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A SEMANTICAL PROOF OF THE ADMISSIBILITY OF THE RULE ASSERTION IN SOME RELEVANT AND MODAL LOGICS

Abstract

It is proved that the rule assertion is admissible in some relevant and modal logics sound and complete in respect of ternary relational models of a certain type.

1. Introduction

The rule Assertion (Asser) is the following:

$$\text{Asser. From } A \text{ to infer } (A \to B) \to B$$

The rule Asser is not derivable in Lewis' modal logic S5. Consider the following set of matrices MSI (2 and 3 are designated values).

Matrix set I (MSI):

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	2
2	0	0	3	3	1
3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

We have:

PROPOSITION 1. *The rule Asser is not derivable in S5.*

PROOF: MSI verifies S5. That is, MSI satisfies the axioms of S5 and the rule Modus ponens formulated in [1] (S5 is axiomatized without $\Box$ and $\Diamond$ as primitive connectives). But it falsifies Asser when $v(A) = v(B) = 2$. $\Box$

Nevertheless, it will be shown that Asser is *admissible* in a series of relevant and modal logics including EW_+ plus the contraposition axiom (Con)

$$\text{Con. } (A \to B) \to (\neg B \to \neg A)$$

and the De Morgan axioms (DM_1 and DM_2)

$$DM_1.\ (\neg A \wedge \neg B) \to \neg(A \vee B)$$
$$DM_2.\ \neg(A \wedge B) \to (\neg A \vee \neg B)$$

The logic EW_+ is the contractionless positive fragment of the logic of entailment E. And the logics in the series referred to above have to present a certain structure.

The proof here provided is based upon the models for E defined in [3].

2. A-models

The expression "A-model" is intended to abbreviate "model in which the rule Assertion can be proved admissible". We begin by defining A-models (cf. [3], p. 411).

DEFINITION 1. *An* A*-model is a structure* $(K, O, P, R, *, \vDash)$ *where* O *and* P *are subsets of* K*,* R *is a ternary relation on* K *and* $*$ *a unary operation on* K *subject to the following definitions and postulates for all* $a, b, c \in K$*:*

d1. $a \leq b =_{df} (\exists x \in O)Rxab$
d2. $a = b =_{df} (a \leq b \ \ \&\ \ b \leq a)$
d3. $R^2abcd =_{df} (\exists x \in K)(Rabx \ \ \&\ \ Rxcd)$

P1. $a \leq a$
P2. $(a \leq b \ \ \& \ \ Rbcd) \Rightarrow Racd$
P3. $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \ \& \ \ Rbxd)$
P4. $(\exists x \in P)Raxa$
P5. $(a \in P \ \ \& \ \ Rabc) \Rightarrow b \leq c$
P6. $Rabc \Rightarrow Rac * b*$

On the other hand, $\vDash$ *is a relation from* K *to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables* p*, wff* A*,* B *and* $a \in K$*:*

(i). $(a \leq b \ \ \& \ \ a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ *iff* $a \vDash A$ *and* $a \vDash B$
(iii). $a \vDash A \vee B$ *iff* $a \vDash A$ *or* $a \vDash B$
(iv). $a \vDash A \to B$ *iff for all* $b, c \in K$ $(Rabc \ \ \& \ \ b \vDash A) \Rightarrow c \vDash B$
(v). $a \vDash \neg A$ *iff* $a* \nvDash A$

Then, validity is defined as follows.

Definition 2 ($\mathcal{A}$-validity). *Let* $\mathcal{A}$ *be a class of* $\mathtt{A}$*-models,* A *is* $\mathcal{A}$*-valid* ($\vDash_{\mathcal{A}} A$) *iff* $a \vDash A$ *for all* $a \in O$ *in all* $\mathtt{A}$*-models.*

Now, the following holds for any model in any class $\mathcal{A}$ of $\mathtt{A}$-models.

Proposition 2. *For any* a*,* $b \in K$ *and wff* A*,* $(a \leq b \ \ \& \ \ a \vDash A) \Rightarrow b \vDash A$*.*

Proof: Induction on the length of A. The conditional case is proved with P2 and the negation case with $a \leq b \Rightarrow b* \leq a*$, an immediate consequence of P6 (cf. [3], p.412). □

Remark 1. *We note that the postulate*

$$P7. \ R^2abcd \Rightarrow (\exists x \in K)(Rbcx \ \ \& \ \ Raxd)$$

holds in any $\mathtt{A}$*-model (cf. [2]).*

Proposition 3. *For any wff* A*,* B*,* $\vDash_{\mathcal{A}} A \to B$ *iff* $(a \vDash A \Rightarrow a \vDash B)$ *for any* $a \in K$ *in all models in* $\mathcal{A}$*.*

PROOF: By P1, d1 and Proposition 2 (cf. [3], p.412). □

3. P-validity

We set:

DEFINITION 3. *Let $\mathcal{A}$ be a class of* $\mathtt{A}$*-models and A a wff. A is $P_{\mathcal{A}}$-valid ($\vDash_{P_{\mathcal{A}}} A$) iff $a \vDash A$ for all $a \in P$ in all* $\mathtt{A}$*-models.*

Consider now the rules *Adjunction* (Adj)

Adj. From A and B to infer $A \wedge B$

and *Modus ponens* (MP)

MP. From A and $A \to B$ to infer B

Let $\mathcal{A}$ be a class of $\mathtt{A}$-models. We have:

LEMMA 1. *Adj. preserves $P_{\mathcal{A}}$-validity. That is, for any wff A, B, if $\vDash_{P_{\mathcal{A}}} A$ and $\vDash_{P_{\mathcal{A}}} B$, then $\vDash_{P_{\mathcal{A}}} A \wedge B$.*

PROOF: Immediate by clause (ii) in Definition 1. □

LEMMA 2. *MP preserves $P_{\mathcal{A}}$-validity. That is, for any wff A, B, if $\vDash_{P_{\mathcal{A}}} A \to B$ and $\vDash_{P_{\mathcal{A}}} A$, then $\vDash_{P_{\mathcal{A}}} B$.*

PROOF: Let $a \in P$ in an arbitrary model in $\mathcal{A}$. By P4, (1) $Raxa$ for some $x \in P$. By hypothesis, (2) $a \vDash A \to B$ and (3) $x \vDash A$. Therefore, (4) $a \vDash B$ by (1), (2), (3) and clause (iv) in Definition 1. □

LEMMA 3. *For any wff A, B, $\vDash_{\mathcal{A}} A \to B \Rightarrow \vDash_{P_{\mathcal{A}}} A \to B$.*

PROOF: Suppose, for reductio, that there is $a \in P$ in some model $\mathcal{A}$ such that for wff A, B, (1) $\vDash_{\mathcal{A}} A \to B$ but (2) $a \nvDash A \to B$. By clause (iv) (Definition 1), there are b, $c \in K$ such that (3) $Rabc$, (4) $b \vDash A$ and (5) $c \nvDash B$. By (3) and P5, (6) $b \leq c$. So, (7) $c \vDash A$ by (4), (6) and Proposition 2. Then, (8) $c \vDash B$ by (1), (7) and Proposition 3. But (5) and (8) contradict each other. □

4. Admissibility of Asser

Let L be a propositional language with the connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). And let S be a logic defined upon L. S is defined in a Hilbert-style way, all axioms being of an implicative form and Adj and MP the sole rules of derivation (A is of implicative form iff A is of the form $B \rightarrow C$ where B and C are wff). Furthermore, let $\mathcal{A}$ be a class of A-models and S be sound and complete with respect to $\mathcal{A}$. That is, $\vdash_{\mathrm{S}} A$ iff $\vDash_{\mathcal{A}} A$, where $\vdash_{\mathrm{S}} A$ is understood in the standard way, i.e., $\vdash_{\mathrm{S}} A$ iff there is a finite sequence of wff $B_1, ..., B_n$ such that each $B_i (1 \leq i \leq n)$ is either an axiom or the result of applying Adj or MP to two previous formulas in the sequence, and A is B_i. And $\vDash_{\mathcal{A}} A$ iff $a \vDash A$ for all $a \in O$ in each model in $\mathcal{A}$ (cf. Definition 2). Then, it is proved:

Lemma 4. *For any wff A, if* $\vdash_S A$ *then* $\vDash_{P_{\mathcal{A}}} A$.

Proof: Induction on the length of the proof of A. And it is immediate by Lemma 1, Lemma 2 and Lemma 3: all axioms are $P_{\mathcal{A}}$-valid (Lemma 3) and rules Adj and MP preserves $P_{\mathcal{A}}$-validity (Lemma 1 and Lemma 2). □

Finally, we have:

Theorem 1 (Admissibility of Asser). *Let S be a logic defined upon the propositional language L, as indicated above. Then, Asser is admissible in S. That is, if A is a theorem of S, then* $(A \rightarrow B) \rightarrow B$ *is a theorem of S.*

Proof: Suppose (1) $\vdash_{\mathrm{S}} A$ and (2) $a \vDash A \rightarrow B$ for $a \in K$ in a given model in $\mathcal{A}$. By P4, there is some $x \in P$ such that (3) *Raxa*. By (1) and Lemma 4, (4) $x \vDash A$. So, (5) $a \vDash B$ by (2), (3), (4) and clause (iv) in Definition 1. Then, $\vDash_{\mathcal{A}} (A \rightarrow B) \rightarrow B$ by (2), (5) and Proposition 3. Finally, $\vdash_{\mathrm{S}} (A \rightarrow B) \rightarrow B$ by completeness of S. □

5. $\mathrm{EW_M}$, the logic sound and complete with respect to the class $\mathcal{A}$ of minimal definable A-models

We set:

Definition 4 ($\mathrm{EW_M}$-models). *An* EW_M*-model is a structure* $(K, O, P, R, *, \vDash)$ *where* $K, O, P, R, *, \vDash$ *are defined exactly as in Definition 1.*

Consider now the following logic EW_M.
Axioms

A1. $A \to A$
A2. $(A \to B) \to [(B \to C) \to (A \to C)]$
A3. $(A \wedge B) \to A \;/\; (A \wedge B) \to B$
A4. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A5. $[[(A \to A) \wedge (B \to B)] \to C] \to C$
A6. $A \to (A \vee B) \;/\; B \to (A \vee B)$
A7. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A8. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A9. $(A \to B) \to (\neg B \to \neg A)$
A10. $(\neg A \wedge \neg B) \to \neg(A \vee B)$
A11. $\neg(A \wedge B) \to (\neg A \vee \neg B)$

Rules: MP and Adj.

EW_M could intuitively be described as the result of introducing negation by means of A9, A10 and A11 in EW_+, the contractionless positive fragment of the logic of entailment E. Now, an easy consequence of the soundness and completeness theorems for E proved in [3] is the following.

PROPOSITION 4. $\vdash_{EW_M} A$ *iff* $\vDash_{EW_M} A$.

That is, EW_M is sound and complete in respect of EW_M-models.

On the other hand, EW_M-models form a class $\mathcal{A}$ of A-models (indeed, the class of minimal definable A-models in the sense that an EW_M-model is an A-model, but A-models to which EW_M-models are not equivalent can be defined —as shown in the following section). Therefore, given the formulation of EW_M, we have:

PROPOSITION 5. *Asser is admissible in EW_M.*

PROOF: By Theorem 1, given that EW_M is sound and complete in respect of a class of A-models (EW_M-models), all its axioms are implicative formulas, and MP and Adj are the sole rules of derivation. □

6. Some extensions of $\mathbf{EW_M}$

It follows from Theorem 1 that if S is a logic fulfilling the requirements for applying Theorem 1, then Asser is admissible in S. In this section we shall consider some extensions of $\mathrm{EW_M}$ in which Asser is admissible.

Consider the following axioms and semantical postulates.

A12. $[A \to (A \to B)] \to (A \to B)$
A13. $A \to (A \to A)$
A14. $B \to (A \to A)$
A15. $A \to \neg\neg A$
A16. $\neg\neg A \to A$
A17. $(A \to \neg A) \to \neg A$
PA12. $Rabc \Rightarrow R^2abbc$
PA13. $Rabc \Rightarrow a \le c$ or $b \le c$
PA14. $Rabc \Rightarrow b \le c$
PA15. $a \le a**$
PA16. $a** \le a$
PA17. $Raa*a$

We have:

Proposition 6. *Given the logic EW_M and EW_M-models, PA12, PA13, PA14, PA15, PA16 and PA17 are the corresponding postulates (cp) to A12, A13, A14, A15, A16 and A17, respectively. That is, given the logic EW_M, each postulate can be shown to hold in the corresponding canonical model by using the respective axiom, and given EW_M-models, each axiom can be shown to be valid in the corresponding extended models by using the respective postulate.*

Proof: It can be found in (or easily derived from) [2]. □

Proposition 7. *Let S be any extension of EW_M with any selection of A12-A17, and Σ-models be defined by adding the cp to the axiom(s) added. Then, (1) any Σ-model is an* A*-model; (2) S is sound and complete in respect of Σ-models.*

Proof: (1) It is obvious as each model is defined by restricting $\mathrm{EW_M}$-models. (2) Immediate by Proposition 4 and Proposition 6. □

Asser is admissible in any of the logics in Proposition 7. That is:

Proposition 8. *Let S be any extension of EW_M with any selection of A12-A17. Then Asser is admissible in S.*

Proof: Given that all axioms of S are implicative formulas and that MP and Adj are the sole rules of inference, Proposition 8 follows from Proposition 7(2) and Theorem 1. □

We note that among the logics described in Proposition 8, the logic of entailment E ($\mathrm{EW_M}$ plus A12, A15, A16 and A17) and Lewis' S4 (E plus A14) are to be found (S4 is axiomatized in [1] without $\Box$ and $\Diamond$ as primitive connectives and without Adj as a primitive rule. But Adj is, of course, admissible in both S4 and S5).

The paper is ended with two remarks. The first is contained in the following proposition.

Proposition 9. *Let S be any extension of EW_M meeting the conditions of Theorem 1. That is, S is sound and complete with respect to a class of* $\mathbf{A}$*-models; all axioms of S are implicative formulas and MP and Adj are the sole rules of derivation. Furthermore, A14 is derivable in S. Then, rule K, i.e.,*

$$K.\ \textit{From } A \textit{ to infer } B \to A$$

is admissible in S.

Proof: Suppose (1) $\vdash_{\mathrm{S}} A$. By Theorem 1, Asser is admissible in S. So, (2) $\vdash_{\mathrm{S}} (A \to A) \to A$. Then, (3) $\vdash_{\mathrm{S}} B \to A$ follows by (2), A2, A14 and MP. □

The second remark is the following. Suppose that S is an extension of $\mathrm{EW_M}$ in which either not all axioms are implicative formulas or else S has one or more rules of derivation in addition to MP and Adj. Then, it may be the case that Asser is not admissible in S. We shall provide an example. Let $\mathrm{EW_{MPEM}}$ be the result of adding the Principle of excluded middle (PEM)

$$\text{PEM. } A \vee \neg A$$

to $\mathrm{EW_M}$. And consider the following postulate

$$\mathrm{P_{PEM}}.\ a \in O \Rightarrow a* \leq a$$

We have:

Proposition 10. *P_{PEM} is the cp to PEM.*

Proof: Similar to that of Proposition 6. □

And, consequently:

Proposition 11. *EW_{MPEM} is sound and complete in respect of EW_{MPEM}-models. (An EW_{MPEM}-model is a EW_M-model in which P_{PEM} holds).*

Proof: Immediate by Proposition 4 and Proposition 10. □

Now, an $\mathrm{EW_{MPEM}}$-model is clearly an A-model, but as PEM is not of implicative form, it turns out that the following is provable.

Proposition 12. *Asser is not admissible in EW_{MPEM}.*

Proof: Consider the following set of matrices MSII (all values but 0 are designated):

Matrix set II (MSII):

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	2
2	0	0	3	3	1
3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Then, the logic $\mathrm{EW_{MPEM}}$ is verified by MSII, but $[(A \vee \neg A) \to (A \vee \neg A)] \to (A \vee \neg A)$ is falsified when $v(A) = 1$. □

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