closed under the rules from $\mathcal{R}^{\Diamond\bullet}$.*

PROOF: "$\Rightarrow$" Let $\varphi \in \mathbf{A}$. Then, by Fact 2.2, there is a sequence $\chi_1, \ldots, \chi_n$ in which $\chi_n = \varphi$ and for any $i \leqslant n$, either $\chi_i \in \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^\Box) \cup \mathrm{Gen}$ or there are $j, k < i$ such that $\chi_k = \ulcorner\chi_j \to \chi_i\urcorner$.

Suppose that γ_1, …, γ_l are all elements in the sequence $\chi_1, \ldots, \chi_n$ that belong to Gen. For every $k \in \{1, \ldots, l\}$ there is $A^{\gamma_k} \in \mathbf{D_2}$ such that $\gamma_k = \ulcorner\Diamond(A^{\gamma_k})^\bullet\urcorner$. Moreover, there is a sequence $A_1^{\gamma_k}, \ldots, A_{m_k}^{\gamma_k}$ of formulae from $\mathbf{D_2}$ in which $A_{m_k}^{\gamma_k} = A^{\gamma_k}$ and for any $i \leqslant m_k$, either $A_i^{\gamma_k} \in \mathcal{A}$ or there are $R \in \mathcal{R}$, $q < m_k$, $t_1, \ldots, t_q < i$ such that $\langle A_{t_1}^{\gamma_k}, \ldots, A_{t_q}^{\gamma_k}, A_i^{\gamma_k}\rangle \in R$. We transform the above sequence into the modal language. Thus, we obtain the sequence $\ulcorner\Diamond(A_1^{\gamma_k})^\bullet\urcorner, \ldots, \ulcorner\Diamond(A_{m_k}^{\gamma_k})^\bullet\urcorner$ of formulae from $\mathrm{For}_{\mathrm{m}}$, where $\ulcorner\Diamond(A_{m_k}^{\gamma_k})^\bullet\urcorner = \gamma_k$ and for any $i \leqslant m_k$, either $\ulcorner\Diamond(A_i^l)^\bullet\urcorner \in \mathcal{A}^{\Diamond\bullet} \subseteq \mathrm{Gen}$ or for some $R \in \mathcal{R}$, $q < m_k$, $t_1, \ldots, t_q < i$: $\langle \Diamond(A_{t_1}^{\gamma_k})^\bullet, \ldots, \Diamond(A_{t_q}^{\gamma_k})^\bullet, \Diamond(A_i^{\gamma_k})^\bullet\rangle \in R^{\Diamond\bullet}$.

Now we build the following sequence. We exchange the elements γ_1, …, γ_l within the sequence $\chi_1, \ldots, \chi_n$ respectively by sequences, $\ulcorner\Diamond(A_1^{\gamma_1})^\bullet\urcorner$, …, $\ulcorner\Diamond(A_{m_1}^{\gamma_1})^\bullet\urcorner$; …; $\ulcorner\Diamond(A_1^{\gamma_l})^\bullet\urcorner$, …, $\ulcorner\Diamond(A_{m_l}^{\gamma_l})^\bullet\urcorner$. Thus, there exists a sequence $\psi_1, \ldots, \psi_s$ in which $\psi_s = \varphi$ and for any $i \leqslant s$, either $\psi_i \in \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^\Box) \cup \mathcal{A}^{\Diamond\bullet}$ or there are $R \in \mathcal{R}$, $q < s$, $j_1, \ldots, j_q < i$ such that $\langle \psi_{j_1}, \ldots, \psi_{j_q}, \varphi_i\rangle \in R^{\Diamond\bullet}$, or there are $j, k < i$ such that $\psi_k = \ulcorner\psi_j \to \psi_i\urcorner$.

"$\Leftarrow$" By Lemma 4.1, if there is a sequence presented in the last paragraph, then $\varphi \in \mathbf{A}$. $\dashv$

One can consider any axiomatization of the logic $\mathbf{D_2}$ including those defined by *modus ponens* for '$\rightarrow^{\mathrm{d}}$' as the only rule:

$$\frac{A \qquad A \rightarrow^{\mathrm{d}} B}{B} \tag{$\mathrm{mp_d}$}$$

The following rule $(\mathrm{mp_d})^{\Diamond\bullet}$ corresponds to $(\mathrm{mp_d})$:

$$\frac{\Diamond A^{\bullet} \qquad \Diamond(\Diamond A^{\bullet} \rightarrow B^{\bullet})}{\Diamond B^{\bullet}} \tag{$(\mathrm{mp_d})^{\Diamond\bullet}$}$$

Of course, $(\mathrm{mp_d})^{\Diamond\bullet} \subsetneq (\mathrm{R}\mathtt{C})$. Thus, by facts 2.9 and 4.2, the logic $\mathbf{A}$ can be axiomatized by two rules: (mp) and either $(\mathrm{mp_d})^{\Diamond\bullet}$ or $(\mathrm{R}\mathtt{C})$.

Corollary 4.3. *Let* $\langle \mathcal{A}, (\mathrm{mp_d})\rangle$ *be an axiomatization of the logic* $\mathbf{D_2}$*. Then* $\langle \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathcal{A}^{\Diamond\bullet}, \{(\mathrm{mp_d})^{\Diamond\bullet}, (\mathrm{mp})\}\rangle$ *and* $\langle \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathcal{A}^{\Diamond\bullet}, \{(\mathrm{R}\mathtt{C}), (\mathrm{mp})\}\rangle$ *are axiomatizations of* $\mathbf{A}$*. Consequently,* $\mathbf{A}$ *is the smallest modal logic which includes the set* $\mathcal{A}^{\Diamond\bullet}$ *and is closed under the rule* $(\mathrm{mp_d})^{\Diamond\bullet}$ *(resp.* $(\mathrm{R}\mathtt{C})$*).*

Proof: For $(\mathrm{mp_d})^{\Diamond\bullet}$ consider Fact 4.2.

For $(\mathrm{R}\mathtt{C})$: We act similarly as in the proof of Fact 4.2. In the part "$\Rightarrow$" we use the inclusion $(\mathrm{mp_d})^{\Diamond\bullet} \subseteq (\mathrm{R}\mathtt{C})$. In the part "$\Leftarrow$" we use Fact 2.9. $\dashv$

It is obvious, that for every logic that has the thesis $(\mathtt{C})$, both rules $(\mathrm{mp_d})^{\Diamond\bullet}$ and $(\mathrm{R}\mathtt{C})$ are derivable. Therefore we obtain:

Corollary 4.4. *Let* $\langle \mathcal{A}, (\mathrm{mp_d})\rangle$ *be an axiomatization of the logic* $\mathbf{D_2}$*. Then* $\langle \mathrm{Ax}_{\mathbf{PL}} \cup (\mathtt{rep}^{\Box}) \cup \mathcal{A}^{\Diamond\bullet} \cup \mathrm{Sub}(\mathtt{C}), \{(\mathrm{mp})\}\rangle$ *is an axiomatization of* $\mathbf{A}_{\vdash}$*. Consequently,* $\mathbf{A}_{\vdash}$ *is the smallest modal logic including the set* $\mathcal{A}^{\Diamond\bullet} \cup \{(\mathtt{C})\}$*.*

References

[1] J. Ciuciura, *A new real axiomatization of the discursive logic D2*, [in:] J. Y. Beziau, W. Carnielli, and D. M. Gabbay (eds.), **Handbook of Paraconsistency**, College Publications London, 2007, pp. 427–437.

[2] T. Furmanowski, *Remarks on discussive propositional calculus*, **Studia Logica** 34 (1975), pp. 39–43.

[3] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis**, Sect. A, vol. I, no. 5 (1948), pp. 57–77. The first English version: *Propositional calculus for contradictory deductive systems*, **Studia Logica** 24 (1969), pp. 143–157.

[4] S. Jaśkowski, *A propositional calculus for inconsistent deductive systems*, **Logic and Logical Philosophy** 7 (1999), pp. 35–56; the English version of [3].

[5] S. Jaśkowski, *O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis**, Sect. A, vol. I, no. 8 (1949), pp. 171–172.

[6] S. Jaśkowski, *On the discussive conjunction in the propositional calculus for inconsistent deductive systems*, **Logic and Logical Philosophy** 7 (1999), pp. 57–59; the English version of [5].

[7] J. Kotas, *The axiomatization of S. Jaśkowski's discussive system*, **Studia Logica** 33 (no 2) (1974), pp. 195–200.

[8] M. Nasieniewski and A. Pietruszczak, *A method of generating modal logics defining Jaśkowski's discussive logic D_2*, **Studia Logica** 97, 1 (2011), pp. 161–182.

[9] M. Nasieniewski and A. Pietruszczak, *On modal logics defining Jaśkowski's D_2-consequence*, chapter 9, pp. 141–161 [in:] K. Tanaka, F. Berto, E. Mares, F. Paoli (eds.), **Paraconsistency: Logic and Applications**, series: "Logic, Epistemology and the Unity of Science", Volume 26, Springer 2013.

[10] M. Nasieniewski and A. Pietruszczak, *A method of generating modal logics defining Jaśkowski's discussive D_2-consequence*, [in:] E. Weber, D. Wouters, J. Meheus (eds.), **Logic, Reasoning & Rationality**, Proceedings of LRR10 Conference, September 20–22, 2010, Ghent, Belgium, [accepted].

[11] J. Perzanowski, *On M-fragments and L-fragments of normal modal propositional logics*, **Reports on Mathematical Logic** 5 (1975), pp. 63–72.

Nicolaus Copernicus University
Department of Logic
ul. Asnyka 2, 87-100 Toruń, Poland
{mnasien,pietrusz}@uni.torun.pl