PARTIAL PROBABILITY FUNCTIONS AND
INTUITIONISTIC LOGIC∗

Abstract
We first present a sound and complete system of intuitionistic logic augmented
with Nelson’s strong negation and interpreted in Kripke’s models’ structure.
Then, we show that intuitionistic logic can naturally be interpreted in a modal
trivalent logic (propositions are true, false or undefined). Secondly, we introduce
a partial probability interpretation, which is inspired by Popper’s conditional
probability functions and are characterized by the fact that conditions are not
propositions but rather sets of propositions. Finally, we define the notion of
probabilistic validity and we show using Kripke’s models that the intuitionistic
system is sound and complete.

1. Motivation and background
One starting point of this paper is partial logic. In [5] and [6], Lapierre
and I showed that partial logic can be interpreted as a three-valued logic,
the third value being the undefined value. Following Thijsse [13], we used
Saturated deductively closed consistent sets (SDCCS) for the completeness
proof. A set is SDCCS if it is deductively closed and if it contains (A ∨ B)
only if it contains either A or B. If we restrict ourselves to partial
propositional logic, the following system is proved to be sound and complete
for Kleene strong 3-valued logic.

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paper.
A derivation of $A$ from $\Gamma$ ($\Gamma \vdash A$) is a sequence of propositions $A_0, \ldots, A_n$ such that each $A_i$ satisfies one of the following clauses:

R. If $A_i \in \Gamma$, then $\Gamma \vdash A_i$
D1. If $\Gamma \vdash B \land \neg B$, then $\Gamma \vdash A_i$
D2. If $\Gamma \vdash \neg\neg A_i$, then $\Gamma \vdash A_i$
D3. If $\Gamma \vdash A_j$ then, $\Gamma \vdash A_i$ (where $A_i$ is $\neg\neg A_j$)
D4. If $\Gamma \vdash A_i \land B$, then $\Gamma \vdash A_i$
D5. If $\Gamma \vdash B \land A_i$, then $\Gamma \vdash A_i$
D6. If $\Gamma \vdash B$ and $\Gamma \vdash C$, then $\Gamma \vdash A_i$ (where $A_i$ is $B \land C$)
D7. If $\Gamma \vdash A_i$, $\Gamma \vdash A_i \lor B$
D8. If $\Gamma \vdash A_i$, $\Gamma \vdash B \lor A_i$
D9. If $\Gamma \vdash \neg (C \land B)$, then $\Gamma \vdash A_i$ (where $A_i$ is $(\neg C \lor \neg B)$)
D10. If $\Gamma \vdash \neg (C \lor B)$, then $\Gamma \vdash A_i$ (where $A_i$ is $(\neg C \land \neg B)$)
D11. If $\Gamma \cup \{B\} \vdash A_i$ and $\Gamma \cup \{C\} \vdash A_i$, then $\Gamma \cup \{B \lor C\} \vdash A_i$

This system has no theorem and the deduction theorem does not hold.

As it has been noted by Frankowski [3], there are strong bounds between partial logic and intuitionistic logic. Actually, it can be shown that the aforementioned deduction rules are valid in Nelson’s intuitionistic logic with strong negation [11]. Moreover, Aczel [1] showed that Nelson’s system is sound and completed for Kripke’s semantics [4] where possible worlds are SDCCS.

The second starting point of this paper is the notion of Probabilistic interpretation. It is Popper [12] who first presented an axiomatization of a probability calculus which takes a two-places probability function as a primitive notion. The two important features of Popper’s approach are that conditionalization on sentences that have a 0 probability is always defined and the classical absolute probability functions can be considered as a special case of conditionalization on a tautology.

However, the great breakthrough in the probabilistic interpretation of the logical calculus is Hartry Field’s [2] axiomatization of the notion of probabilistic validity without making use of the notion of truth-value and more generally without using any classical semantic notion that makes use of truth functions. Field’s axiomatization of conditional probability for both propositional calculus and first order predicate calculus can be used to prove the soundness and completeness of their usual axiomatization.

There are many ways to provide axiomatizations of conditional probability functions. The most general is probably that of Morgan [8], [9] which
uses two-place probability functions where the second place is a set of sentences instead of a single sentence. Morgan showed that every consistent extension of classical logic is complete for a given class of probabilistic interpretation. Moreover, Morgan and Leblanc [10] showed that intuitionistic logic is sound and complete for a given class of probabilistic interpretations. The problem is that for their interpretations, the probability is always defined and any non-consequence of $\Gamma$ has a probability 0 in the canonical model that they used for the completeness proof, which is quite unnatural.

Besides, Morgan and I [7] showed that partial logic (which is not an extension of classical logic) is complete for a given class of partial probability functions.

In the rest of this paper, I will show how these partial probability functions can be used to provide sound and complete interpretations for Nelson’s logic. The two main features of my proposal are

1. $\Pr(A \lor \sim A, \Gamma) = 1$ iff $\Pr(A, \Gamma)$ is defined (and $A$ is not $B \rightarrow C$)\(^1\) and $\Pr(A \lor \sim A, \Gamma)$ is undefined iff $\Pr(A, \Gamma)$ is undefined.
2. The probabilistic interpretation of the intuitionistic conditional is a partial conditional probability, i.e. $\Pr(A \rightarrow B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$ when defined.

In [10], $\Pr(A \rightarrow B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$, but it is always defined.

2. Syntax

Let PI (for Partial Intuitionistic) be the following system, which is based on four connectives $\sim$, $\land$, $\lor$ and $\rightarrow$) and where $\sim$ is Nelson’s strong negation. The axioms are

1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
3. $A \land B \rightarrow A$
4. $A \land B \rightarrow B$
5. $A \rightarrow A \lor B$
6. $B \rightarrow A \lor B$
7. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \lor B \rightarrow C))$
8. $F \rightarrow A$

\(^1\)For the justification of this restriction, see the closing remarks of the paper.
PI1 \( \sim A \rightarrow A \)
PI2 \( A \rightarrow \sim \sim A \)
PI3 \( \sim (A \land B) \rightarrow \sim (A \lor \sim B) \)
PI4 \( \sim (A \lor B) \rightarrow \sim A \land \sim B \)
PI5 \( A \land \sim A \rightarrow F \)
PI6 \( \sim (A \rightarrow B) \rightarrow (A \land \sim B) \)
PI7 \( (A \land \sim B) \rightarrow \sim (A \rightarrow B) \)
PI8 \( \sim \neg A \rightarrow A \)
PI9 \( A \rightarrow \sim \neg A \)
PI20 \( \neg A \rightarrow (A \rightarrow B) \)
PI21 \( A \rightarrow (B \rightarrow (A \land B)) \)
PI22 \( A \lor \sim B \rightarrow \sim (A \land B) \)
PI23 \( A \land \sim B \rightarrow \sim (A \lor B) \)

where \( \neg A \) stand for \( A \rightarrow F \), \( F \) is \( \sim (A \rightarrow (B \rightarrow A)) \) and MP is the only rule. The deduction theorem holds in PI since its proof needs only I.1 and I.2.

A Saturated Deductively Closed Consistent Set (SDCCS) \( \Gamma \) is a set which is:

1. Saturated, i.e. \( (A \lor B) \in \Gamma \) iff \( A \in \Gamma \) or \( B \in \Gamma \) ;
2. Deductively closed, i.e. \( A \in \Gamma \) iff \( \Gamma A \);
3. Consistent, i.e. there is an \( A \) such that \( \Gamma A \).

**Proposition 1.** For any consistent set \( \Delta \), there is a SDCCS \( \Gamma \) such that \( \Delta \subseteq \Gamma \). By extension, if \( \Gamma \) is inconsistent, \( S(\Gamma) \) will be the set of all sentences.

### 3. Semantics

We now turn our attention to the question of probability distributions. Of course the main idea is that a partial probability distribution should be just like a total ordinary probability distribution but undefined for some arguments. But while we want to allow gaps in the distributions, these gaps are not totally arbitrary. Essentially, while we allow gaps, we require that any value that can be computed from values that are given cannot be left undefined. More precisely,
**Definition 1.** A partial probability function \( \Pr \) is a partial function

\[
\Pr : L \times 2^L \to [0, 1]
\]

such that the following constraints are always satisfied:

P.1 If \( A \in \Gamma \), then \( \Pr(A, \Gamma) \) is defined;
P.2 If \( \Pr(A, \Gamma) \) is defined, then \( \Pr(\sim A, \Gamma) \) is defined;
P.3 If \( \Pr(\sim A, \Gamma) \) is defined, then \( \Pr(A, \Gamma) \) is defined;
P.4 If \( \Pr(A \land B, \Gamma) \) is defined, then \( \Pr(B \land A, \Gamma) \) is defined;
P.5 If \( \Pr(A, \Gamma) \) and \( \Pr(B, \Gamma) \) are undefined \( \Pr(A \land B, \Gamma) \) is undefined;
P.6 If \( \Pr(A, \Gamma) = 0 \), then \( \Pr(A \land B, \Gamma) = 0 \);
P.7 If \( \Pr(A, \Gamma) = 1 \), then \( \Pr(A \lor B, \Gamma) = 1 \);
P.8 If \( \Pr(A \land B, \Gamma) \) is defined, and \( \Pr(A, \Gamma) \) is undefined, then \( \Pr(B, \Gamma) = 0 \);
P.9 If \( \Pr(A \lor B, \Gamma) \) is defined, and \( \Pr(A, \Gamma) \) is undefined, then \( \Pr(B, \Gamma) = 1 \).

And when all the appropriate values of \( \Pr \) are defined, the following constraints are satisfied:

NP.1 \( 0 \leq \Pr(A, \Gamma) \leq 1 \);
NP.2 If \( A \in \Gamma \), then \( \Pr(A, \Gamma) = 1 \);
NP.3 \( \Pr(A \lor B, \Gamma) = \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(\sim A, \Gamma) \);
NP.4 \( \Pr(A \land B, \Gamma) = \Pr(A, \Gamma) \times \Pr(B, \Gamma \cup \{A\}) \);
NP.5 \( \Pr(\sim A, \Gamma) = 1 \)- \( \Pr(A, \Gamma) \) provided \( \Gamma \) is \( \Pr \)-normal (i.e., there is at least an \( A \) such that \( \Pr(A, \Gamma) = 0 \));
NP.6 \( \Pr(A \land B, \Gamma) = \Pr(B \land A, \Gamma) \);
NP.7 \( \Pr(C, \Gamma \cup \{A \land B\}) = \Pr(C, \Gamma \cup \{A, B\}) \);
NP.8 \( \Pr(A \rightarrow B, \Gamma) = \Pr(B, \Gamma \cup \{A\}) \).

The following elementary results will be useful.

**Proposition 2.**

E.1 If \( \Pr(A, \Gamma) = 1 \), then \( \Pr(A \land B, \Gamma) = \Pr(B, \Gamma) \) if \( \Pr(B, \Gamma) \) is defined, undefined otherwise.

E.2 If \( \Pr(A, \Gamma) = 1 \), then \( \Pr(B, \Gamma) = \Pr(B, \Gamma \cup \{A\}) \) if \( \Pr(B, \Gamma) \) is defined, undefined otherwise.

E.3 If \( \Pr(C, \Gamma \cup \{A\}) = 1 \) and \( \Pr(C, \Gamma \cup \{B\}) = 1 \) and \( \Pr(A \lor B, \Gamma) = 1 \), then \( \Pr(C, \Gamma) = 1 \).

E.4 For all \( A, B \) and \( \Gamma \), \( \Pr(B, \Gamma \cup \{A\}) = 1 \) iff for all \( \Delta \), if \( \Pr(A, \Gamma \cup \Delta) = 1 \), then \( \Pr(B, \Gamma \cup \Delta) = 1 \).
E.5 If Γ is Pr-normal but Γ ∪ {A} is Pr-abnormal, then Pr(A, Γ) = 0.
E.6 Pr(A ∨ B, Γ) ≥ Pr(A, Γ) when Pr(A ∨ B, Γ) if defined.

We now define the semantic consequence relation, based on partial probability distributions. We use the standard intuition that Γ semantically implies A if and only if no matter what your belief system is, there is nothing we could add to Γ that would make you doubt A.

**Definition 2.** A is a semantical consequence of Γ (written Γ ⊩ A) if and only if for all probability distributions Pr, Pr(A, Γ ∪ ∆) = 1 for all ∆.

4. **Soundness**

We will need the following lemma (the semantical counterpart of DT).

**Lemma 1.** Γ ⊩ A → B iff Γ ∪ {A} ⊩ B

**Proof:** Γ ⊩ A → B iff for any Pr and any ∆, Pr(A → B, Γ ∪ ∆) = 1 iff for any Pr and any ∆, Pr(B, Γ ∪ ∆ ∪ {A}) = 1 iff for any Pr and any ∆, Pr(B, Γ ∪ {A} ∪ ∆) = 1 iff Γ ∪ {A} ⊩ B.

**Proposition 3.** PI is sound according to probabilistic interpretation, i.e. if Γ ⊬ A, then Γ ⊩ A

**Proof:** Let us suppose that Γ ⊬ A. Then there is a proof from Γ

\{A_0, ..., A_n\} ⊩ A

We prove by induction on i that Γ ⊩ A_i. Basis: we have to check that (i) if A_0 ∈ Γ or (ii) A_0 is an axiom, then Pr(A_0, Γ ∪ ∆) for all Pr, Γ and ∆.

(i) follows from NP.2.

(ii) We have to check that every axiom is valid.

(iii) MP transmits the value 1.

We will give three examples, the other cases are left to the readers.

\[ A → (B → A) \]

By definition of ⊩, Γ ⊩ A → (B → A). Using DT twice we have Γ ∪ {A} ∪ {B} ⊩ A. But Γ ∪ {A} ∪ {B} ⊩ A, i.e. for any Pr and ∆,
\( \Pr(A, \Gamma \cup \{ A \} \cup \{ B \} \cup \Delta) = 1 \) which is always the case by NP.2. By applying the lemma at the beginning of this section \( \Gamma \models A \rightarrow (B \rightarrow A) \).

\( \text{PI6} \sim (A \rightarrow B) \rightarrow (A \land \sim B) \)

We have to show that for all \( \text{Pr}, A \) and \( \Gamma \), if \( \Pr(\sim (A \rightarrow B), \Gamma \cup \Delta) = 1 \), then \( \Pr(A \land \sim B, \Gamma \cup \Delta) = 1 \) for all \( \Delta \). Again, we have two cases:

(i) \( \Gamma \cup \Delta \) is \( \text{Pr} \)-abnormal. Then \( \Pr(A \land \sim B, \Gamma \cup \Delta) = 1 \).

(ii) \( \Gamma \cup \Delta \) is \( \text{Pr} \)-normal. Let us suppose that \( \Pr(\sim (A \rightarrow B), \Gamma \cup \Delta) = 1 \). Then, by NP.5, \( \Pr(A \rightarrow B, \Gamma \cup \Delta) = 0 \) and \( \Pr(\sim B, \Gamma \cup \Delta \cup \{ A \}) = 0 \). Now, \( \Pr(A \land \sim B, \Gamma \cup \Delta) = \Pr(A, \Gamma \cup \Delta) \times \Pr(\sim B, \Gamma \cup \Delta \cup \{ A \}) = \Pr(A, \Gamma \cup \Delta) \).

So we have to prove that \( \Pr(A, \Gamma \cup \Delta) = 1 \).

Suppose that \( \Pr(\sim A, \Gamma \cup \Delta) \neq 0 \). By E.5, \( \Gamma \cup \Delta \cup \{ \sim A \} \) is \( \text{Pr} \)-normal and \( \Pr(A \rightarrow B, \Gamma \cup \Delta \cup \{ \sim A \}) = 0 \) and thus \( \Pr(\sim B, \Gamma \cup \Delta \cup \{ A \}) = 0 \) which is impossible because \( \Gamma \cup \Delta \cup \{ \sim A, A \} \) is \( \text{Pr} \)-abnormal. So \( \Pr(\sim A, \Gamma \cup \Delta) = 0 \) and by NP.5, \( \Pr(A, \Gamma \cup \Delta) = 1 \).

\( \text{PI9} \vdash A \rightarrow \sim \neg A \)

By E.4, we have to show that for all \( \text{Pr}, A \) and \( \Gamma \), if \( \Pr(A, \Gamma \cup \Delta) = 1 \), then \( \Pr(\sim \neg A, \Gamma \cup \Delta) = 1 \) for all \( \Delta \). We have two cases:

(i) \( \Gamma \) is \( \text{Pr} \)-abnormal. Then \( \Pr(\sim \neg A, \Gamma \cup \Delta) = 1 \).

(ii) \( \Gamma \cup \Delta \) is \( \text{Pr} \)-normal. Let us suppose that \( \Pr(A, \Gamma \cup \Delta) = 1 \). We know that \( \Pr(F, \Gamma \cup \Delta) = 0 \) by E.1, \( \Pr(A \rightarrow F, \Gamma \cup \Delta) = \Pr(F, \Gamma \cup \Delta \cup \{ A \}) = 0 \) unless \( \Gamma \cup \Delta \cup \{ A \} \) is \( \text{Pr} \)-abnormal. But in that case by E.5, \( \Pr(A, \Gamma) = 0 \) which contradicts the hypothesis. So \( \Pr(\sim A, \Gamma \cup \Delta) = 0 \) and \( \Pr(\sim \neg A, \Gamma \cup \Delta) = 1 \).

\[ \square \]

5. Completeness

In order to show that if \( \Gamma \models A \) then \( \Gamma \vdash A \), we have to show that if \( \Gamma \models A \) and \( \not\vdash A \), we have a model of \( \Gamma \), for which \( A \) is not true, i.e. is false or undefined. Let \( W \) be the set of all SDCCS sets of PI. The following propositions will be useful.

**Proposition 4.** \( W \) is partially ordered by \( \subseteq \) and for any \( \Delta \in W \), if \( A \rightarrow B \in \Delta \), then for any \( \Delta' \) such that \( \Delta \subseteq \Delta' \), if \( A \in \Delta' \) then \( B \in \Delta' \). (We do not show it because it is nowadays a classic.)
PROPOSITION 5. Let $\Gamma$ be a consistent set such that $\forall p : p \rightarrow B$. There is a saturate set $\Gamma'$ such that $\Gamma \subseteq \Gamma'$ and $A \in \Gamma'$ and $B \notin \Gamma'$.

PROOF: Clearly, $\Gamma \cup \{A\} \vdash p$ (the deduction theorem holds in $PI$). So, starting from $\Gamma \cup \{A\}$ we define $\Gamma'$ in the following way. Let $E_1, ..., E_n, ...$ be a canonical enumeration of the formulas of $PI$ in which each formula appears denumerably many times. Let $\Gamma_0, ..., \Gamma_i, ...$ be the following sequence of sets:

$\Gamma_0 = \Gamma \cup \{A\}$

$\vdots$

$\Gamma_{i+1} = \begin{cases} 
\Gamma_i & \text{if } \Gamma_i \vdash p \\
\Gamma_i \cup \{E_i\} & \text{if } \Gamma_i \vdash p \text{ and } E_i \text{ is not } C \lor D \\
\Gamma_i \cup \{E_i, C\} & \text{if } \Gamma_i \vdash p \text{ and } E_i \text{ is } C \lor D \text{ and } \Gamma_i \cup \{E_i, C\} \vdash p \\
\Gamma_i \cup \{E_i, D\} & \text{if } \Gamma_i \vdash p \text{ and } E_i \text{ is } C \lor D \text{ and } \Gamma_i \cup \{E_i, C\} \vdash p \\
\Gamma' = \bigcup_i \Gamma_i & \text{is a saturated set such that } \Gamma \subseteq \Gamma' \text{ and } B \notin \Gamma'.
\end{cases}$

Let $\Gamma$ be any set. We define $U(\Gamma) = \{\Delta : \Delta \text{ is a SDCCS and } \Gamma \subseteq \Delta\}$. We will call $\langle W, \subseteq \rangle$ a canonical model structure. We define a function $Pr_{\langle W, \subseteq \rangle}$ such that for any $A$

$$Pr_{\langle W, \subseteq \rangle}(A, \Gamma) = \begin{cases} 
1 & \text{iff } A \in \Delta \text{ for all } \Delta \in U(\Gamma) \text{ such that } \Gamma \subseteq \Delta \\
0 & \text{iff } \neg A \in \Delta \text{ for all } \Delta \in U(\Gamma) \text{ such that } \Gamma \subseteq \Delta \\
\text{undefined otherwise}
\end{cases}$$

PROPOSITION 6. $Pr_{\langle W, \subseteq \rangle}$ satisfy P.1-8 and NP.1-8

PROOF: We drop the index $\langle W, \subseteq \rangle$.

P.1 If $A \in \Gamma$, then $Pr(A, \Gamma)$ is defined. Trivial.

P.2 If $Pr(A, \Gamma)$ is defined, then $Pr(\neg A, \Gamma)$ is defined. Trivial.

P.3 If $Pr(\neg A, \Gamma)$ is defined, then $Pr(A, \Gamma)$ is defined. Trivial.
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P.4 If \( \Pr(A \land B, \Gamma) \) is defined, then \( \Pr(B \land A, \Gamma) \) is defined. Two cases:

(a) \( \Pr(A \land B, \Gamma) = 1 \). \( A \land B \in \Delta \) and by I3 and I4, \( A \in \Delta \) and \( B \in \Delta \) and by PI21 and finally, \( B \land A \in \Delta \).

(b) \( \Pr(A \land B, \Gamma) = 0 \). We have \( \Pr(\neg (A \land B), \Gamma) = 1 \) and \( (A \land B) \in \Delta \in U(\Gamma) \). By PI13, \( \sim A \lor \sim B \in \Delta \), and thus \( \sim A \in \Delta \) or \( \sim B \in \Delta \) by I5 and I6, in both cases \( B \lor \sim A \in \Delta \) and finally by PI22, \( (B \land A) \in \Delta \) implying that \( \Pr(B \land A, \Gamma) = 0 \).

P.6 If \( \Pr(A, \Gamma) = 0 \), then \( \Pr(A \land B, \Gamma) = 0 \). If \( \Pr(A, \Gamma) = 0 \), then \( \sim A \in \Delta \in U(\Gamma) \). Thus \( \sim A \lor \sim B \in \Delta \) and \( A \land B \in \Delta \).

P.7 If \( \Pr(A, \Gamma) = 1 \), then \( \Pr(A \lor B, \Gamma) = 1 \). If \( \Pr(A, \Gamma) = 1 \), \( \Pr(\sim A, \Gamma) = 0 \), \( \Pr(\sim A \land \sim B, \Gamma) = 0 \) and \( \Pr(A \lor B, \Gamma) = 1 \).

NP.1 \( 0 \leq \Pr(A, \Gamma) \leq 1 \). Trivial.

NP.2 If \( A \in \Gamma \), then \( \Pr(A, \Gamma) = 1 \) (and is defined for all \( \Pr \) and \( \Gamma \)). Trivial.

NP.3 \( \Pr(A \lor B, \Gamma) = \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A \land B, \Gamma) \). Let us suppose that \( \Pr(A \lor B, \Gamma) = 1 \). \( A \lor B \in S(\Gamma) \) and \( A \in \Delta \) or \( B \in \Delta \). If \( A \in \Delta \), we have two possibilities:

(i) \( B \in \Delta \) or
(ii) \( \sim B \in \Delta \).

(i) In that case, by PI21, \( A \land B \in \Delta \).
(ii) In that case, \( \sim A \lor \sim B \in \Delta \) and \( (A \land B) \in \Delta \) and thus \( \Pr(A \land B, \Gamma) = 0 \).

In both cases, NP.3 holds.

NP.4 \( \Pr(A \land B, \Gamma) = \Pr(A, \Gamma) \times \Pr(B, \Gamma \cup \{A\}) \).

If \( \Pr(A \land B, \Gamma) = 1 \) then \( A \land B \in \Delta \in U(\Gamma) \) and \( A \in \Delta \), \( B \in \Delta \). Thus \( \Pr(A, \Gamma) = \Pr(B, \Gamma \cup \{A\}) = 1 \).

If \( \Pr(A \land B, \Gamma) = 0 \) then \( (A \land B) \in \Delta \in U(\Gamma) \) and \( A \in \Delta \) or \( \sim B \in \Delta \). In the first case \( \Pr(A, \Gamma) = 0 \). In the second case, because \( \Pr(A, \Gamma) \) is defined, \( \Pr(A, \Gamma) = 1 \) and thus \( \Pr(B, \Gamma \cup \{A\}) = \Pr(B, \Gamma) = 0 \) and \( \sim B \in \Delta \).

NP.5 \( \Pr(\sim A, \Gamma) = 1- \Pr(A, \Gamma) \) provided \( \Gamma \) is \( \Pr \)-normal (i.e., there is at least an \( A \) such that \( \Pr(A, \Gamma) = 0 \)). Let us suppose that \( \Pr(\sim A, \Gamma) = 1 \). This means that \( \sim A \in \Delta \in S(\Gamma) \) and thus \( \Pr(A, \Gamma) = 0 \) and \( 1 = 1 - 0 \). Let us suppose that \( \Pr(\sim A, \Gamma) = 0 \). This means that \( \sim A \in \Delta \in U(\Gamma) \).
and thus $A \in \Delta$ and $\Pr(A, \Gamma) = 1$. But $0 = 1 - 1$.

NP.6 $\Pr(A \land B, \Gamma) = \Pr(B \land A, \Gamma)$. Trivial.

NP.7 $\Pr(C, \Gamma \cup \{A \land C\}) = \Pr(C, \Gamma \cup \{A, C\})$. Trivial because $U(\Gamma \cup \{A \land C\}) = U(\Gamma \cup \{A, C\})$.

NP.8 $\Pr(A \rightarrow B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$. We show that $\Pr(A \rightarrow B, \Gamma) = 1$ iff $\Pr(B, \Gamma \cup \{A\}) = 1$ and $\Pr(A, \Gamma) = 1$. But $0 = 1 - 1$.

Now, suppose that $\Pr(B, \Gamma \cup \{A\}) = 1$. This means that $B \in \Delta$ for any $\Delta$ such that $\Delta \in U(\Gamma \cup \{A\})$. Thus $\Delta \subseteq \Delta'$, $A \in \Delta'$ and $B \notin \Delta'$, we have for any $\Delta \in U(\Gamma \cup \{A\})$, $B \in \Delta$. So, $\Pr(B, \Gamma \cup \{A\}) = 1$.

Let us suppose that $\Pr(A \rightarrow B, \Gamma) = 0$. This means that $\neg (A \rightarrow B) \in \Delta$ for all $\Delta$ such that $\Delta \in U(\Gamma \cup \{A\})$. Thus $A \in \Delta$ and $B \notin \Delta$. Thus $\Pr(B, \Gamma \cup \{A\}) = \Pr(B, \Gamma) = 0$.

Now, suppose that $\Pr(B, \Gamma \cup \{A\}) = 0$. For any $\Delta \in U(\Gamma \cup \{A\})$, $\neg B \in \Delta$. But $U(\Gamma \cup \{A\}) \subseteq U(\Gamma)$ and any $\Delta \in U(\Gamma)$ such that $A \in \Delta$ is such that $\Delta \in U(\Gamma \cup \{A\})$. So, $A, \neg B \in \Delta$ and thus $\neg (A \rightarrow B) \in \Delta$ and $\Pr(A \rightarrow B, \Gamma) = 0$.

Proposition 7. If $\Gamma \nvdash A$ then $\Gamma \vdash A$

Proof: If $\Gamma \nvdash A$ there is a SCDDS $\Delta$ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$. So $\Pr(A, \Gamma) \neq 1$ and $\Gamma \nvdash A$.

6. Closing remarks

Intuitionistic logic was introduced by Brouwer as the logic for mathematical reasoning. $A$ is true if there is a proof of $A$, otherwise it is not true. Nelson introduced a notion of falsity: among the non-truths there are falsity and propositions that are not yet true or false.
The probabilistic interpretation is a generalization to knowledge in general, knowledge which is not restricted to mathematical knowledge. \( \Pr(A, \Gamma) \) can be interpreted as the subjective chance that \( A \) is the case given \( \Gamma \) and \( 1 - \Pr(A, \Gamma) \) the chance that \( A \) is not the case given \( \Gamma \).

But sometimes, an agent can have no idea of the likelihood of \( A \) given the evidence \( \Gamma \). In that case, \( \Pr(A, \Gamma) \) is undefined and so is \( \Pr(\sim A, \Gamma) \) is undefined.

The probabilistic approach is not very intuitive for the interpretation of mathematical knowledge. One reason is that the most natural interpretation for “having a set \( \Gamma \) of evidence for \( A \)” is having a proof of \( A \) from the set of premises \( \Gamma \) and this is what show the canonical model. However, if one thinks that natural logic is intuitionistic logic, then the probabilistic approach is natural. Most of our beliefs are not absolutely certain. Worst, some propositions are so uncertain that they escape the law of excluded middle. “It will rain on Mars or it will not rain on Mars tomorrow” is an example; at least for me because I do not know if the concept of “raining on Mars” makes sense. In other words, for what I know about Mars, I cannot imagine what is an evidence for raining on Mars or not.

My result gives an answer to a question that has haunted philosophers for many years. Is there a logical connective * such that \( \Pr(A * B, \Gamma) = \Pr(B, \Gamma \cup \{A\}) ? \) We know that there is no classical truth functional connective having this property. What we have put forward is that when the probability of \( A \rightarrow B \) is defined, it is \( \Pr(B, \Gamma \cup \{A\}) \). My system also has the following property. Let \( A \) be a sentence that does not contain any \( \rightarrow \). 

\[
\Pr(A \vee \sim A, \Gamma) = \Pr(A, \Gamma) + \Pr(\sim A, \Gamma) - \Pr(A \land \sim A, \Gamma) \text{ which is 1 when defined. This is a very nice property.}
\]

A final word regarding the restriction that \( A \) should not contain any \( \rightarrow \) in order to have \( \Pr(A \vee \sim A, \Gamma) = 1 \) when \( \Pr(A, \Gamma) \) is defined. Just consider the following counterexample. \( \Pr((B \rightarrow C) \vee \sim (B \rightarrow C), \Gamma) = \Pr((B \rightarrow C), \Gamma) + \Pr(\sim (B \rightarrow C), \Gamma) - \Pr((B \rightarrow C) \land \sim (B \rightarrow C), \Gamma) = \Pr(C, \Gamma \cup \{B\}) + \Pr(B \land \sim C, \Gamma) \) when defined. One can easily verify that it is \( \leq 1 \) in general. It is 1 just in the limiting case where \( \Pr(C, \Gamma \cup \{B\}) = 1 \).

These results give more plausibility to both thesis that intuitionistic logic is the general logic of discovery and that probabilistic interpretations are the semantic foundation of logic.
References


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