ON DEDUCTIVE BASES FOR PARTIAL EQUILIBRIUM LOGIC

1. Introduction

The well founded semantics (WFS) of Van Gelder, Ross and Schlipf [7] has provided one of the most popular approaches to understanding and implementing default negation in logic programming. The partial stable model semantics suggested by Przymusinski [12] provides a natural bridge between the well founded and the stable model semantics [8] that now forms the basis for the paradigm of answer set programming.

The different semantics for default negation generate non-monotonic consequence relations and this is the main source of problems in studying semantics of this kind. These problems can be effectively overcome if we point out a monotonic logic such that the required class of models (stable or partial stable) can be distinguished as minimal (in a suitable sense) models of this monotonic logic. For the first time Pearce [10, 11] used this approach to obtain a logical framework for the semantics of stable models and answer sets. The logic of here-and-there, HT, (also known as Gödel's 3-valued logic) can be used to represent stable models as minimal models and can be shown to be a maximal logic with the property that equivalent theories have the same (stable model) semantics. From the logical point of view Pearce proved that HT is a deductive base [5] for non-monotonic consequence over stable models and that this base is maximal in the class of intermediate logics.

In [2] (see also [3]) the logic N* combining intuitionistic positive connectives with Routley style negation was introduced as a logical framework for...
for investigation of the well founded semantics and the partial stable model semantics. It was proved in [2] that the logic $HT^2$, a finite valued extension of $N^*$, is a deductive base for the non-monotonic consequence over partial stable and well-founded models. In the present paper, we study the question on maximality of this deductive base in the class of $N^*$-extensions.

2. Preliminaries

We work with the propositional language $\mathcal{L} = \{\lor, \land, \rightarrow, \neg\}$. The set $\text{For}$ of formulas is constructed from atoms with the help of connectives $\lor$, $\land$, $\rightarrow$, $\neg$ in a standard way. By a logic in the language $\mathcal{L}$ we mean a subset of $\text{For}$ closed under the rule of substitution, modus ponens, and the contraposition rule $\frac{\varphi \rightarrow \psi}{\neg \psi \rightarrow \neg \varphi}$. For a logic $L$, we denote by $\mathcal{E}L$ the lattice of its extensions. With every logic $L$ we associate local and global consequence relations, $\vdash_L$ and $\vdash^*_L$. For $X \cup \{\varphi\} \subseteq \text{For}$, the relation $X \vdash_L \varphi$ holds iff $\varphi$ can be obtained from elements of $L \cup X$ with the help of modus ponens only. At the same time, $X \vdash^*_L \varphi$ means that $\varphi$ can be obtained from elements of $L \cup X$ with the help of modus ponens and contraposition rules.

The logic $N^*$ was introduced in [2] as an extension of Došen’s logic $N$ [6] with negation treated as the modal operator of impossibility. Later (see [9]) it was noted that the negation of $N^*$ shares the properties of both negative modal operators of impossibility and unnecessity. The logic $N^*$ can be defined as the least logic in the language $\mathcal{L}$ containing the axioms of positive intuitionistic logic

- $P1. \, p \rightarrow (q \rightarrow p)$
- $P2. \, (p \land q) \rightarrow p$
- $P3. \, (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $P4. \, (p \land q) \rightarrow q$
- $P5. \, (p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow (q \land r)))$
- $P6. \, p \rightarrow (p \lor q)$
- $P7. \, (p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \lor q) \rightarrow r))$
- $P8. \, q \rightarrow (p \lor q)$

together with the following axioms for negation

- $N1. \, \neg p \land \neg q \rightarrow \neg (p \lor q)$
- $N2. \, \neg (p \rightarrow p) \rightarrow q$
- $N3. \, \neg (p \land q) \rightarrow \neg p \lor \neg q$
- $N4. \, \neg \neg (p \rightarrow p)$

Notice that both De Morgan laws are provable in $N^*$

$$N^* \vdash \neg (p \land q) \leftrightarrow \neg p \lor \neg q, \, \neg (p \lor q) \leftrightarrow \neg p \land \neg q$$

since axioms $N1$ and $N3$ explicitly provide one direction of each law, whereas the opposite implications can be inferred via the rule of contrapo-
sition. Moreover, axiom N2 allows to define intuitionistic negation, ‘−’, in N∗ as: −ϕ := ϕ → ¬(p₀ → p₀).

The semantics for N∗ can be given via the following class of frames.

**Definition 2.1.** A Routley frame is a triple W = ⟨W, ≤, ∗⟩, where W is a set, ≤ a partial order on W and ∗ : W → W is such that x ≤ y iff y* ≤ x*.

A Routley model M = ⟨W, V⟩ is a Routley frame W together with a valuation V : At × W → {0, 1} such that V(p, w) = 1 and w ≤ w' imply V(p, w') = 1. In this case we say that M is a model over W.

The validity relation M, x |= ϕ of formulas in the language L on worlds of Routley models is defined in a similar way to the validity of intuitionistic formulas on Kripke frames. The only exception is in the negation connective, for which the validity condition is defined by the clause:

M, x |= ¬ϕ iff M, x* |= ϕ.

As usual, ϕ is true at M, M |= ϕ, iff M, x |= ϕ for all worlds x of M.

A formula ϕ is true at a frame W if it is true at all models over W. For Γ ⊆ For, we write M(W) |= Γ if M(W) |= ϕ for all ϕ ∈ Γ.

For a Routley frame W (model µ) and a class of frames (models) K we put

\[ LW(\mu) = \{ \varphi \mid W(\mu) |= \varphi \} \] and \[ LK = \bigcap \{ LW(L\mu) \mid W(\mu) \in K \}. \]

Obviously, sets LW, Lµ, and LK constitute logics.

With a class of models (frames) K we associate local and global semantical consequences, ⊨K and ⊨∗K. For Γ ⊆ For, Γ ⊨K ϕ holds iff M, x |= Γ implies M, x |= ϕ for every world x of M ∈ K (of a model M over a frame in K). And the relation Γ ⊨∗K ϕ holds iff M |= Γ implies M |= ϕ for every M ∈ K (for every model M over a frame in K).

It was proved in [2] that the logic N∗ is strongly complete wrt the class R of all Routley frames, i.e., that ⊨N∗ = ⊨R.

Put HT² = LWHT², where WHT² = ⟨WHT², ≤, ∗⟩ is such that (i) W = {h, h', t, t'}, (ii) ≤ is a partial ordering on W satisfying h ≤ t, h ≤ h', h' ≤ t' and t ≤ t', (iii) h* = t* = t', (h')* = (t')* = t. The ordering of WHT² and the action of ∗ are presented at the following diagram.
See [2] and [4] for axiomatization of $HT^2$ over $N^*$. Consider an $HT^2$-model $\mathcal{M} = \langle V^{HT^2}, W \rangle$ denoting by $H, H', T, T'$ the four sets of atoms respectively verified at each corresponding world $h, h', t, t'$. Notice that $H \subseteq H', H \subseteq T$, and $T \subseteq T'$. We represent $\mathcal{M}$ as a pair $\langle H, T \rangle$, where $H = (H, H')$ and $T = (T, T')$.

For two pairs of sets of atoms $S = (S, S')$ and $R = (R, R')$ we write $S \leq R$ iff $S \subseteq R$ and $S' \subseteq R'$. If $\mathcal{M} = (H, T)$ is an $HT^2$-model, then $H \leq T$.

Now we define a partial ordering $\leq$ among $HT^2$-models. We set $\langle H_1, T_1 \rangle \leq \langle H_2, T_2 \rangle$ if (i) $T_1 = T_2$; (ii) $H_1 \leq H_2$. A model $\langle H, T \rangle$ in which $H = T$ is said to be total.

**Definition 2.2 (Partial equilibrium model).** An $HT^2$-model $\mathcal{M}$ is said to be a partial equilibrium ($p$-equilibrium) model of a set of formulas $\Pi$ iff (i) $\mathcal{M} \models \Pi$, (ii) $\mathcal{M}$ is total; (iii) $\mathcal{M}$ is minimal among $HT^2$-models of $\Pi$ under the ordering $\leq$.

In [2], it was established that for every disjunctive or normal program $\Pi$, a total model $\langle T, T \rangle$ is a $p$-equilibrium model of $\Pi$ iff $T$ is a partial stable model of $\Pi$ in the sense of [12].

By a **disjunctive program** we mean a set of formulas of the form

$$(p_1 \land \ldots \land p_k \land \neg q_1 \land \ldots \land \neg q_m) \rightarrow (r_1 \lor \ldots \lor r_n),$$

where $k, m \geq 0$ and $n > 0$. A disjunctive program called **normal** if $n = 1$ for all formulas in the program.

### 3. Algebraic semantics for $N^*$ and $HT^2$

The algebraic semantics for $N^*$ and the lattice of $HT^2$-extensions were described in [9]. Here we recall the main facts.
Definition 3.1. An algebra $A = \langle A, \wedge, \vee, \to, \neg, 0, 1 \rangle$ is called a Heyting-Ockham algebra (HO-algebra) if the following conditions are satisfied:

1. The $\neg$-free reduct of $A$, $A^H := \langle A, \wedge, \vee, \to, 0, 1 \rangle$, is a Heyting algebra, i.e. it is a restricted lattice with the least element $0$, the greatest element $1$, and the implication operation $\to$ satisfying the equivalence
   \[ x \leq a \to b \text{ iff } a \land x \leq b. \]

2. The $\to$-free reduct of $A$, $A^O := \langle A, \wedge, \vee, \neg, 0, 1 \rangle$, is an Ockham lattice \[13\], i.e. it is a bounded distributive lattice satisfying the identities:
   \[ \neg(x \lor y) = \neg x \land \neg y, \quad \neg(x \land y) = \neg x \lor \neg y, \quad \neg 0 = 1, \quad \neg 1 = 0. \]

A formula $\varphi$ is said to be true on an HO-algebra $A$, symbolically $A \models \varphi$, if $v(\varphi) = 1$ for any $A$-valuation $v$, or, equivalently, if the identity $\varphi = 1$ holds on $A$. Put $LA := \{ \varphi \mid A \models \varphi \}$ and $LK := \bigcap \{ LA \mid A \in K \}$ for a class $K$ of HO-algebras. It is easy to see that the sets $LA$ and $LK$ are logics extending $N^*$.

Since Heyting algebras as well as Ockham lattices form varieties, the class $V^*$ of all HO-algebras also forms a variety. For a variety of algebras $V$, denote by $Sub(V)$ the lattice of its subvarieties and by $Eq(V)$ the equational theory of this variety, i.e. the set of all identities that hold on all algebras of $V$.

For a logic $L \in EN^*$ and a subvariety $V \in Sub(V^*)$, put
\[ V(L) := \{ A \mid L \subseteq LA \} \text{ and } L(V) := \{ \varphi \mid \varphi = 1 \in Eq(V) \}. \]

Theorem 3.2. \[9\] For any $L \in EN^*$, we have $V(L) \in Sub(V^*)$. For any $V \in Sub(V^*)$, holds $L(V) \in EN^*$. Moreover, the mappings $V : EN^* \to Sub(V^*)$ and $L : Sub(V^*) \to EN^*$ are mutually inverse dual lattice isomorphisms between $EN^*$ and $Sub(V^*)$.

Following \[9\] we define the duality between HO-algebras and Routley frames. Let $W = (W, \leq, *)$ be a Routley frame. A cone of $W$ is a subset $K$ of $W$ such that $x \in K$ and $x \leq y$ imply $y \in K$ for every $y \in W$. Denote by $(W, \leq)^+$ the set of all cones of $W$. Define the algebra of cones $A(W)$ of the frame $W$ as follows:
\[ A(W) := \langle (W, \leq)^+, \cap, \cup, \to, \neg, \varnothing, W \rangle; \]
where

- \((W, \leq)^+\) is the set of cones of the partial ordering \((W, \leq)\);
- \(\cap\) and \(\cup\) are the intersection and the sum of sets;
- \(X \Rightarrow Y := \{w \in W \mid \forall u \geq w(u \in X \implies u \in Y)\}\);
- \(\neg X := \{w \in W \mid w^* \not\in X\}\).

For an arbitrary \(HO\)-algebra \(A\), we construct now a frame \(W^A\) as follows.

Recall that a prime filter \(F\) on \(A\) is non-empty subset of \(A\) such that:

(i) \(a \land b \in F\) whenever \(a \in F\) and \(b \in F\);
(ii) if \(a \in F\) and \(a \leq b\), then \(b \in F\);
(iii) if \(a \lor b \in F\), then \(a \in F\) or \(b \in F\). Denote by \(W(A)\) the set of all prime filters on \(A\) and put

\[
W^A := (W(A), \subseteq, *),
\]

where

- \(\subseteq\) is a set theoretical inclusion;
- \(F^* := \{a \mid \neg a \not\in F\}\).

It can be easily checked that \(A(W)\) is an \(HO\)-algebra for every Routley frame \(W\) and that \(W^A\) is a Routley frame for every \(HO\)-algebra \(A\).

**Proposition 3.3.** [9] For any \(HO\)-algebra \(A\), the mapping \(a \mapsto \{F \in W(A) \mid a \in F\}, a \in A\), is an embedding of \(A\) into \(A(W^A)\). If \(A\) is finite, then it is an isomorphism.

Let \(A\) be a finite \(HO\)-algebra. For an \(A\)-valuation \(v\), define a \(W^A\)_valuation \(V\) as follows: \(V(p, F) = 1\) iff \(v(p) \in F\). In view of the last proposition, we established a one-to-one correspondence between \(A\)-valuations and \(W^A\)-valuations. This allows us to prove

**Proposition 3.4.** For any finite \(HO\)-algebra and set of formulas \(\Gamma \cup \{\varphi\}\), holds the equivalence:

\[
\Gamma \models_{W^A} \varphi \iff \Gamma \models_A \varphi.
\]

The algebra of cones of \(W^{HT^2}\) is isomorphic to the six-element \(HO\)-algebra \(6 := \{0, a, b, c, d, 1\}, \land, \lor, \rightarrow, \neg, 0, 1\). The lattice structure of \(6\) and the truth table for negation look as follows.
It is easy to see that the frame $W^6$ is isomorphic to $W^{HT^2}$. Consequently, $HT^2 = L6$, moreover, $\vdash^* 6 = \vdash^* W^{HT^2}$.

All subdirectly irreducible algebras of the variety $V(HT^2)$ are isomorphic either to $6$, or to one of the following algebras:

- $1$
  - $x | \neg x$
  - $\begin{array}{c|c}
    0 & 1 \\
    1 & 0 \\
  \end{array}$

- $0$
  - $2$
  - $d$
  - $\begin{array}{c|c}
    0 & 1 \\
    d & 0 \\
    1 & 0 \\
  \end{array}$

- $0$
  - $3$
  - $c$
  - $\begin{array}{c|c}
    0 & 1 \\
    c & c \\
    1 & 0 \\
  \end{array}$

- $0$
  - $4$
  - $b$
  - $\begin{array}{c|c}
    0 & 1 \\
    b & 0 \\
    1 & 0 \\
  \end{array}$

- $0$
  - $4'$
  - $c$
  - $\begin{array}{c|c}
    0 & 1 \\
    c & c \\
    1 & 0 \\
  \end{array}$
The diagram of Figure 1 presents the ordering of logics corresponding to subdirectly irreducible algebraic models of $HT^2$.

Here $Cl$ denotes the classical logic and $HT$ the logic of here-and-there, the maximal intermediate logic different from $Cl$. This logic also plays an important part in the foundations of logic programming being a deductive base for answer set semantics [12]. The lattice $EHT^2$ of $HT^2$-extensions is isomorphic to the lattice of cones of the ordering presented at Figure 1. Thus, the lattice of non-trivial $HT^2$-extensions contains 13 logics:

$$Cl, HT, L3', HT \cap L3', L4, L4', L5', L4 \cap L3', L4 \cap L4', L4 \cap L5', L4 \cap L4' \cap L5', HT^2.$$
Now we can construct Routley frames defining the same logics as the algebras $2^2$, $3^2$, $3^3$, $4^4$, $4^4'$, and $5^5'$. Below we give the diagrams presenting the orderings of these frames and tables presenting the action of $*$-function.

Models over all these frames except for $W^{5'}$ can be identified with $HT^2$-models in the following way:

- **$Cl$-models** have the form $\langle(T,T),(T,T)\rangle$
- **$HT$-models** have the form $\langle(H,H),(T,T)\rangle$
- **$L3^3$-models** have the form $\langle(T,T'),(T,T')\rangle$
- **$L4^4$-models** have the form $\langle(H,H'),(T,T)\rangle$
- **$L4^4'$-models** have the form $\langle(H,T'),(T,T')\rangle$
It looks natural at first glance to identify models of $L_5'$ with $HT^2$-models of the form $\langle (T, H'), (T, T') \rangle$. The problem is that $T$ need not be a subset of $H'$, moreover, the validity of implication is calculated according to different rules at worlds $h'$ and $t$. Therefore, we represent $L_5'$-models with pairs $\langle H', T \rangle$, where $T = (T, T')$ and $H'$, $T$, $T'$ are sets of atoms validated respectively at worlds $h'$, $t$, and $t'$ of the frame $W^{5'}$.

With every $L$ from the set $\{Cl, HT, L_3', LA, LA', L_5', HT^2 \}$ we associate a semantical consequence $\models_L$, which coincides with global consequence over the respective defining frame. For example, $\models_{L_3'} = \models_{W^{3'}}^*$, $\models_{HT^2} = \models_{W^{HT^2}}^*$, etc. An intersection of logics, for example, $L = LA \cap LA' \cap L_5'$ is characterized by three frames $W^4$, $W^4'$, and $W^{5'}$, and we define $\models_L$ as $\models_{(W^4, W^{4'}, W^{5'})}$. In this way we associated a semantical consequence relation $\models_L$ with every logic $L \in \mathcal{E}HT^2$.

4. Monotonic bases for PEL

**Definition 4.1.** Let $\Pi \cup \{\varphi\} \subseteq \text{For}$ and $\mathcal{PEL}(\Pi)$ be the collection of all $\rho$-equilibrium models of $\Pi$. We say $\Pi$ entails $\varphi$ in partial equilibrium logic (PEL), in symbols $\Pi \models_\text{pet} \varphi$, if $M \models \varphi$ for every $M \in \mathcal{PEL}(\Pi)$. For $\Pi_1 \cup \Pi_2 \subseteq \text{For}$, the relation $\Pi_1 \approx_\text{pet} \Pi_2$ means that $\mathcal{PEL}(\Pi_1) = \mathcal{PEL}(\Pi_2)$.

According to this definition all sets of formulas $\Pi$ with empty set of $\rho$-equilibrium models are explosive, in a sense that $\Pi \models_\text{pet} \varphi$ for any $\varphi$. This means that we are not interested in PEL-consequences of theories without $\rho$-equilibrium models, and at this point our definition of PEL-entailment differs from that of [3].

We adopt the definition of a monotonic base [5] for a non-monotonic consequence relation to the considered situation.

**Definition 4.2.** Let $L \in \mathcal{E}N^*$ and $\models_L$ be the associated consequence relation. We say that the consequence relation $\models_L$ is a deductive base for $\models_\text{pet}$ iff the following conditions hold for every $\Pi, \Pi_1, \Pi_2 \subseteq \text{For}$: (i) $\Pi \models_L \subseteq \models_\text{pet}$; (ii) if $\Pi \models_\text{pet} \varphi$ and $\varphi \models_L \psi$, then $\Pi \models_\text{pet} \psi$; (iii) if $\Pi_1 \equiv_\text{pet} \Pi_2$, then $\Pi_1 \models_L \Pi_2$.

Here the equivalence $\Pi_1 \equiv_\text{pet} \Pi_2$ means that $\Pi_1 \equiv_\text{pet} \Pi_2$ and $\Pi_2 \equiv_\text{pet} \Pi_1$, or equivalently, that $\Pi_1$ and $\Pi_2$ have the same models over respective frames.

Now we are ready to formulate the main result of the paper.
Theorem 4.3. The consequence relation $\vdash_{HT^2}$ is a deductive base for $\models_{pel}$.
At the same time, for every proper extension $L \in \mathcal{E}HT^2$ of the logic $HT^2$, the consequence relation $\vdash_L$ is not a deductive base for $\models_{pel}$.

Proof. That $HT^2$ is a deductive base of $\models_{pel}$ follows immediately from the definition of $\models_{pel}$.

Let us consider $HT^2$-extensions contained in $L^3'$.

Lemma 4.4. Let $L \in \mathcal{E}HT^2$ and $L \subseteq L^3'$, then conditions (i) and (ii) hold for $\vdash_L$.

Proof. (i) Let $\Pi \models_L \varphi$. If $\Pi$ has no $p$-equilibrium models, then obviously $\Pi \models_{pel} \varphi$.

Let $M \in \mathcal{PEL}(\Pi)$. Then $M \models L^3'$ since all $p$-equilibrium models are total, and so $M \models L$. Consequently, $M \models \varphi$ in view of $\Pi \models_L \varphi$. Thus, $\Pi \models_{pel} \varphi$.

(ii) Assume that $\Pi \models_{pel} \varphi$ and $\varphi \models_L \psi$. Let $M \in \mathcal{PEL}(\Pi)$. Then $M \models \varphi$ by $\Pi \models_{pel} \varphi$. At the same time $M \models L$ in view of $L \subseteq L^3'$ and the totality of $M$. The relation $\varphi \models_L \psi$ implies $M \models \psi$, which proves $\Pi \models_{pel} \psi$.

□

Now we check whether condition (iii) holds for logics $L$ with $L \subseteq L^3'$.

Lemma 4.5. Let $L \in \mathcal{E}HT^2 \setminus \{HT^2\}$ and $L \subseteq L^3'$. There are $\Pi_1, \Pi_2 \subseteq \mathcal{F}$ such that $\Pi_1 \equiv_{L} \Pi_2$, but $\Pi_1 \not\models_{pel} \Pi_2$.

Proof. We have to prove this lemma for 9 logics: $L^3'$, $HT \cap L^3'$, $L^4'$, $L^5'$, $L^6 \cap L^3'$, $L^6 \cap L^4'$, $L^6 \cap L^5'$, $L^6 \cap L^7'$, and $L^6 \cap L^4' \cap L^5'$.

First we consider the case of logic $L^4'$. Let $\Pi_1 := \{p \leftrightarrow \neg p\}$ and $\Pi_2 := \{p \rightarrow \neg p, \neg p \rightarrow \neg \neg p\}$. It is easy to check that for an arbitrary $HT^2$-model $M = \langle (H,T'), (T,T') \rangle$ the following equivalences hold:

$$
M \models \Pi_1 \iff p \in H', T' \iff M \models \Pi_2 \iff p \notin T' \iff p \notin T
$$

Taking into account that $L^4'$-models have the form $\langle (H,T'), (T,T') \rangle$, i.e., they are distinguished by the condition $H' = T'$, we obtain that $\Pi_1$ and $\Pi_2$ have the same $L^4'$-models. Consequently, $\Pi_1 \equiv_{L^4'} \Pi_2$. Put $M_0 := \langle (\emptyset, \{p\}), (\emptyset, \{p\}) \rangle$. Obviously, $M_0$ is a $p$-equilibrium model of $\Pi_1$. But it
is not a \( p \)-equilibrium model of \( \Pi_2 \), because \( \langle (\emptyset, \emptyset), (\emptyset, \{ p \}) \rangle \models \Pi_2 \). Thus \( \Pi_1 \not\equiv_{pel} \Pi_2 \).

We have \( \Pi_1 \equiv_{L3'} \Pi_2 \), because \( L4' \subseteq L3' \). This implies the conclusion of this lemma for \( L3' \).

Let us consider logics \( HT \) and \( L4 \). Models of these logics have the property \( T = T' \). At the same time \( p \in T \setminus T \) in models of both \( \Pi_1 \) and \( \Pi_2 \). This means that \( \Pi_1 \) and \( \Pi_2 \) have no \( HT \)- or \( L4 \)-models, i.e., they are equivalent in \( HT \) and \( L4 \). Due to this reason the conclusion of the lemma holds for these logics as well as for each of the intersections \( HT \cap L3' \), \( L4 \cap L3' \), \( L4 \cap L4' \).

Now we turn to the logic \( L5' \). Consider the theories
\[
\Pi_1 := \{ p \lor q, q \leftrightarrow \neg q \} \quad \text{and} \quad \Pi_2 := \{ \neg
\neg p, q \leftrightarrow \neg q \}.
\]
Let \( \langle H', T \rangle \models \Pi_1 \). It is easy to see that \( q \in H', T' \), but \( q \not\in T \). Since \( p \lor q \) is true at the world \( t \) of this model, we have \( p \in T, T' \). Obviously, the above conditions are also sufficient for \( \langle H', T \rangle \models \Pi_1 \). The validity \( \langle H', T \rangle \models \neg\neg p \) implies \( p \in T, T' \), which leads to the equivalence
\[
\langle H', T \rangle \models \Pi_2 \iff p \in T, T', q \in H', T', q \not\in T.
\]
Thus, \( \Pi_1 \) and \( \Pi_2 \) have the same \( L5' \)-models, i.e., \( \Pi_1 \equiv_{L5'} \Pi_2 \).

Let \( T_0 := \langle \{ p \}, \{ p, q \} \rangle \). It is easy to see that \( M_0 := \langle T_0, T_0 \rangle \) is a \( p \)-equilibrium model of \( \Pi_1 \). Indeed, it is a model of \( \Pi_1 \). If \( \langle H, T_0 \rangle \) is a model of \( \Pi_1 \), then \( q \in H' \) and \( q \not\in H \) by \( q \leftrightarrow \neg q \in \Pi_1 \). Since \( p \lor q \in \Pi_1 \), we have \( p \in H, H' \), and so \( H = T_0 \). At the same time \( M_0 \not\in \mathcal{PEL}(\Pi_2) \), because \( \langle (\emptyset, \{ q \}), T_0 \rangle \models \Pi_2 \). Thus, the lemma is proved for \( L5' \).

For the case of logic \( L5 := L4' \cap L5' \), consider the same theory \( \Pi_1 \) with \( p \)-equilibrium model \( M_0 = \langle T_0, T_0 \rangle \) and the theory
\[
\Pi_3 := \Pi_2 \cup \{ (\neg q \rightarrow (p \land q)) \rightarrow p \}.
\]
It is easy to see that \( (\neg q \rightarrow (p \land q)) \rightarrow p \) holds in all \( L5' \)-models of \( \Pi_2 \). In this way, \( L5' \)-equivalence of \( \Pi_1 \) and \( \Pi_2 \) implies that \( \Pi_1 \equiv_{L5'} \Pi_3 \). Now we look at \( L4' \)-models of \( \Pi_1 \) and \( \Pi_3 \), i.e., at models of the form \( \langle (H, T'), T \rangle \).

The validity of \( q \leftrightarrow \neg q \) is equivalent to \( q \in T' \) and \( q \not\in T \). In this case, the validity of \( p \lor q \) implies \( p \in H \). Thus,
\[
\langle (H, T'), T \rangle \models \Pi_1 \iff p \in H, q \not\in T, q \in T'.
\]
It is clear that formulas of \( \Pi_3 \) are true in models of this form. Let \( \langle (H, T'), T \rangle \) be a model of \( \Pi_3 \). Check that it satisfies the above conditions. We have
q \not\in T$ and $q \in T'$ since $q \leftrightarrow \neg q \in \Pi_3$, and $p \in T, T'$ by $\neg \neg p \in \Pi_3$. Since $q \in T'$, the negation $\neg q$ is false at worlds $h$ and $t$, at the same time $p, q \in T'$. These conditions guarantee that the implication $\neg q \rightarrow (p \land q)$ holds at world $h$. Therefore, $p \in H$ by $(\neg q \rightarrow (p \land q)) \rightarrow p \in \Pi_3$. Thus, $\Pi_1$ and $\Pi_3$ are equivalent in $L_4$ and in $L_5^\circ$. To prove the lemma for $L_5^\circ$ it remains to note that $\langle (\emptyset, \{q\}), (\{p\}, \{p, q\}) \rangle \models \Pi_3$. Indeed, the consequence of the implication $(\neg q \rightarrow (p \land q)) \rightarrow p$ is false at the worlds $h$ and $h'$ of this model, but the same holds for the premiss, since $h' \models \neg q$, but $h' \not\models p \land q$.

We have thus proved that $M_0$ is not a $p$-equilibrium model of $\Pi_3$. All models of theories $\Pi_1$, $\Pi_2$, or $\Pi_3$ have the property $q \in T' \setminus T$. Therefore, these theories have no $L_4$-models, which satisfy the condition $T = T'$. Thus, all theories $\Pi_1$, $\Pi_2$, and $\Pi_3$ are $L_4$-equivalent. We can use $\Pi_1$ and $\Pi_2$ to prove the conclusion of the lemma for $L_4 \cap L_5^\circ$, and $\Pi_1$ and $\Pi_3$ for $L_4 \cap L_5^\circ$.

We have thus considered all 9 logics. □

Now we are ready to finish the proof of the theorem. The last lemma implies that all proper $HT^2$-extensions contained in $L_3'$ are not deductive bases for $\vdash_{pel}$. It remains to consider three logics $Cl$, $HT$, and $L_4$. Models of these logics have the property $T = T'$. The formula $q \leftrightarrow \neg q$ has no such models. Therefore, $q \leftrightarrow \neg q \vdash p$ in each of these logics, but $p$ does not hold in $\langle (\emptyset, \{q\}), (\emptyset, \{q\}) \rangle$, the only $p$-equilibrium model of $q \leftrightarrow \neg q$. Thus the condition (i) of Definition 4.2 fails for $Cl$, $HT$, and $L_4$. □

We have identified with each logic $L \in EHT^2$ the consequence relation $\vdash_L$ in a way typical for logic programming, where consequence relations usually are defined semantically with the help of the most simple models of a logic. So, the last theorem may be considered as a result on the maximality of $\vdash_{HT^2}$ as a deductive base for $\vdash_{pel}$ in the class of $N^*$ extensions.

Notice also that it can be proved that for all logics $L \in EHT^2 \setminus \{HT^2\}$ the semantical consequence $\vdash_L$ coincides with the global syntactic consequence $\vdash^*_L$. At the same time, the consequence $\vdash_{HT^2}$ differs from $\vdash^*_HT^2$. The first relation is closed under the rule $\frac{\varphi \lor \bot}{\varphi}$, where as the second does not. Moreover, syntactically the relation $\vdash_{HT^2}$ can be characterized as the extension of $\vdash^*_HT^2$ via adding the rule $\frac{\varphi \lor \bot}{\varphi}$. Due to the lack of space we only claim these results here.

Concluding the paper let us mention several questions, which we plan to study in subsequent works. First of all, it is interesting to prove the
result on maximality of $\vdash_{HT^2}$ in a more general form, namely that for every proper extension $\vdash$ of $\vdash_{HT^2}$ via a finite number of axioms and rules, $\vdash$ is not a deductive base for $\models_{pel}$. Further, well founded models [2] of a theory form a subclass of its $p$-equilibrium models and we can define a non-monotonic well-founded entailment (WF-entailment) in the same way as PEL-entailment. So, it would be interesting to find out the maximal deductive base for WF-entailment.

References


