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SEMANTICAL INVESTIGATIONS ON SOME WEAK MODAL LOGICS. Part II*

Abstract

In both parts of this paper[1] we examine weak logics similar to $\mathbf{S0.5}[\Box\Phi]$, where $\Phi \subseteq \mathbf{S0.5}$. We also examine their versions (one of which is $\mathbf{S0.5_{rte}}[\Box\Phi]$) that are closed under replacement of tautological equivalents (rte). We have that: $\mathbf{S0.5_{rte}}[\Box(\mathtt{K}), \Box(\mathtt{T})] \subsetneq \mathbf{S0.9}$, $\mathbf{S0.5_{rte}}[\Box(\mathtt{X}), \Box(\mathtt{T})] \subsetneq \mathbf{S1}$, and in general, if $\Phi \subseteq \mathbf{E1}$, then $\mathbf{S0.5_{rte}}[\Box\Phi] \subsetneq \mathbf{S2}$.

In the present part we give simplified semantics for these logics, formulated by means of some Kripke-style models. We prove that the logics in question are determined by some classes of these models.

Key words: Very weak modal logics, simplified Kripke-style semantics.

3. Simplified Kripke-style semantics for weak t-normal and t-regular systems

3.1. Models for the logics $\mathbf{S0.5^{\circ}}$, $\mathbf{S0.5^{\circ}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}}[\mathtt{T_q}]$ and $\mathbf{S0.5}$

For very weak t-normal modal systems (e.g. for the logics $\mathbf{S0.5^{\circ}}$, $\mathbf{S0.5^{\circ}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}}[\mathtt{T_q}]$ and $\mathbf{S0.5}$) in [3] are used the following semantics, which consists of "t-normal models". A *model for very weak t-normal systems* (or *t-normal model*) is any triple $\langle w, A, V\rangle$ in which

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[1]For its first part see [4].

1. w is a «distinguished» (normal) world,
2. A is a set of *worlds* which are alternatives to the world w,
3. V is a valuation from For $\times$ ($\{w\} \cup A$) to $\{0,1\}$ such that:
   (i) for any world $x \in A \cup \{w\}$, the function $V(\cdot, x)$ belongs to $\mathsf{Val}^{\mathsf{cl}}$;
   (ii) for the world w and any $\varphi \in$ For
   ($V_w^{\Box}$) $\quad V(\Box\varphi, w) = 1$ iff $\forall_{x \in A}\, V(\varphi, x) = 1$.

   Besides for any world from $A \setminus \{w\}$ and any $\varphi \in$ For, the formula $\ulcorner\Box\varphi\urcorner$ may have an arbitrary value.

A formula φ is *true* in a t-normal model $\langle w, A, V\rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t-normal valid* iff it is true in all t-normal models. Of course, the set of all formulae which are true in a t-model (resp. t-normal valid) is closed under (MP).

Notice that all formulae from the sets **PL**, $\Box$**PL**, $\mathrm{M_{PL}}$ $\mathrm{R_{PL}}$ and $\mathrm{E_{PL}}$ are t-normal valid. Moreover, for any t-normal model $\langle w, A, V\rangle$, for any $\tau \in \mathbf{PL}$ and any $x \in \{w\} \cup A$ we have that $V(\tau, x) = 1$. Besides we have the following obvious fact.

FACT 3.1. *Let w be any object and A be any set. Then*:

1. $w \in A$ *iff for any* $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ *such that* $\langle w, A, V\rangle$ *is a t-model we have that* $V((\mathtt{T}), w) = 1$.
2. *If* $A \neq \emptyset$*, then for any* $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ *such that* $\langle w, A, V\rangle$ *is a t-model we have that* $V((\mathtt{D}), w) = 1$.
3. *If* $A = \emptyset$*, then for any* $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ *such that* $\langle w, A, V\rangle$ *is a t-model we have that* $V((\mathtt{D}), w) = 0$.
4. *Ether* $w \in A$ *or* $A = \emptyset$ *iff for any* $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ *such that* $\langle w, A, V\rangle$ *is a t-model we have that* $V((\mathtt{T_q}), w) = 1$.

PROOF: 1. "$\Rightarrow$" Obvious. "$\Leftarrow$" If $w \notin A$, let v_w be any assignment such that $v_w(p) = 0$ and for any $x \in A$ let v_x be any assignment such that $v_x(p) = 1$. Let $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0,1\}$ be the unique extension of v_w and v_x, for $x \in A$, as in Lemma 3.2(2). Then $\langle w, A, V\rangle$ is a t-normal model such that $V(\Box p, w) = 1$. So $V((\mathtt{T}), w) = 0$.

2 and 3. Obvious.

4. "$\Rightarrow$" Obvious (see 1 and 3). "$\Leftarrow$" If $w \notin A \neq \emptyset$, then as in 1, wskazujemy t-model such that $V((\mathtt{T}), w) = 0$. Moreover, $V(\Diamond(q \supset q), w) = 1$, since $A \neq \emptyset$. ⊣

The lemma below shows that the notion of *a t-normal model* can be defined in a different, but equivalent, way.

LEMMA 3.2. 1. *Let* $\langle w, A, v_w, \{V_x\}_{x\in A\setminus\{w\}}\rangle$ *be a structure in which* w *and* A *are such as in t-normal models,* $v_w\colon \mathrm{At} \to \{0,1\}$*, and for any* x *in* $A\setminus\{w\}$*,* $V_x \in \mathsf{Val}^{\mathsf{cl}}$*. Then there is the unique* $V\colon \mathrm{For}\times(\{w\}\cup A)\to\{0,1\}$ *such that*:

- $\forall_{\alpha\in\mathrm{At}}\colon V(\alpha,w) = v_w(\alpha)$ *and* $\forall_{\varphi\in\mathrm{For}}\forall_{x\in A\setminus\{w\}}\colon V(\varphi,x) = V_x(\varphi)$,
- V *satisfies conditions* (i) *and* (ii) *from definition of t-normal models.*

Thus, $\langle w, A, V\rangle$ *is a t-normal model. Moreover, if* $w \in A$*, then this model is self-associate.*

2. *Let* $\langle w, A, v_w, \{v_x\}_{x\in A\setminus\{w\}}\rangle$ *be a structure in which* w *and* A *are such as in t-normal models,* $v_w\colon \mathrm{At}\to\{0,1\}$*, and for any* $x \in A\setminus\{w\}$*,* $v_x\colon \mathrm{PAt}\to\{0,1\}$*. Then there is the unique* $V\colon \mathrm{For}\times(\{w\}\cup A)\to\{0,1\}$ *such that*:

- $\forall_{\alpha\in\mathrm{At}}\colon V(\alpha,w) = v_w(\alpha)$ *and* $\forall_{\varphi\in\mathrm{PAt}}\forall_{x\in A\setminus\{w\}}\colon V(\varphi,x) = v_x(\varphi)$,
- V *satisfies conditions* (i) *and* (ii) *from definition of t-normal models.*

Thus, $\langle w, A, V\rangle$ *is a t-normal model. Moreover, if* $w \in A$*, then this model is self-associate.*

PROOF: 1. Obvious.

2. By Lemma 1.1(1), from the first part [4], for every $x \in A\setminus\{w\}$ there is the unique extension $V_x\colon \mathrm{For}\to\{0,1\}$ of v_x by classical truth conditions for truth-value operators (i.e. e.g. $V_x \in \mathsf{Val}^{\mathsf{cl}}$ and $\forall_{\chi\in\mathrm{For}}\colon V_x(\Box\chi) = v_x(\Box\chi)$). The rest as by 1. ⊣

REMARK 3.1. *1. We can see then that structures* $\langle w, A, v_w, \{v_x\}_{x\in A\setminus\{w\}}\rangle$ *satisfying the conditions from the above lemma can be taken as t-normal models. Again we say that in such model a formula* φ *is true iff* $V(\varphi,w) = 1$.

2. However, the latter approach is not general enough while considering week t-normal logics with a set $\Box\Phi$ *of additional axioms, where* $\Phi \subseteq$ **S0.5** *(see the condition (iii) in Section 3.3). In these not always can we use Lemma 3.2 while constructing t-normal models.* ⊣

FACT 3.3. *Let* $\ulcorner\varphi \equiv \psi\urcorner \in$ **PL**. *Then for any classical formula* χ *(i.e. without the modal operator) for any world* x *from* $A \cup \{w\}$ *in any t-normal model* $\langle w, A, V\rangle$ *we have that* $V(\chi, x) = V(\chi[^{\varphi}/_{\psi}], x)$.

However, when we analyze t-normal rte-logics we need to have such a notion of model, for which an analogous fact will hold for any formula χ from For.

We say that a t-normal model $\langle w, A, V\rangle$ is *self-associate* (resp. *empty*, *non-empty*) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$). Let $\mathbf{nM}$ be the class of all t-normal models. Moreover, let $\mathbf{nM^{sa}}$ (resp. $\mathbf{nM^{\varnothing}}$, $\mathbf{nM^{+}}$) be the class of t-normal models which are self-associate (resp. empty, non-empty). Of course, $\mathbf{nM^{sa}} \subsetneq \mathbf{nM^{+}}$ and $\mathbf{nM^{\varnothing}} \cap \mathbf{nM^{+}} = \emptyset$.

Let $\boldsymbol{C}$ be any class of considered models. We say that a formula φ is $\boldsymbol{C}$*-valid* (written $\models_{\boldsymbol{C}} \varphi$) iff φ is true in all models from $\boldsymbol{C}$.

Let $\boldsymbol{\Sigma}$ be an arbitrary modal system. We say that $\boldsymbol{\Sigma}$ is *sound* with respect to $\boldsymbol{C}$ iff $\boldsymbol{\Sigma} \subseteq \{\varphi \in \mathrm{For} : \ \models_{\boldsymbol{C}} \varphi\}$. We say that $\boldsymbol{\Sigma}$ is *complete* with respect to $\boldsymbol{C}$ iff $\boldsymbol{\Sigma} \supseteq \{\varphi \in \mathrm{For} : \ \models_{\boldsymbol{C}} \varphi\}$. We say that $\boldsymbol{\Sigma}$ is *determined* by $\boldsymbol{C}$ iff $\boldsymbol{\Sigma} = \{\varphi \in \mathrm{For} : \ \models_{\boldsymbol{C}} \varphi\}$, i.e., $\boldsymbol{\Sigma}$ is sound and complete with respect to $\boldsymbol{C}$.

In [3] we proved the following determination theorems for the logics $\mathbf{S0.5^{\circ}}$, $\mathbf{S0.5^{\circ}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}}[\mathtt{T_q}]$ and $\mathbf{S0.5}$:

THEOREM 3.4 ([3]). 1. $\mathbf{S0.5^{\circ}}$ *is determined by the class* $\mathbf{nM}$.
2. $\mathbf{S0.5^{\circ}}[\mathtt{D}]$ *is determined by the class* $\mathbf{nM^{+}}$.
3. $\mathbf{S0.5^{\circ}}[\mathtt{T_q}]$ *is determined by the class* $\mathbf{nM^{sa}} \cup \mathbf{nM^{\varnothing}}$.
4. $\mathbf{S0.5}$ *is determined by the class* $\mathbf{nM^{sa}}$.[2]

From the above theorem and Fact 3.1(1) we obtain:

COROLLARY 3.5. $(\mathtt{T}) \notin \mathbf{S0.5^{\circ}}$. *Hence* $\mathbf{S0.5^{\circ}} \subsetneq \mathbf{S0.5}$.

From Theorem 3.4 (or from Fact 3.1 likewise), we obtain:

FACT 3.6 ([3]). *The formulae* (†) *from the first part do not belong to* $\mathbf{S0.5}$. *Consequently,* $\mathbf{PL}_{\mathrm{rte}} \nsubseteq \mathbf{S0.5}$ *and* $\mathbf{S0.5}$ *is not an rte-system.*

[2]See also [2, Exercise 11.8].

PROOF: For ($\dagger_{\rm a}$): For $w \neq a$ and $A := \{w, a\}$, let v_w and v_a be assignments such that $v_w(p) = v_a(p) = v_a(\Box p) = 1$, $v_a(\Box\neg\neg p) = 0$. Let, as in Lemma 3.2(2), $V\colon \mathrm{For} \times A \to \{0,1\}$ be the unique extension of v_w and v_a. Thus, $\langle w, A, V\rangle$ is a self-associate t-normal model such that $V(\Box\Box p, w) = 1$ and $V(\Box\Box\neg\neg p, w) = 0$. So $V((\dagger_{\rm a}), w) = 0$. Similarly for ($\dagger_{\rm b}$): let $v_a(\Box p) = 0$ and $v_a(\Box\neg\neg p) = 1$. ⊣

FACT 3.7. $\Diamond\mathrm{For} \cap \mathbf{S0.5^\circ} = \emptyset = \Diamond\mathrm{For} \cap \mathbf{S0.5^\circ}[\mathtt{T_q}]$.

PROOF: For any empty t-normal model $\langle w, \emptyset, V\rangle$, we have $V(\Diamond\varphi, w) = 0$, for any $\varphi \in \mathrm{For}$. Hence $\ulcorner\Diamond\varphi\urcorner \notin \mathbf{S0.5^\circ}[\mathtt{T_q}]$, by Theorem 3.4(3). ⊣

FACT 3.8. *For any* $\varphi \in \mathrm{For}$:

$$\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5^\circ} \textit{ iff } \varphi \in \mathbf{PL} \textit{ iff } \ulcorner\Box\varphi\urcorner \in \mathbf{S0.5}.$$

So $\mathbf{S0.5^\circ}$*,* $\mathbf{S0.5^\circ}[\mathtt{D}]$*,* $\mathbf{S0.5^\circ}[\mathtt{T_q}]$ *and* $\mathbf{S0.5}$ *are closed under* (RN_*) *and* (SMP).

PROOF: Firstly, $\Box\mathbf{PL} \subseteq \mathbf{S0.5^\circ} \subseteq \mathbf{S0.5}$.

Secondly, let $\varphi \notin \mathbf{PL}$, $w \neq a$, $A := \{w, a\}$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have that $V_a(\varphi) = 0$. By Lemma 3.2(1), for V_a and any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ there is a self-associate t-normal model $\langle w, \{w, a\}, V\rangle$, for which $V(\Box\varphi, w) = 0$. Hence $\ulcorner\Box\varphi\urcorner \notin \mathbf{S0.5_{rte}}$, by Theorem 3.4(4). ⊣

FACT 3.9. *For any* $n > 0$ *and* $\varphi_1, \ldots, \varphi_n, \psi \in \mathrm{For}$:

$$\ulcorner(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \supset \Box\psi\urcorner \in \mathbf{S0.5^\circ} \textit{ iff } \ulcorner(\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi\urcorner \in \mathbf{PL}$$
$$\textit{iff } \ulcorner(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \supset \Box\psi\urcorner \in \mathbf{S0.5^\circ}[\mathtt{D}]\,.$$

PROOF: Firstly, $\mathrm{R}_{\mathrm{PL}} \subseteq \mathbf{S0.5^\circ} \subseteq \mathbf{S0.5^\circ}[\mathtt{D}]$.

Let $\ulcorner(\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi\urcorner \notin \mathbf{PL}$, $w \neq a$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have that $V_a(\varphi_1) = \cdots = V_a(\varphi_n) = 1$ and $V_a(\psi) = 0$. By Lemma 3.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a there is a non-empty t-normal model $\langle w, \{a\}, V\rangle$, for which $V(\Box\varphi_1, w) = V(\Box\varphi_n, w) = 1$ and $V(\Box\psi, w) = 0$. Hence $\ulcorner(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \supset \Box\psi\urcorner \notin \mathbf{S0.5^\circ}[\mathtt{D}]$, by Theorem 3.4(2).[3] ⊣

[3]Notice that, by Theorem 3.4(3), '$\Box\Box p \supset \Box p$' $\in \mathbf{S0.5^\circ}[\mathtt{T_q}] \subseteq \mathbf{S0.5}$.

3.2. Models for C1, D1, C1[$\mathsf{T_q}$] and E1

In the case of very weak t-regular systems we broaden the class of t-normal models by the class of *queer* models of the form $\langle w, V\rangle$ with only one (queer) world w and a valuation $V\colon \mathrm{For} \times \{w\} \to \{0,1\}$ which satisfies classical conditions for truth-value operators, i.e. $V(\cdot, w) \in \mathsf{Val}^{\mathsf{cl}}$, and

(ii′) for any $\varphi \in \mathrm{For}$, $V(\Box\varphi, w) = 0$.

LEMMA 3.10. *Let $\langle w, v_w\rangle$ be a structure, where v_w is an assignment from* At *to* $\{0,1\}$. *Then there is the unique function* $V\colon \mathrm{For}\times\{w\} \to \{0,1\}$ *such that*:

- $\forall_{\alpha\in\mathrm{At}}$: $V(\alpha, w) = v_w(\alpha)$,
- V *satisfies conditions* (i) *and* (ii′) *from definition of queer models.*

Thus, $\langle w, V\rangle$ *is queer model.*

Let qM be the class of all queer models and we put $\mathsf{rM} := \mathsf{nM} \cup \mathsf{qM}$, i.e. rM is the class of models for very weak t-regular systems.

A formula φ is *true* in a queer model $\langle w, V\rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t-regular valid* iff it is true in all models from rM. Notice that all formulae from the sets **PL**, $\mathrm{M}_{\mathbf{PL}}$ $\mathrm{R}_{\mathbf{PL}}$ and $\mathrm{E}_{\mathbf{PL}}$ are t-regular valid.

In [3] we proved the following determination theorems for the logics **C1**, **D1**, **C1**[$\mathsf{T_q}$] and **E1**:

THEOREM 3.11 ([3]). 1. **C1** *is determined by the class* rM.
2. **D1** *is determined by the class* $\mathsf{nM}^{+} \cup \mathsf{qM}$.
3. **C1**[$\mathsf{T_q}$] *is determined by the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{nM}^{\varnothing} \cup \mathsf{qM}$.
4. **E1** *is determined by the class* $\mathsf{nM}^{\mathsf{sa}} \cup \mathsf{qM}$.

We will now give a semantical proof of facts (2.1)–(2.3), about which we wrote in the first part [4]:

FACT 3.12. $\mathbf{C1} = \mathbf{C2} \cap \mathbf{S0.5}^{\circ}$, $\mathbf{C1} \subsetneq \mathbf{C2} \cap \mathbf{S0.5} \nsubseteq \mathbf{S0.5}^{\circ}$ *and* $\mathbf{E1} = \mathbf{E2} \cap \mathbf{S0.5}$.

PROOF: For "$\subseteq$": See the first part [4].

For "$\mathbf{C2} \cap \mathbf{S0.5}^{\circ} \subseteq \mathbf{C1}$" (resp. "$\mathbf{E2} \cap \mathbf{S0.5} \subseteq \mathbf{E1}$"): Let $\varphi \in \mathbf{C2} \cap \mathbf{S0.5}^{\circ}$ (resp. $\varphi \in \mathbf{E2} \cap \mathbf{S0.5}$) and $\mathscr{M} \in \mathsf{rM}$ (resp. $\mathscr{M} \in \mathsf{nM}^{\mathsf{sa}} \cup \mathsf{qM}$). If $\mathscr{M} \in \mathsf{nM}$ (resp. $\mathscr{M} \in \mathsf{nM}^{\mathsf{sa}}$), then since $\varphi \in \mathbf{S0.5}^{\circ}$ (resp. $\varphi \in \mathbf{S0.5}$), φ is true in

$\mathscr{M}$, by Theorem 3.4. If $\mathscr{M} = \langle w, V\rangle \in \mathsf{qM}$, then we can identify it with the following relational model $\langle\{w_0\}, \emptyset, \emptyset, V_0\rangle$ used for (strictly) regular logics.[4] Since $\varphi \in \mathbf{C2}$, so from soundness of **C2** with Kripke relational model semantics we obtain that $V(\varphi, w) = 1$. Hence φ is also true in $\mathscr{M}$. Them, by Theorem 3.11, we obtain that $\varphi \in \mathbf{C1}$ (resp. $\varphi \in \mathbf{E1}$).

For "$\mathbf{C1} \subsetneq \mathbf{C2} \cap \mathbf{S0.5} \nsubseteq \mathbf{S0.5}^\circ$": Since '$\Box r \supset \Box(\mathtt{K})$' $\in \mathbf{C2}$', so '$(\mathtt{T}) \vee (\Box r \supset \Box(\mathtt{K}))$' belongs to $\mathbf{C2} \cap \mathbf{S0.5}$. But the latest formula does belong to $\mathbf{S0.5}^\circ$ (and so it does not belong to **C1**). Indeed, for $w \neq a$, let v_w and v_a be assignments such that $v_w(p) = 0$, $v_a(p) = v_a(r) = 1 = v_a(\Box p) = v_a(\Box(p \supset q)$ and $v_a(\Box q) = 0$. $V\colon \mathrm{For} \times \{w, a\} \to \{0, 1\}$ be the unique extension of v_w and v_a, as in Lemma 3.2(2). Then $\langle w, \{a\}, V\rangle$ is a t-normal model such that $V(\Box p, w) = 1$. So we have that $V((\mathtt{T}), w) = 0 = V(\Box(\mathtt{K}), w)$ and $V(\Box r, w) = 1$. ⊣

3.3. Models for weak t-normal systems with additional axioms of the form $\ulcorner\Box\varphi\urcorner$

While considering very week t-normal systems with an additional axiom of the form $\ulcorner\Box\varphi\urcorner$, where $\varphi \in \mathbf{S0.5}$, we will take into account such t-normal models $\mathscr{M} = \langle w, A, V\rangle$ which satisfy the following additional condition:

(iii$_\varphi$) for all $x \in A \setminus \{w\}$ and uniform substitution s, $V(s(\varphi), x) = 1$.

A model of this kind will be called *a t-normal model for* $\Box\varphi$.

Let $\Phi \subseteq \mathbf{S0.5}$. If for every $\varphi \in \Phi$, $\mathscr{M}$ is a t-normal model for $\Box\varphi$, then we say that $\mathscr{M}$ is a *t-normal model for* $\Box\Phi$. Thus such models satisfy the following additional condition:

(iii) for any $x \in A \setminus \{w\}$ and any ψ which is an instance of some formula from Φ, $V(\psi, x) = 1$.

Remark 3.2. *For any* $\Phi \subseteq \mathbf{S0.5}$ *we put* $\Phi^\star := \{\psi : \psi$ *is an instance of some formula from* $\Phi\}$. *Of course,* $\Phi^\star \subseteq \mathbf{S0.5}$. *The logic* **S0.5** *is consistent, so* **S0.5** *is* **PL**-*consistent; i.e.* $\mathbf{S0.5} \not\models_{\mathbf{PL}} p \wedge \neg p$. *Therefore,* $\Phi^\star$ *is also* **PL**-*consistent, i.e.* $\Phi^\star \not\models_{\mathbf{PL}} p \wedge \neg p$. *Hence there is a valuation* $V \in \mathsf{Val}^{\mathsf{cl}}$ *such that* $V(\Phi^\star) = \{1\}$. ⊣

[4] For (strictly) regular logics we use Kripke models of the form $\langle W, N, R, V\rangle$, where W is a non-empty set of possible worlds, N is a subset of W (is a set of *normal worlds*), $R \subseteq W \times W$ and $V\colon \mathrm{For} \times W \to \{0, 1\}$ such that for any $x \in W$: $V(\cdot, x) \in \mathsf{Val}^{\mathsf{cl}}$ and for any $\varphi \in \mathrm{For}$, $V(\Box\varphi, x) = 1$ iff both $x \in N$ and $\forall_{y \in R(x)}\ V(\varphi, x) = 1$. If $N = \emptyset$, then $V(\Box\varphi, w) = 0$, for any $\varphi \in \mathrm{For}$.

Let $\mathsf{nM}[\Box\Phi]$ be the class of all t-normal models for $\Box\Phi$. Moreover, let $\mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$ (resp. $\mathsf{nM}^{\emptyset}[\Box\Phi]$, $\mathsf{nM}^{+}[\Box\Phi]$) be the class of t-normal models which are self-associate (resp. empty, non-empty) for $\Box\Phi$. Of course, $\mathsf{nM}^{\mathsf{sa}}[\Box\Phi] \subsetneq \mathsf{nM}^{+}[\Box\Phi]$ and $\mathsf{nM}^{\emptyset}[\Box\Phi] \cap \mathsf{nM}^{+}[\Box\Phi] = \emptyset$.

FACT 3.13. *For any* $\Phi \subseteq \mathbf{S0.5}$:
1. $\mathbf{S0.5}^{\circ}[\Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}[\Box\Phi]$.
2. $\mathbf{S0.5}^{\circ}[\mathtt{D}, \Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}^{+}[\Box\Phi]$.
3. $\mathbf{S0.5}^{\circ}[\mathtt{T_q}, \Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}^{\emptyset}[\Box\Phi] \cup \mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.
4. $\mathbf{S0.5}[\Box\Phi]$ *is sound with respect to the class of* $\mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.

PROOF: 1. Let $\mathscr{M} = \langle w, A, V\rangle$ be any t-normal model for $\Box\Phi$. All members of the sets $\mathbf{PL}$, $\Box\mathbf{PL}$ and $\mathrm{sub}(\mathtt{K})$ are true in $\mathscr{M}$. Moreover, suppose that $\varphi \in \mathrm{sub}(\Phi)$. Then for any x from $A \setminus \{w\}$ we have that $V(\varphi, x) = 1$, by the condition (iii). Now we consider two cases.

(a) $w \notin A$: Then $V(\Box\varphi, w) = 1$, by the conditions (ii) and (iii).

(b) $w \in A$: Since $\varphi \in \mathbf{S0.5}$, so $V(\varphi, w) = 1$, by Theorem 3.4(4). Thus, $V(\Box\varphi, w) = 1$, by the conditions (ii) and (iii).

2. Let $\mathscr{M} = \langle w, A, V\rangle$ be any non-empty t-normal model for $\Box\Phi$. All instances of $(\mathtt{D})$ are true in $\mathscr{M}$. The rest as in 1.

3. Let $\mathscr{M} = \langle w, A, V\rangle$ be any self-associate t-normal model for $\Box\Phi$. All instances of $(\mathtt{T_q})$ are true in $\mathscr{M}$. The rest as in the case (b) of 1.

Let $\mathscr{M} = \langle w, \emptyset, V\rangle$ be any empty t-normal model for $\Box\Phi$. All instances of $(\mathtt{T_q})$ and all formulae of the form $\ulcorner\Box\psi\urcorner$ are true in $\mathscr{M}$.

4. Let $\mathscr{M} = \langle w, A, V\rangle$ be any self-associate t-normal model for $\Box\Phi$. All instances of $(\mathtt{T})$ are true in $\mathscr{M}$. The rest as in the case (b) of 1. ⊣

4. Simplified Kripke-style semantics for weak t-normal rte-systems

4.1. Models for very weak t-normal rte-systems

For very weak t-normal systems which are closed under (rte) in [3] we use *t-normal rte-models* which are t-normal models $\langle w, A, V\rangle$ satisfing the following condition:

(iv) $\forall_{\varphi,\psi,\chi\in\mathrm{For}}$: if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$, then $V(\chi, w) = V(\chi[{}^{\varphi}/_{\psi}], w)$.

Theorem 4.1 gives other equivalent ways of expressing the condition (iv). The most interesting of them is the one that follows:

(iv′) $\forall_{\varphi,\psi,\chi\in\mathrm{For}}$: if $\ulcorner\varphi\equiv\psi\urcorner\in\mathbf{PL}$, then
$$\forall_{x\in A\setminus\{w\}}\colon V(\Box\chi,x)=V(\Box\chi[{}^{\varphi}/_{\psi}],x).$$

Thus, the conditions (i) and (iv) in definition of t-normal rte-models say that for any such model $\langle w,A,V\rangle$, the function $V(\cdot,w)$ belongs to $\mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

THEOREM 4.1. *Suppose that* $\ulcorner\varphi\equiv\psi\urcorner\in\mathbf{PL}$. *Then for any t-normal model* $\langle w,A,V\rangle$ *the following conditions are equivalent*:
(a) $\forall_{\chi\in\mathrm{For}}\colon V(\chi,w)=V(\chi[{}^{\varphi}/_{\psi}],w)$,
(b) $\forall_{\chi\in\mathrm{For}}\colon V(\Box\chi,w)=V(\Box\chi[{}^{\varphi}/_{\psi}],w)$,
(c) $\forall_{\chi\in\mathrm{For}}\colon \forall_{x\in A}\,V(\chi,x)=1$ *iff* $\forall_{x\in A}\,V(\chi[{}^{\varphi}/_{\psi}],x)=1$,
(d) $\forall_{\chi\in\mathrm{For}}\forall_{x\in A}\colon V(\chi,x)=V(\chi[{}^{\varphi}/_{\psi}],x)$,
(e) $\forall_{\chi\in\mathrm{For}}\forall_{x\in A}\colon V(\Box\chi,x)=V(\Box\chi[{}^{\varphi}/_{\psi}],x)$,
(f) $\forall_{\chi\in\mathrm{For}}\forall_{x\in A\setminus\{w\}}\colon V(\chi,x)=V(\chi[{}^{\varphi}/_{\psi}],x)$,
(g) $\forall_{\chi\in\mathrm{For}}\forall_{x\in A\setminus\{w\}}\colon V(\Box\chi,x)=V(\Box\chi[{}^{\varphi}/_{\psi}],x)$.

PROOF: Let $\langle w,A,V\rangle$ be a t-normal model and suppose (throughout the proof) that $\ulcorner\varphi\equiv\psi\urcorner\in\mathbf{PL}$.

"(a) $\Rightarrow$ (b)", "(d) $\Rightarrow$ (c)", "(d) $\Rightarrow$ (e)", "(d) $\Rightarrow$ (f)" and "(f) $\Rightarrow$ (g)": Obvious.

"(b) $\Leftrightarrow$ (c)" By the condition ($V^{\Box}_{w}$).

"(b) $\Rightarrow$ (d)" Since $V(\Box(\chi\equiv\chi),w)=1$, so $V(\Box(\chi\equiv\chi[{}^{\varphi}/_{\psi}]),w)=1$, by (b). Hence for any $x\in A$: $V(\chi,x)=V(\chi[{}^{\varphi}/_{\psi}],x)$, by ($V^{\Box}_{w}$).

"(b) $\Rightarrow$ (a)" As the proof of the part "$\Leftarrow$" of Lemma 1.21(1), for the valuations $V:=V(\cdot,w)$ and $v:=V(\cdot,w)|_{\mathrm{PAt}}$; so (a) is ($\star$) and (b) is ($\star_{\mathrm{PAt}}$).

"(e) $\Rightarrow$ (d)" and "(g) $\Rightarrow$ (f)": Similarly as in "(b) $\Rightarrow$ (a)". The difference is in taking a world x from A (resp. from $A\setminus\{w\}$) instead of w.

"(f) $\Rightarrow$ (b)" We consider two cases.

Firstly, $w\notin A$: By (f) we obtain (c); so we have also (b).

Secondly, $w\in A$: We show that $V(\chi,w)=V(\chi[{}^{\varphi}/_{\psi}],w)$, i.e. we prove (a), hence we also obtain (b).

First we consider the possibility that $\chi=\varphi$, as for "(b) $\Rightarrow$ (a)", i.e. as in the proof of the part "$\Leftarrow$" of Lemma 1.21(1). Thus, we may assume henceforth that $\chi\neq\varphi$. The proof proceeds now by induction on the complexity of χ. We give it for the cases in which χ is ($*$) atomic; ($**$) $\ulcorner\neg\chi_1\urcorner$ or $\ulcorner\chi_1\circ\chi_2\urcorner$, for $\circ=\vee,\wedge,\supset,\equiv$; and ($***$) a necessitation, $\ulcorner\Box\chi_1\urcorner$.

For ($*$) and ($**$): As for "(b) $\Rightarrow$ (a)", i.e. as in the proof of the part "$\Leftarrow$" of Lemma 1.21(1).

For (∗∗∗): We make the inductive hypothesis that the result holds for all sentences shorter than χ. So $V(\chi_1, w) = V(\chi_1[{}^{\varphi}/_{\psi}], w)$. Moreover, by the assumption (f) we have that $V(\chi_1, x) = V(\chi_1[{}^{\varphi}/_{\psi}], x)$, for any $x \in A \setminus \{w\}$. Thus, by $(V^{\Box}_{w})$, we obtain that $V(\Box\chi_1, w) = V(\Box\chi_1[{}^{\varphi}/_{\psi}], w)$, which ends the inductive proof. ⊣

The lemma below — analogous to Lemma 3.2 — shows that the notion of a *t-normal rte-model* could be defined in different albeit equivalent way.

Lemma 4.2. 1. *Let* $\langle w, A, v_w, \{V_x\}_{x\in A\setminus\{w\}}\rangle$ *be a structure in which* w *and* A *are such as in t-normal models,* $v_w\colon \mathrm{At} \to \{0,1\}$*, and for any* x *in* $A\setminus\{w\}$*,* $V_x \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$*. Then there is the unique* $V\colon \mathrm{For}\times(\{w\}\cup A) \to \{0,1\}$ *such that*:

- $\forall_{\alpha\in\mathrm{At}}\colon V(\alpha, w) = v_w(\alpha)$ *and* $\forall_{\varphi\in\mathrm{For}}\forall_{x\in A\setminus\{w\}}\colon V(\varphi, x) = V_x(\varphi)$,
- V *satisfies conditions* (i)*,* (ii) *and* (iv) *from definition of t-normal rte-models.*

Thus, $\langle w, A, V\rangle$ *is a t-normal rte-model. Moreover, if* $w \in A$*, then this model is self- associate.*

2. *Let* $\langle w, A, v_w, \{v_x\}_{x\in A\setminus\{w\}}\rangle$ *be a structure in which* w *and* A *are such as in t-normal models,* v_w *is an assignment from* At *to* $\{0,1\}$*, and for any* $x \in A\setminus\{w\}$*,* v_x *is an assignment from* PAt *to* $\{0,1\}$ *such that*:

$(\mathrm{iv}_{\mathrm{PAt}})$ $\quad\forall_{\chi,\varphi,\psi\in\mathrm{For}}\colon$ *if* $\ulcorner\varphi\equiv\psi\urcorner \in \mathbf{PL}$*, then* $v_x(\Box\chi) = v_x(\Box\chi[{}^{\varphi}/_{\psi}])$.

Then there is the unique function $V\colon \mathrm{For}\times(\{w\}\cup A) \to \{0,1\}$ *such that*:

- $\forall_{\alpha\in\mathrm{At}}\ V(\alpha, w) = v_w(\alpha)$ *and* $\forall_{\varphi\in\mathrm{PAt}}\forall_{x\in A\setminus\{w\}}\colon V(\varphi, x) = v_x(\varphi)$,
- V *satisfies conditions* (i)*,* (ii) *and* (iv) *from definition of t-normal rte-models.*

Thus, $\langle w, A, V\rangle$ *is a t-normal rte-model. Moreover, if* $w \in A$*, then this model is self-associate.*

Proof: 1. By Theorem 4.1.

2. By Lemma 1.21(1), for every $x \in A\setminus\{w\}$ there is the unique extension $V_x\colon \mathrm{For} \to \{0,1\}$ of v_x by classical truth conditions for truth-value operators (i.e. e.g. $V_x \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ and $\forall_{\chi\in\mathrm{For}}\colon V_x(\Box\chi) = v_x(\Box\chi)$). The rest as in 1. ⊣

REMARK 4.1. *In the light of the above results the structures of the form* $\langle w, A, v_w, \{v_x\}_{x\in A\setminus\{w\}}\rangle$ *which satisfy the conditions from Lemma 4.2 can serve as t-normal rte-models. In a similar way, we assume that in such a model a formula* φ *is true iff* $V(\varphi, w) = 1$. $\dashv$

Let $\mathbf{nM_{rte}}$ be the class of all t-normal rte-models. Moreover, let $\mathbf{nM^{sa}_{rte}}$ (resp. $\mathbf{nM^{\varnothing}_{rte}}$, $\mathbf{nM^{+}_{rte}}$) be the class of t-normal rte-models which are self-associate (resp. empty, non-empty).

We have the following facts.

FACT 4.3. 1. *All members of* $\mathbf{PL}_{\mathrm{rte}}$ *are true in all models from* $\mathbf{nM_{rte}} \cup \mathbf{qM}$.
2. *All members of* $\Box\mathbf{PL}_{\mathrm{rte}}$ *are true in all models from* $\mathbf{nM_{rte}}$.

PROOF: 1. For any $\tau \in \mathbf{PL}$, we have that $V(\tau, w) = 1$, for any model from $\mathbf{nM_{rte}} \cup \mathbf{qM}$. Thus we use the conditions (iv), (ii′) and induction.

2. For any $\tau \in \mathbf{PL}$, we have that $V(\Box\tau, w) = 1$, for any model from $\mathbf{nM_{rte}}$. Therefore it is enough to use the condition (iv). $\dashv$

In [3] we proved the following determination theorems for the logics $\mathbf{S0.5^{\circ}_{rte}}$, $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}]$ and $\mathbf{S0.5_{rte}}$:

THEOREM 4.4 ([3]). 1. $\mathbf{S0.5^{\circ}_{rte}}$ *is determined by the class* $\mathbf{nM_{rte}}$.
2. $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}]$ *is determined by the class* $\mathbf{nM^{+}_{rte}}$.
3. $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}]$ *is determined by the class* $\mathbf{nM^{sa}_{rte}} \cup \mathbf{nM^{\varnothing}_{rte}}$.
4. $\mathbf{S0.5_{rte}}$ *is determined by the class* $\mathbf{nM^{sa}_{rte}}$.

For logic $\mathbf{S0.5^{\circ}_{rte}}$ and $\mathbf{S0.5_{rte}}$ there holds a fact which is analogous to Fact 3.8 for logics $\mathbf{S0.5^{\circ}}$ and $\mathbf{S0.5}$.

FACT 4.5. *For any* $\varphi \in \mathrm{For}$:

$$\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5^{\circ}_{rte}} \textit{ iff } \varphi \in \mathbf{PL}_{\mathrm{rte}} \textit{ iff } \ulcorner\Box\varphi\urcorner \in \mathbf{S0.5_{rte}}.$$

So $\mathbf{S0.5^{\circ}_{rte}}$ *and* $\mathbf{S0.5_{rte}}$ *are closed under* (RN_*) *and* (SMP).

PROOF: Firstly, by Corollary 1.19, $\Box\mathbf{PL}_{\mathrm{rte}} \subseteq \mathbf{S0.5^{\circ}_{rte}} \subseteq \mathbf{S0.5_{rte}}$.

Secondly, let $\varphi \notin \mathbf{PL}_{\mathrm{rte}}$, $w \neq a$, $A := \{w, a\}$. Then, by Lemma 1.21, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ we have that $V_a(\varphi) = 0$. As in Lemma 4.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0, 1\}$ and V_a, we construct a self-associate t-normal rte-model $\langle w, \{w, a\}, V\rangle$ such that $V(\Box\varphi, w) = 0$. Hence $\ulcorner\Box\varphi\urcorner \notin \mathbf{S0.5}_{\mathsf{rte}}$, by Theorem 4.4(4). $\dashv$

4.2. Models for weak t-normal rte-systems with additional axioms of the form $\ulcorner\Box\varphi\urcorner$

While considering week t-normal rte-systems with an additional axiom of the form $\ulcorner\Box\varphi\urcorner$, where $\varphi \in \mathbf{S0.5}$, we will take t-normal rte-models for $\ulcorner\Box\varphi\urcorner$, that is these that satisfy (iii_φ). More generally, for systems with additional axioms from a set $\Box\Phi$, where $\Phi \subseteq \mathbf{S0.5}$, we will use t-normal rte-models for $\Box\Phi$, that is such that satisfy (iii).

Let $\mathsf{nM}_{\mathsf{rte}}[\Box\Phi]$ be the class of all t-normal rte-models for $\Box\Phi$. Moreover, let $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}[\Box\Phi]$ (resp. $\mathsf{nM}^{\varnothing}_{\mathsf{rte}}[\Box\Phi]$, $\mathsf{nM}^{+}_{\mathsf{rte}}[\Box\Phi]$) be the class of t-normal rte-models which are self-associate (resp. empty, non-empty) for $\Box\Phi$.

Similarly to Fact 3.13 we prove the following:

FACT 4.6. *For any* $\Phi \subseteq \mathbf{S0.5}$:
1. $\mathbf{S0.5}^{\circ}_{\mathsf{rte}}[\Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}_{\mathsf{rte}}[\Box\Phi]$.
2. $\mathbf{S0.5}^{\circ}_{\mathsf{rte}}[\mathtt{D}, \Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}^{+}_{\mathsf{rte}}[\Box\Phi]$.
3. $\mathbf{S0.5}^{\circ}_{\mathsf{rte}}[\mathtt{T_q}, \Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}^{\varnothing}_{\mathsf{rte}}[\Box\Phi] \cup \mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}[\Box\Phi]$.
4. $\mathbf{S0.5}_{\mathsf{rte}}[\Box\Phi]$ *is sound with respect to the class* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}[\Box\Phi]$.

5. Completeness and determination theorems

For completeness of considered weak t-normal and t-normal rte-logics we use the method of canonical models.

5.1. Notions and facts concerning maximal consistent sets

Let $\boldsymbol{\Sigma}$ be any modal system and $\Gamma \subseteq \mathrm{For}$. A set Γ is $\boldsymbol{\Sigma}$*-consistent* iff for some $\varphi \in \mathrm{For}$, $\Gamma \nvdash_{\boldsymbol{\Sigma}} \varphi$; equivalently in the light of **PL**, iff $\Gamma \nvdash_{\boldsymbol{\Sigma}} p \wedge \neg p$. We have (see e.g. [1]):

- If Γ is $\boldsymbol{\Sigma}$-consistent, then $\boldsymbol{\Sigma}$ is consistent.
- $\boldsymbol{\Sigma}$ is consistent iff $\boldsymbol{\Sigma}$ is $\boldsymbol{\Sigma}$-consistent.
- If Γ is $\boldsymbol{\Sigma}$-consistent and $\boldsymbol{\Sigma}'$ is a modal system such that $\boldsymbol{\Sigma}' \subseteq \boldsymbol{\Sigma}$, then Γ is $\boldsymbol{\Sigma}'$-consistent; so, Γ is **PL**-consistent.

We say that Γ is $\boldsymbol{\Sigma}$-*maximal* iff Γ is $\boldsymbol{\Sigma}$-consistent and Γ has only $\boldsymbol{\Sigma}$-inconsistent proper extensions. Let $\mathrm{Max}_{\boldsymbol{\Sigma}}$ be the set of all $\boldsymbol{\Sigma}$-maximal sets.

LEMMA 5.1 ([1]). *Let* $\Gamma \in \mathrm{Max}_{\boldsymbol{\Sigma}}$. *Then*
1. $\boldsymbol{\Sigma} \subseteq \Gamma$ *and* Γ *is a modal system.*
2. $\Gamma \vdash_{\boldsymbol{\Sigma}} \varphi$ *iff* $\varphi \in \Gamma$.
3. $\ulcorner \neg\varphi \urcorner \in \Gamma$ *iff* $\varphi \notin \Gamma$.
4. $\ulcorner \varphi \wedge \psi \urcorner \in \Gamma$ *iff both* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
5. $\ulcorner \varphi \vee \psi \urcorner \in \Gamma$ *iff either* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
6. $\ulcorner \varphi \supset \psi \urcorner \in \Gamma$ *iff either* $\varphi \notin \Gamma$ *or* $\psi \in \Gamma$.
7. $\ulcorner \varphi \equiv \psi \urcorner \in \Gamma$ *iff either* $\varphi, \psi \in \Gamma$ *or* $\varphi, \psi \notin \Gamma$.

LEMMA 5.2. *If* $\Gamma \in \mathrm{Max}_{\boldsymbol{\Sigma}}$, *then* $\Gamma \in \mathrm{Max}_{\mathbf{PL}}$.

LEMMA 5.3 ([1]). 1. $\Gamma \vdash_{\boldsymbol{\Sigma}} \varphi$ *iff* $\varphi \in \Delta$, *for any* Δ *such that* $\Delta \in \mathrm{Max}_{\boldsymbol{\Sigma}}$ *and* $\Gamma \subseteq \Delta$.
2. $\varphi \in \boldsymbol{\Sigma}$ *iff* $\varphi \in \Delta$, *for any* $\Delta \in \mathrm{Max}_{\boldsymbol{\Sigma}}$.

We also need the following auxiliary lemma.

LEMMA 5.4 ([3]). *Let* $\boldsymbol{\Sigma}$ *be a t-normal consistent system and* $\Gamma \in \mathrm{Max}_{\boldsymbol{\Sigma}}$. *Then for every* $\varphi \in \mathrm{For}$ *the following conditions are equivalent:*
(a) $\ulcorner \Box\varphi \urcorner \in \Gamma$.
(b) $\Gamma \vdash_{\boldsymbol{\Sigma}} \Box\varphi$.
(c) $\{\psi : \ulcorner \Box\psi \urcorner \in \Gamma\} \vdash_{\mathbf{PL}} \varphi$.
(d) $\varphi \in \Delta$, *for any* **PL**-*maximal set* Δ *such that* $\{\, \psi : \ulcorner \Box\psi \urcorner \in \Gamma \,\} \subseteq \Delta$.

5.2. Canonical models and completeness

Let $\boldsymbol{\Sigma}$ be a t-normal consistent system and $\Gamma \in \mathrm{Max}_{\boldsymbol{\Sigma}}$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is a *canonical model for* $\boldsymbol{\Sigma}$ *and* Γ iff it satisfies the following conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \big\{\, \Delta \in \mathrm{Max}_{\mathbf{PL}} : \forall_{\psi \in \mathrm{For}} (\ulcorner \Box\psi \urcorner \in \Gamma \Rightarrow \psi \in \Delta \,) \,\big\}$,
- $V_\Gamma \colon \mathrm{For} \times (\{w_\Gamma\} \cup A_\Gamma) \to \{0, 1\}$ is a valuation such that for all $\varphi \in \mathrm{For}$ and $\Delta \in \{w_\Gamma\} \cup A_\Gamma$

$$V_\Gamma(\varphi, \Delta) := \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 5.5. *For any t-normal system* $\boldsymbol{\Sigma}$ *and* $\Gamma \in \mathrm{Max}_{\boldsymbol{\Sigma}}$ *it holds that*:

1. $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is a t-normal model.*
2. *For any set* Φ*, if* $\Phi \subseteq \mathbf{S0.5}$ *and* $\mathrm{sub}(\Box\Phi) \subseteq \boldsymbol{\Sigma}$*, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is a t-normal model for* $\Box\Phi$.
3. *If* $\mathrm{sub}(\mathtt{T}) \subseteq \boldsymbol{\Sigma}$*, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is self-associate.*
4. *If* $\mathrm{sub}(\mathtt{D}) \subseteq \boldsymbol{\Sigma}$*, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is non-empty.*
5. *If* $\mathrm{sub}(\mathtt{T_q}) \subseteq \boldsymbol{\Sigma}$*, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is either empty or self-associate.*
6. *If* $\boldsymbol{\Sigma}$ *is an rte-system, then* $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ *is a t-normal rte-model.*

PROOF: Let $\Gamma \in \mathrm{Max}_{\boldsymbol{\Lambda}[\Box\Phi]}$; hence $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}[\Box\Phi]$ are consistent.

1. Thanks to properties of maximal sets (see Lemma 5.1), for every $\Delta \in \{w_\Gamma\} \cup A_\Gamma$, $V_\Gamma(\cdot, \Delta) \in \mathsf{Val}^{\mathsf{cl}}$. We prove that for w_Γ the assignment $V_\Gamma(\cdot, w_\Gamma)$ satisfies the condition $(V^{\Box}_{w_\Gamma})$ for any $\varphi \in \mathrm{For}$: $V_\Gamma(\Box\varphi, w_\Gamma) = 1$ iff $\ulcorner\Box\varphi\urcorner \in \Gamma$ (by definition of V_Γ) iff $\varphi \in \Delta$, for every $\Delta \in \mathrm{Max}_{\mathbf{PL}}$ such that $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$ (by Lemma 5.4) iff $\varphi \in \Delta$, for every $\Delta \in A_\Gamma$ (by definition of A_Γ) iff $V_\Gamma(\varphi, \Delta) = 1$, for every $\Delta \in A_\Gamma$ (by the definition of V_Γ).

2. Let $\Phi^\star := \mathrm{sub}(\Phi)$. By definitions of A_Γ and V_Γ, for any world from $A_\Gamma \setminus \{w_\Gamma\}$, all formulae from $\Phi^\star$ have the value 1, since $\Box\Phi^\star \subseteq \boldsymbol{\Sigma} \subseteq \Gamma$, by Lemma 5.1(1).

3. We show that $w_\Gamma \in A_\Gamma$. Firstly, by Lemma 5.2, $\Gamma \in \mathrm{Max}_{\mathbf{PL}}$. Secondly, by Lemma 5.1(1), for any $\psi \in \mathrm{For}$, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. So, by Lemma 5.1(6), if $\ulcorner\Box\psi\urcorner \in \Gamma$, then $\psi \in \Gamma$, i.e. $\Gamma \in A_\Gamma$.

4. By Lemma 5.1, $\ulcorner\Diamond\top\urcorner \in \Gamma$, i.e., $\ulcorner\neg\Box\neg\top\urcorner \in \Gamma$; so and $\ulcorner\Box\neg\top\urcorner \notin \Gamma$. Therefore, by Lemma 5.4, $\ulcorner\neg\top\urcorner \notin \Delta_0$, for some Δ_0 such that Δ_0 is **PL**-maximal and $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta_0$. Hence $\Delta_0 \in A_\Gamma$. Thus, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle \in \mathsf{nM}^+$.

5. We show that $w_\Gamma \in A_\Gamma$ or $A_\Gamma = \emptyset$. Notice that, by Lemma 5.1, $\ulcorner\neg\Box(p \wedge \neg p) \supset (\Box\psi \supset \psi)\urcorner \in \Gamma$, for any formula ψ. Suppose that $A_\Gamma \neq \emptyset$. Then '$\Box(p \wedge \neg p)$' $\notin \Gamma$, by Lemma 5.4, since '$p \wedge \neg p$' $\notin \Delta$, for any Δ which is **PL**-consistent. So, '$\neg\Box(p \wedge \neg p)$' $\in \Gamma$. Therefore $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. Hence $w_\Gamma \in A_\Gamma$, as in 3.

6. Since $\mathrm{REP}_{\mathrm{PL}} \subseteq \boldsymbol{\Sigma}$, so if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$, then $\ulcorner\chi \equiv \chi[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Sigma}$. Hence $\ulcorner\chi \equiv \chi[^{\varphi}/_{\psi}]\urcorner \in \Gamma$, by Lemma 5.1(1). Thus, by definitions of w_Γ and V_Γ, $V(\chi, w_\Gamma) = V(\chi[^{\varphi}/_{\psi}], w_\Gamma)$. ⊣

By lemmas 5.3 and 5.5 we obtain the completeness of considered logics.

THEOREM 5.6. *Let* $\boldsymbol{\Lambda}$ *be a t-normal consistent logic and* $\Phi \subseteq \mathbf{S0.5}$. *Then*
1. $\boldsymbol{\Lambda}[\Box\Phi]$ *is complete with respect to the class* $\mathsf{nM}[\Box\Phi]$.
2. *If* $(\mathtt{T}) \in \boldsymbol{\Lambda}$, *then* $\boldsymbol{\Lambda}[\Box\Phi]$ *is complete with respect to the class* $\mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.
3. *If* $(\mathtt{D}) \in \boldsymbol{\Lambda}$, *then* $\boldsymbol{\Lambda}[\Box\Phi]$ *is complete with respect to the class* $\mathsf{nM}^{+}[\Box\Phi]$.
4. *If* $(\mathtt{T_q}) \in \boldsymbol{\Lambda}$, *then* $\boldsymbol{\Lambda}[\Box\Phi]$ *is complete with respect to the class* $\mathsf{nM}^{\varnothing}[\Box\Phi] \cup \mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.
5. *If* $\boldsymbol{\Lambda}$ *is an rte-logic, then* $\boldsymbol{\Lambda}[\Box\Phi]$ *is complete with respect to the class* $\mathsf{nM}_{\mathsf{rte}}[\Box\Phi]$.

PROOF: All considered logics are consistent.

1. Let φ be an arbitrary formula which is true in all t-normal models for $\Box\Phi$. Let Γ be an arbitrary $\boldsymbol{\Lambda}[\Box\Phi]$-maximal set. By Lemma 5.5(1)(2), $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a t-normal model for $\Box\Phi$. So $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of w_Γ and V_Γ. So, we have shown that φ belongs to all $\boldsymbol{\Lambda}[\Box\Phi]$-maximal sets. Hence $\varphi \in \boldsymbol{\Lambda}[\Box\Phi]$, by Lemma 5.3(2).

2. By Lemma 5.5(3), $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is self-associate. The rest as in 1.

3. By Lemma 5.5(4), $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is non-empty. The rest as in 1.

4. By Lemma 5.5(5), $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is self-associate or empty. The rest as in 1.

5. If $\boldsymbol{\Lambda}$ is an rte-logic, then $\boldsymbol{\Lambda}[\Box\Phi]$ is also an rte-logic. By Lemma 5.5(6), $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a t-normal rte-model. The rest as in 1. ⊣

5.3. Determination theorems

By facts 3.13 and 4.6, and Theorem 5.6 we obtain:[5]

THEOREM 5.7. *For any* $\Phi \subseteq \mathbf{S0.5}$:
1. $\mathbf{S0.5}^{\circ}[\Box\Phi]$ *is determined by the class* $\mathsf{nM}[\Box\Phi]$.
2. $\mathbf{S0.5}^{\circ}[\mathtt{D}, \Box\Phi]$ *is determined by the class* $\mathsf{nM}^{+}[\Box\Phi]$.
3. $\mathbf{S0.5}^{\circ}[\mathtt{T_q}, \Box\Phi]$ *is determined by the class* $\mathsf{nM}^{\varnothing}[\Box\Phi] \cup \mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.

[5]Of course, if $\Phi \subseteq \mathbf{PL}$ (so also if $\Phi = \emptyset$), then we obtain theorems 3.4 and 4.4.

4. $\mathbf{S0.5}[\Box\Phi]$ *is determined by the class* $\mathsf{nM}^{\mathsf{sa}}[\Box\Phi]$.
5. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}[\Box\Phi]$ *is determined by the class* $\mathsf{nM}_{\mathsf{rte}}[\Box\Phi]$.
6. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}[\mathtt{D}, \Box\Phi]$ *is determined by the class* $\mathsf{nM}^{+}_{\mathsf{rte}}[\Box\Phi]$.
7. $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}[\mathtt{T_q}, \Box\Phi]$ *is determined by the class* $\mathsf{nM}^{\varnothing}_{\mathsf{rte}}[\Box\Phi] \cup \mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}[\Box\Phi]$.
8. $\mathbf{S0.5}_{\mathbf{rte}}[\Box\Phi]$ *is determined by the class of* $\mathsf{nM}^{\mathsf{sa}}_{\mathsf{rte}}[\Box\Phi]$.

6. Mutual dependencies among very weak t-normal logics. Very weak t-normal logics vs. S0.9°, S0.9, S1° and S1

Firstly notice that the following lemma holds.

LEMMA 6.1. *Let a logic* $\boldsymbol{\Lambda}$ *be one from* $\mathbf{S0.9}^{\circ}$, $\mathbf{S0.9}$, $\mathbf{S1}^{\circ}$, $\mathbf{S1}$. *Then for all* $\varphi, \psi \in \mathrm{For}$, *if* $\ulcorner\Box\varphi\urcorner$ *and* $\ulcorner\Box\psi\urcorner \in \boldsymbol{\Lambda}$, *then* $\ulcorner\Box(\Box\varphi \equiv \Box\psi)\urcorner \in \boldsymbol{\Lambda}$. *Consequently,* $\ulcorner\Box(\Box(\mathtt{K}) \equiv \Box\top)\urcorner \in \mathbf{S0.9}^{\circ}$.

PROOF: Since $\mathrm{R_{PL}} \subseteq \boldsymbol{\Lambda}$, so $\ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box(\varphi \equiv \psi)\urcorner \in \boldsymbol{\Lambda}$ and so $\ulcorner\Box(\varphi \equiv \psi)\urcorner \in \boldsymbol{\Lambda}$. Hence, by $(\mathrm{RRSE_T})$, $\ulcorner\Box(\Box\varphi \equiv \Box\psi)\urcorner \in \boldsymbol{\Lambda}$. Finally, $\Box(\mathtt{K}), \Box\top \in \mathbf{S0.9}^{\circ}$. ⊣

FACT 6.2. *For any* $\varphi \notin \mathbf{PL}_{\mathrm{rte}}$ *and* $\psi \in \mathbf{PL}_{\mathrm{rte}}$,

$$\ulcorner\Box(\Box\varphi \equiv \Box\psi)\urcorner \notin \mathbf{S0.5}_{\mathbf{rte}}[\Box\mathbf{S0.5}].$$

Consequently, $\ulcorner\Box(\Box(\mathtt{K}) \equiv \Box\top)\urcorner \notin \mathbf{S0.5}_{\mathbf{rte}}[\Box\mathbf{S0.5}]$.

PROOF: Let $w \neq a$, $A := \{w, a\}$. Let $v_a \colon \mathrm{PAt} \to \{0,1\}$ be any assignment such that for any $\chi \in \mathrm{For}$: $v_a(\Box\chi) = 1$ iff $\chi \in \mathbf{PL}_{\mathrm{rte}}$. The assignment v_a satisfies the condition $(\star_{\mathrm{PAt}})$ from Lemma 1.21. Indeed, for any $\chi, \chi_1, \chi_2 \in \mathrm{For}$ such that $\ulcorner\chi_1 \equiv \chi_2\urcorner \in \mathbf{PL}$: $v_a(\Box\chi) = 1$ iff $\chi \in \mathbf{PL}_{\mathrm{rte}}$ iff $\chi[{}^{\chi_1}/{}_{\chi_2}] \in \mathbf{PL}_{\mathrm{rte}}$ iff $v_a(\Box\chi[{}^{\chi_1}/{}_{\chi_2}]) = 1$. Let $V_a \colon \mathrm{For} \to \{0,1\}$ be the unique extension of v_a by classical truth conditions for truth- value operators. By Lemma 1.21(1), $V_a \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

Notice that $V_a(\mathrm{sub}(\mathtt{T})) = \{1\} = V_a(\mathrm{sub}(\mathtt{K}))$. Indeed, if $V_a(\Box\chi) = 1$, then $\chi \in \mathbf{PL}_{\mathrm{rte}}$. So $V_a(\chi) = 1$, by Lemma 1.21(2). Moreover, if $V_a(\Box(\chi_1 \supset \chi_2)) = 1 = V_a(\Box\chi_1)$, then $\ulcorner\chi_1 \supset \chi_2\urcorner \in \mathbf{PL}_{\mathrm{rte}}$ and $\chi_1 \in \mathbf{PL}_{\mathrm{rte}}$. Hence, by Lemma 1.21(3), for any $V \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$: $V(\chi_1 \supset \chi_2) = 1 = V(\chi_1)$, so also $V(\chi_2) = 1$. Hence $\chi_2 \in \mathbf{PL}_{\mathrm{rte}}$ and consequently, $V_a(\Box\chi_2) = 1$.

Thus, $V_a(\mathbf{S0.5}) = \{1\}$, since all theses of **S0.5** are derivable in **PL**, $\Box$**PL**, sub(K) and sub(T) by (MP), and for all formulae derivable in this way the function V_a has the value 1.

Now, as in Lemma 4.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a we construct a self-associate t-normal rte-model $\langle w, \{w,a\}, V\rangle$ for $\Box$**S0.5**. For any $\varphi \notin \mathbf{PL}_{\mathrm{rte}}$ and $\psi \in \mathbf{PL}_{\mathrm{rte}}$ we have that $V(\Box(\Box\varphi \equiv \Box\psi), w) = 0$, since $V(\Box\varphi \equiv \Box\psi), a) = 0$. Thus, $\ulcorner\Box(\Box\varphi \equiv \Box\psi)\urcorner \notin \mathbf{S0.5_{rte}}[\Box\mathbf{S0.5}]$, by Fact 4.6.

Finally notice that (K) $\notin \mathbf{PL}_{\mathrm{rte}}$ and $\top \in \mathbf{PL}_{\mathrm{rte}}$. ⊣

By the above facts, Fact 2.2 and Corollary 2.6 we obtain:

Corollary 6.3. 1. $\mathbf{S0.5^{\circ}_{rte}}[\Box\mathtt{K}] \subsetneq \mathbf{S0.9^{\circ}}$.
2. $\mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.9}$.
3. $\mathbf{S0.5^{\circ}_{rte}}[\Box\mathtt{X}] \subsetneq \mathbf{S1^{\circ}}$.
4. $\mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}] \subsetneq \mathbf{S1}$.
5. *If* $\Phi \subseteq \mathbf{C2} \cap \mathbf{S0.5}$, *then* $\mathbf{S0.5^{\circ}_{rte}}[\Box\Phi] \subsetneq \mathbf{S2^{\circ}}$.
6. *If* $\Phi \subseteq \mathbf{E1}$, *then* $\mathbf{S0.5_{rte}}[\Box\Phi] \subsetneq \mathbf{S2}$.

Fact 6.4. *The formulae* $\Box(\dagger)$ *and* $(\ddagger)$ *from the first part do not belong to* $\mathbf{S0.5}[\Box\mathbf{S0.5}]$. *Consequently,* $\mathbf{S0.5}[\Box\mathbf{S0.5}]$ *is not an rte-system.*

Proof: Let $w \neq a$ and $A := \{w,a\}$.

First way:[6] Since $(\dagger_{\mathrm{a}}) \notin \mathbf{S0.5}$, so $\mathbf{S0.5} \not\models_{\mathbf{PL}} (\dagger_{\mathrm{a}})$. Hence there is $V_a \in \mathsf{Val}^{\mathsf{cl}}$ such that $V_a(\mathbf{S0.5}) = \{1\}$ and $V_a(\dagger_{\mathrm{a}}) = 0$. So $V_a(\Box\Box p) = 1$, $V_a(\Box\Box\neg\neg p) = 0$ and $V_a(\mathrm{sub}(\mathtt{T})) = \{1\}$. Consequently, $V_a(\Box p) = 1 = V_a(p)$.

Now, as in Lemma 3.2(1), for V_a and any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ such that $v_w(p) = 1$, we build a self-associate t-normal model $\langle w, \{w,a\}, V\rangle$ for $\Box$**S0.5**. We have: $V(\Box\Box p, a) = 1$, $V(\Box\Box\neg\neg p, a) = 0$, $V(\Box p, w) = V(\Box\Box p, w) = V(\Box\Box\Box p, w) = 1$ and $V(\Box\Box\Box\neg\neg p, w) = 0$. So $V(\Box(\dagger_{\mathrm{a}}), w) = 0$ and $V((\ddagger_{\mathrm{a}}), w) = 0$. Thus, by Fact 3.13(4), $\Box(\dagger_{\mathrm{a}})$ and $(\ddagger_{\mathrm{a}})$ do not belong to $\mathbf{S0.5}[\Box\mathbf{S0.5}]$. Similarly for $\Box(\dagger_{\mathrm{b}})$ and $(\ddagger_{\mathrm{b}})$.

Second way: Let $v_a\colon \mathrm{PAt} \to \{0,1\}$ be any assignment such that $v_a(p) = 1$ and for any $\varphi \in \mathrm{For}$:

[6] We will present two different ways in order to show different methods of construction of countermodels.

$$v_a(\Box\varphi) = \begin{cases} 1 & \text{if } \ulcorner p \supset \varphi \urcorner \in \mathbf{PL} \\ 1 & \text{if } \varphi = \text{`}\Box p\text{'} \\ 0 & \text{otherwise} \end{cases}$$

Let $V_a\colon \mathrm{For} \to \{0,1\}$ be the unique extension of v_a by classical truth conditions for truth-value operators. Evidently $V_a(\Box\Box\neg\neg p) = 0$. Notice that $V_a(\mathrm{sub}(\mathtt{K})) = \{1\}$ and $V_a(\mathrm{sub}(\mathtt{T})) = \{1\}$. Indeed, suppose that $V_a(\Box(\varphi \supset \psi)) = 1$ and $V_a(\Box\varphi) = 1$. Hence both $\ulcorner p \supset (\varphi \supset \psi)\urcorner \in \mathbf{PL}$ and either $\ulcorner p \supset \varphi\urcorner \in \mathbf{PL}$ or $\varphi = $ '$\Box p$'. So either $\ulcorner p \supset \psi\urcorner \in \mathbf{PL}$ or $\psi = $ '$\Box p$'. Consequently, $V_a(\Box\psi) = 1$. Moreover, if $V_a(\Box\varphi) = 1$, then either $\ulcorner p \supset \varphi\urcorner \in \mathbf{PL}$ or $\varphi = $ '$\Box p$'. So $V_a(\varphi) = 1$, since $V \in \mathsf{Val}^{\mathsf{cl}}$ and $V_a(p) = 1 = V_a(\Box p)$.

Thus, $V_a(\mathbf{S0.5}) = \{1\}$, since all theses of **S0.5** are derivable from $\Box\mathbf{PL}$, $\mathrm{sub}(\mathtt{K})$ and $\mathrm{sub}(\mathtt{T})$ by **PL** and (MP).

Now, as in Lemma 3.2(1), for V_a and any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ such that $v_w(p) = 1$, we build a self-associate t-normal model $\langle w, \{w,a\}, V\rangle$ for $\Box\mathbf{S0.5}$. We have: $V(\Box\Box p, a) = 1$, $V(\Box\Box\neg\neg p, a) = 0$, $V(\Box\Box p, w) = V(\Box\Box\Box p, w) = 1$, $V(\Box\Box\Box\neg\neg p, w) = 0$. So $V(\Box(\dagger_{\mathrm{a}}), w) = 0$ and $V((\ddagger_{\mathrm{a}}), w) = 0$. Thus, by Fact 3.13(4), $\Box(\dagger_{\mathrm{a}})$ and $(\ddagger_{\mathrm{a}})$ do not belong to $\mathbf{S0.5}[\Box\mathbf{S0.5}]$. Similarly for $\Box(\dagger_{\mathrm{b}})$ and $(\ddagger_{\mathrm{b}})$. ⊣

Fact 6.5. $\Box(\mathtt{X}) \notin \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}]$.

Proof: Since $\Box(\mathtt{X}) \notin \mathbf{S0.9}$ and $\mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.9}$. ⊣

Fact 6.6. $\Box(\mathtt{K}) \notin \mathbf{S0.5_{rte}}[\Box\mathtt{T}]$.[7]

Proof: Let $w \neq a$, $A := \{w,a\}$. Let $v_a\colon \mathrm{PAt} \to \{0,1\}$ be any assignment such that $v_a(p) = 1 = v_a(q)$ and for any $\chi \in \mathrm{For}$: $\varphi \in \mathrm{For}$:

$$v_a(\Box\chi) = \begin{cases} 1 & \text{if } \ulcorner \chi \equiv p \urcorner \in \mathbf{PL} \\ 1 & \text{if } \ulcorner \chi \equiv (p \supset q) \urcorner \in \mathbf{PL} \\ 0 & \text{otherwise} \end{cases}$$

The assignment v_a satisfies the condition $(\star_{\mathrm{PAt}})$ from Lemma 1.21. Indeed, for any $\chi, \chi_1, \chi_2 \in \mathrm{For}$ such that $\ulcorner \chi_1 \equiv \chi_2 \urcorner \in \mathbf{PL}$: $v_a(\Box\chi) = 1$ iff either

[7]Notice that, by Fact 4.5, $\Box(\mathtt{K}) \notin \mathbf{S0.5_{rte}}$.

$\ulcorner p \equiv \chi \urcorner \in \mathbf{PL}$ or $\ulcorner (p \supset q) \equiv \chi \urcorner \in \mathbf{PL}$ iff either $\ulcorner p \equiv \chi[{}^{\chi_1}/_{\chi_2}] \urcorner \in \mathbf{PL}$ or $\ulcorner (p \supset q) \equiv \chi[{}^{\chi_1}/_{\chi_2}] \urcorner \in \mathbf{PL}$ iff $v_a(\Box\chi[{}^{\chi_1}/_{\chi_2}]) = 1$. Let $V_a\colon \mathrm{For} \to \{0,1\}$ be the unique extension of v_a by classical truth conditions for truth- value operators. By Lemma 1.21(1), $V_a \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

Notice that $V_a(\mathrm{sub}(\mathtt{T})) = \{1\}$. Indeed, if $V_a(\Box\chi) = 1$, then either $\ulcorner p \equiv \chi \urcorner \in \mathbf{PL}$ or $\ulcorner (p \supset q) \equiv \chi \urcorner \in \mathbf{PL}$. So $V_a(\chi) = 1$, by Lemma 1.21(2).

Since $V_a(\Box(p \supset q)) = 1 = V_a(\Box p)$ and $V_a(\Box q) = 0$, so $V_a(\mathtt{K}) = 0$.

Now, as in Lemma 4.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a, we construct a self-associate t-normal rte-model $\langle w, \{w,a\}, V\rangle$ for $\{\Box(\mathtt{T})\}$, since $V(\mathrm{sub}(\mathtt{T}), w) = \{1\}$. Since $V(\Box(\mathtt{K}), w) = 0$, so $\Box(\mathtt{K}) \notin \mathbf{S0.5_{rte}}[\Box\mathtt{T}]$, by Theorem 5.7(8). ⊣

Fact 6.7. $\Box(\mathtt{T}) \notin \mathbf{S0.5_{rte}}[\Box\mathtt{X}]$.

Proof: Let $w \neq a$, $A := \{w,a\}$. Let $v_a\colon \mathrm{PAt} \to \{0,1\}$ be any assignment such that $v_a(0)$ and for any $\chi \in \mathrm{For}$: $\varphi \in \mathrm{For}$:

$$v_a(\Box\chi) = \begin{cases} 1 & \text{if } \ulcorner \chi \equiv p \urcorner \in \mathbf{PL} \\ 0 & \text{otherwise} \end{cases}$$

The assignment v_a satisfies the condition $(\star_{\mathrm{PAt}})$ from Lemma 1.21. Indeed, for any $\chi, \chi_1, \chi_2 \in \mathrm{For}$ such that $\ulcorner \chi_1 \equiv \chi_2 \urcorner \in \mathbf{PL}$: $v_a(\Box\chi) = 1$ iff $\ulcorner p \equiv \chi \urcorner \in \mathbf{PL}$ iff $\ulcorner p \equiv \chi[{}^{\chi_1}/_{\chi_2}] \urcorner \in \mathbf{PL}$ iff $v_a(\Box\chi[{}^{\chi_1}/_{\chi_2}]) = 1$. Let $V_a\colon \mathrm{For} \to \{0,1\}$ be the unique extension of v_a by classical truth conditions for truth- value operators. By Lemma 1.21(1), $V_a \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

Notice that $V_a(\mathrm{sub}(\mathtt{X})) = \{1\}$. Indeed, suppose that $V_a(\Box(\varphi_1 \supset \varphi_2)) = 1 = V_a(\Box(\varphi_2 \supset \varphi_3))$. Then (i) $\ulcorner p \equiv (\varphi_1 \supset \varphi_2) \urcorner \in \mathbf{PL}$ and (ii) $\ulcorner p \equiv (\varphi_2 \supset \varphi_3) \urcorner \in \mathbf{PL}$. From (i): either both $\ulcorner \varphi_1 \equiv \neg p \urcorner \in \mathbf{PL}$ and $\ulcorner \varphi_2 \equiv p \urcorner \in \mathbf{PL}$, or both $\varphi_1 \in \mathbf{PL}$ and $\ulcorner \varphi_2 \equiv p \urcorner \in \mathbf{PL}$, or both $\ulcorner \varphi_1 \equiv \neg p \urcorner \in \mathbf{PL}$ and $\ulcorner \neg\varphi_2 \urcorner \in \mathbf{PL}$. From (ii): either both $\ulcorner \varphi_2 \equiv \neg p \urcorner \in \mathbf{PL}$ and $\ulcorner \varphi_3 \equiv p \urcorner \in \mathbf{PL}$, or both $\varphi_2 \in \mathbf{PL}$ and $\ulcorner \varphi_3 \equiv p \urcorner \in \mathbf{PL}$, or both $\ulcorner \varphi_2 \equiv \neg p \urcorner \in \mathbf{PL}$ and $\ulcorner \neg\varphi_3 \urcorner \in \mathbf{PL}$. Contradiction.

Moreover, $V_a(\mathtt{T}) = 0$, since $V_a(\Box p)$ and $V_a(p) = 0$.

Now, as in Lemma 4.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a, we build a self-associate t-normal rte-model $\langle w, \{w,a\}, V\rangle$ for $\{\Box(\mathtt{X})\}$, since $V(\mathrm{sub}(\mathtt{X}), w) = \{1\}$. Since $V(\Box(\mathtt{T}), w) = 0$, so $\Box(\mathtt{T}) \notin \mathbf{S0.5_{rte}}[\Box\mathtt{T}]$, by Theorem 5.7(8). ⊣

Fact 6.8. $\Box(\mathtt{K}) \notin \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{X}, \Box\mathtt{R}]$.

Proof: Let $w \neq a$ and $A := \{w, a\}$. Let $v_a\colon \mathrm{PAt} \to \{0,1\}$ such that $v_a(p) = 1 = v_a(q)$ and for any $\varphi \in \mathrm{For}$: $v_a(\Box\varphi) = 1$ iff either $\varphi =$ 'p', or $\varphi =$ '$p \wedge p$', or $\varphi =$ '$p \supset q$', or $\varphi =$ '$(p \supset q) \wedge (p \supset q)$'.

Let V_a be the unique extension of v_a by classical truth conditions for truth-value operators. Then $V_a(\Box q) = 0$ and $V_a(\mathrm{sub}(\mathtt{T})) = V_a(\mathrm{sub}(\mathtt{X})) = V_a(\mathrm{sub}(\mathtt{R})) = \{1\}$.

Now, as in Lemma 3.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a, we build a self-associate t-normal model $\langle w, \{w,a\}, V\rangle$. By Theorem 3.4, $V(\mathrm{sub}(\mathtt{T}), w) = V(\mathrm{sub}(\mathtt{X}), w) = V(\mathrm{sub}(\mathtt{C}), w) = V(\mathrm{sub}(\mathtt{M}), w) = \{1\}$. So we have a model for $\{\Box(\mathtt{T}), \Box(\mathtt{X}), \Box(\mathtt{R})\}$, in which $V(\Box(\mathtt{K}), w) = 0$. Thus, by Fact 3.13(4), $\Box(\mathtt{K})$ does not belong to $\mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{X}, \Box\mathtt{R}]$. $\dashv$

If we are only interested in formulae $\Box(\mathtt{K})$, $\Box(\mathtt{X})$ and $\Box(\mathtt{T})$, as in the case of $\mathbf{S0.9^\circ}$, $\mathbf{S0.9}$, $\mathbf{S1^\circ}$ and $\mathbf{S1}$, by the above facts and Fact 2.2 we obtain.

Corollary 6.9. 1. $\mathbf{S0.5^\circ}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{K}]$,
$\mathbf{S0.5^\circ}[\Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{X}]$,
$\mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}]$,
$\mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$.

2. $\mathbf{S0.5^\circ} \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}]$,
$\mathbf{S0.5^\circ} \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{K}, \Box\mathtt{X}]$,
$\mathbf{S0.5^\circ}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}]$,
$\mathbf{S0.5^\circ}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}]$.

3. $\mathbf{S0.5^\circ} \subsetneq \mathbf{S0.5^\circ_{rte}} \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{X}]$.

4. $\mathbf{S0.5^\circ}[\Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5^\circ_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$.

5. $\mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.9}$.

6. $\mathbf{S0.5} \subsetneq \mathbf{S0.5}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5}[\Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$,
$\mathbf{S0.5} \subsetneq \mathbf{S0.5}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5}[\Box\mathtt{K}, \Box\mathtt{X}]$,
$\mathbf{S0.5}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}]$,
$\mathbf{S0.5}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}]$.

7. $\mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$.

8. $\mathbf{S0.5} \subsetneq \mathbf{S0.5_{rte}} \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$,
$\mathbf{S0.5}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{K}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{K}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{T}, \Box\mathtt{X}]$,
$\mathbf{S0.5}[\Box\mathtt{X}] \subsetneq \mathbf{S0.5_{rte}}[\Box\mathtt{X}]$.

Fact 6.7 can be strengthen to the following:

FACT 6.10. $\Box(\mathtt{T}) \notin \mathbf{S0.5}_{\mathbf{rte}}[\Box\mathbf{S0.5^\circ}]$.

PROOF: Let $w \neq a$, $A := \{w, a\}$. Let $v_a\colon \mathrm{PAt} \to \{0,1\}$ be any assignment such that $v_a(p) = 0$ and for any $\chi \in \mathrm{For}$:

$$v_a(\Box\chi) = \begin{cases} 1 & \text{if } \chi \in \mathbf{PL}_{\mathrm{rte}} \\ 1 & \text{if } \ulcorner p \supset \chi \urcorner \in \mathbf{PL}_{\mathrm{rte}} \\ 0 & \text{otherwise} \end{cases}$$

The assignment v_a satisfies the condition $(\star_{\mathrm{PAt}})$ from Lemma 1.21. Indeed, we have two cases. In the first one the situation is analogous to that of in the proof of Fact 6.2. In the second one for $\chi, \chi_1, \chi_2 \in \mathrm{For}$ such that $\ulcorner \chi_1 \equiv \chi_2 \urcorner \in \mathbf{PL}$: $v_a(\Box\chi) = 1$ iff $\ulcorner p \supset \chi \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ iff $\ulcorner p \supset \chi[{}^{\chi_1}/_{\chi_2}] \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ iff $v_a(\Box\chi[{}^{\chi_1}/_{\chi_2}]) = 1$. Let $V_a\colon \mathrm{For} \to \{0,1\}$ be the unique extension of v_a by classical truth conditions for truth-value operators. By Lemma 1.21(1), $V_a \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

Notice that $V_a(\mathrm{sub}(\mathtt{K})) = \{1\}$. Indeed, suppose that $V_a(\Box(\varphi_1 \supset \varphi_2)) = 1 = V_a(\Box\varphi_1)$. Then both either $\ulcorner \varphi_1 \supset \varphi_2 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ or $\ulcorner p \supset (\varphi_1 \supset \varphi_2) \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ and either $\varphi_1 \in \mathbf{PL}_{\mathrm{rte}}$ or $\ulcorner p \supset \varphi_1 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$. Hence, either (i) both $\varphi_1 \in \mathbf{PL}_{\mathrm{rte}}$ and $\ulcorner \varphi_1 \supset \varphi_2 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$, or (ii) both $\ulcorner p \supset \varphi_1 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ and $\ulcorner \varphi_1 \supset \varphi_2 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$, or (iii) both $\varphi_1 \in \mathbf{PL}_{\mathrm{rte}}$ and $\ulcorner p \supset (\varphi_1 \supset \varphi_2) \urcorner \in \mathbf{PL}_{\mathrm{rte}}$, or (iv) both $\ulcorner p \supset \varphi_1 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$ and $\ulcorner p \supset (\varphi_1 \supset \varphi_2) \urcorner \in \mathbf{PL}_{\mathrm{rte}}$. Hence, by Lemma 1.21(3), $\varphi_2 \in \mathbf{PL}_{\mathrm{rte}}$ or $\ulcorner p \supset \varphi_2 \urcorner \in \mathbf{PL}_{\mathrm{rte}}$, and consequently, $V_a(\Box\varphi_2) = 1$.

Thus, $V_a(\mathbf{S0.5^\circ}) = \{1\}$, since all theses of $\mathbf{S0.5^\circ}$ are derivable from $\mathbf{PL}$, $\Box\mathbf{PL}$ and $\mathrm{sub}(\mathtt{K})$ by (MP), and for all formulae derivable in this way the function V_a takes the value 1.

We also have that $V_a(\mathtt{T}) = 0$, since $V_a(\Box p)$ and $V_a(p) = 0$.

Now, as in Lemma 4.2(1), for any assignment $v_w\colon \mathrm{At} \to \{0,1\}$ and V_a, we build a self-associate t-normal rte-model $\langle w, \{w,a\}, V\rangle$ for $\Box\mathbf{S0.5^\circ}$ such that $V(\Box(\mathtt{T}), w) = 0$. ⊣

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