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SEMANTICAL INVESTIGATIONS ON SOME
WEAK MODAL LOGICS. Part I

Abstract
In this paper we examine weak logics similar to $S_{0.5}[\Box \Phi]$, where $\Phi \subseteq S_{0.5}$. We also examine their versions (one of which is $S_{0.5_{\text{rte}}}[\Box \Phi]$) that are closed under replacement of tautological equivalents (rte). We have that: $S_{0.5_{\text{rte}}}[\Box (\text{K})] \subseteq S_{0.9}$, $S_{0.5_{\text{rte}}}[\Box (\text{X})] \subseteq S_{1}$, and in general, if $\Phi \subseteq E_{1}$, then $S_{0.5_{\text{rte}}}[\Box \Phi] \subseteq S_{2}$.

In the second part we shall give simplified semantics for these logics, formulated by means of some Kripke-style models. We shall also prove that the logics in question are determined by some classes of these models.

Key words: Very weak modal logics, simplified Kripke-style semantics.

1. Preliminaries

1.1. Basic notions

Modal formulae are formed in the standard way from the set $\text{At}$ of propositional letters: ‘$p$’, ‘$q$’, ‘$r$’, ‘$p_0$’, ‘$p_1$’, ‘$p_2$’, . . . ; truth-value operators: ‘$\neg$’, ‘$\lor$’, ‘$\land$’, ‘$\supset$’, and ‘$\equiv$’ (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); the modal operator ‘$\Box$’ (necessity; the possibility sign ‘$\Diamond$’ is the abbreviation of ‘$\neg \Box \neg$’); and brackets. Let $\text{For}$ be the set of all modal formulae.

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In original Lewis’ works (see e.g. [5]) the primitive modal operator is the possibility sign ‘◊’. The necessity sign ‘□’ is the abbreviation of ‘¬◊¬’. Moreover, ◊φ ≺ ψ (the strict implication) was used as an abbreviation of ◊¬(φ ∧ ¬ψ).

In this paper, as in [4], the primitive modal operator is ‘□’ and ◱ψ is an abbreviation of □(ψ ⊃ phi). Moreover, similarly as in [5, 4], the strict equivalence ◱ψ ≻ ≻ ϕ is an abbreviation of ◱(ϕ ≺ ψ) ∧ (ψ ≺ φ).

For any formula ϕ let sub(ϕ) be the set of all instances of ϕ. For any set Φ of formulae we put: sub(Φ) := \bigcup ϕ∈Φ sub(ϕ), □ Φ := {⌜□ϕ⌝: ϕ ∈ Φ} and ◊ Φ := {⌜◊ϕ⌝: ϕ ∈ Φ}.

Let Taut be the set of all classical tautologies (without the modal operator). We put \top := ‘p ⊃ p’. Moreover, let PL be the set of modal formulae which are instances of classical tautologies. Of course, PL = sub(Taut).

A formula ϕ is propositionally atomic iff ϕ ∈ At or ϕ ∈ □ For. Let PAt be the set of all propositionally atomic formulae, i.e. PAt := At ∪ □ For.

Let Valcl be the set of all valuations V: For → {0, 1} which preserve classical truth conditions for truth-value operators.

Lemma 1.1. 1. V ∈ Valcl iff for some assignment v: PAt → {0, 1}, V is the unique extension of v by classical truth conditions for truth-value operators.

2. For any ϕ ∈ For: ϕ ∈ PL iff for every assignment v: PAt → {0, 1} we have that V(ϕ) = 1, where V is the unique extension of v by classical truth conditions for truth-value operators.

3. For any ϕ ∈ For: ϕ ∈ PL iff V(ϕ) = 1, for any V ∈ Valcl.

For any Ψ ⊆ For and ϕ ∈ For we write Ψ ⊨ PL ϕ iff for any V from Valcl: if V |Ψ |= 1, then V(ϕ) = 1. Of course, ψ ⊨ PL ϕ iff for some {ψ1, ..., ψn} ⊆ Ψ, n ≥ 0, we have that ◊(ψ1 ∧ ... ∧ ψn) ⊃ ϕ ∈ PL. We also write Ψ |= PL Φ iff Ψ |= PL ϕ, for any ϕ ∈ Φ.

A set Σ of modal formulae is a modal system if PL ⊆ Σ and Σ is closed under the rule of detachment for ‘⊃’ (modus ponens), i.e., for any ϕ, ψ ∈ For:

\[ \text{if } \varphi \text{ and } □ \varphi ⊃ \psi \text{ are members of } \Sigma, \text{ so is } \psi. \]

A set of modal formulae is a logic iff it is a modal system and it is closed under the rule of uniform substitution. Of course, PL is the smallest modal system and it is a logic.
For any modal system $\Sigma$, any $\Psi \subseteq \text{For}$ and any $\varphi \in \text{For}$: $\varphi$ is *deducible from $\Psi$ in $\Sigma$* (written: $\Psi \vdash_{\Sigma} \varphi$) iff for some $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$, $n \geq 0$, we have that $\Gamma (\psi_1 \land \cdots \land \psi_n) \supset \varphi^\gamma \in \Sigma$. Of course, $\models_{\text{PL}} \supset \models_{\Sigma} \varphi$.

Moreover, $\Sigma \vdash_{\Sigma} \varphi$ iff $\varphi \in \Sigma$, and

$\Sigma \vdash_{\Sigma} \varphi$ iff $\varphi \in \Sigma$ iff $\emptyset \vdash \varphi$.

A system $\Sigma$ is *consistent* iff $\Sigma \neq \text{For}$; equivalently in the light of propositional logic $\text{PL}$, iff '$p \land \neg p$' does not belong to $\Sigma$.

To simplify notation of logics we use the following code. If $\Lambda$ is a logic and $\Phi \subseteq \text{For}$, then $\Lambda[\Phi]$ denotes the smallest logic which includes the set $\Lambda \cup \Phi$. We write $\Lambda[\varphi_1, \ldots, \varphi_n]$ instead of $\Lambda[\{\varphi_1, \ldots, \varphi_n\}]$, and $\Lambda[\Phi_1, \ldots, \Phi_n]$ instead of $\Lambda[\Phi_1 \cup \cdots \cup \Phi_n]$.

We say that a modal system is *congruential* (or *classical*) iff it is closed under the following rule of congruence, i.e., for any $\varphi, \psi \in \text{For}$:

$$\Gamma \varphi \equiv \psi^\gamma \in \Sigma, \text{ then } \Gamma \Box \varphi \equiv \Box \psi^\gamma \in \Sigma.$$  \hfill (RE)

**Fact 1.2.** A modal system $\Sigma$ is congruential iff it is closed under replacement, i.e., for any $\varphi, \psi, \chi \in \text{For}$:

if $\Gamma \varphi \equiv \psi^\gamma \in \Sigma$ and $\chi \in \Sigma$, then $\chi[^\varphi/\psi] \in \Sigma$, \hfill (RRE)

or equivalently:

if $\Gamma \varphi \equiv \psi^\gamma \in \Sigma$, then $\Gamma \chi[^\varphi/\psi] \equiv \chi^\gamma \in \Sigma$, \hfill (RRE′)

where $\chi[^\varphi/\psi]$ is any formula that results from $\chi$ by replacing zero, one or more occurrences of $\varphi$, in $\chi$, by $\psi$.

A modal system $\Sigma$ is called *monotonic* iff $\Sigma$ is closed under the following rule of monotonicity, i.e., for any $\varphi, \psi, \chi \in \text{For}$:

if $\Gamma \varphi \supset \psi^\gamma \in \Sigma$ then $\Gamma \Box \varphi \supset \Box \psi^\gamma \in \Sigma$. \hfill (RM)

**Fact 1.3.** A modal system is monotonic iff it is congruential and contains all instances of the following formula:

$$\Box (p \land q) \supset (\Box p \land \Box q)$$ \hfill (M)

A modal system $\Sigma$ is called *regular* iff $\Sigma$ is closed under the following regularity rule, i.e., for any $\varphi, \psi, \chi \in \text{For}$:

if $\Gamma (\varphi \land \psi) \supset \chi^\gamma \in \Sigma$, then $\Gamma (\Box \varphi \land \Box \psi) \supset \Box \chi^\gamma \in \Sigma$. \hfill (RR)
FACT 1.4. For any modal system the following conditions are equivalent:
(a) the system is regular,
(b) it is monotonic and contains all instances of
\[ \Box(p \supset q) \supset (\Box p \supset \Box q) \] (K)
(c) it is monotonic and contains all instances of
\[ (\Box p \land \Box q) \supset \Box(p \land q) \] (C)
(d) it is monotonic and contains all instances of
\[ (\Box(p \supset q) \land \Box(q \supset r)) \supset \Box(p \supset r) \] (X)
(e) it is congruential and contains all instances of
\[ \Box(p \land q) \equiv (\Box p \land \Box q) \] (R)

To simplify notation of logics we use the following code. If \( \Lambda \) is a regular
logic and \( \Phi \subseteq \text{For} \), then \( \Lambda \oplus \Phi \) denotes the smallest regular logic which
includes the set \( \Lambda \cup \Phi \). We write \( \Lambda \oplus \varphi_1 \ldots \varphi_n \) instead of \( \Lambda \oplus \{\varphi_1, \ldots, \varphi_n\} \).
\text{C2} \ is the smallest regular logic and \text{E2} \ is the smallest regular logic
which contains \( (T) \), i.e. \text{E2} = \text{C2} \oplus (T).

We say that a modal system \( \Sigma \) is normal iff it contains all instances of \( (K) \) and is closed under the following rule:
\[ \text{if } \varphi \in \Sigma, \text{ then } \Box \varphi \in \Sigma. \] (RN)

FACT 1.5. For any modal system the following conditions are equivalent:
(a) it is normal,
(b) it is regular and contains \( \Box \top \),
(c) it is congruential, contains \( \Box \top \) and includes \( \text{sub}(K) \).

By the above fact, if \( \Lambda \) is a normal logic, then \( \Lambda \oplus \Gamma \) is as well. Indeed, \( \Lambda \) is regular and contains \( \Box \top \). Hence \( \Lambda \oplus \Gamma \) is also regular and contains \( \Box \top \). So \( \Lambda \oplus \Gamma \) is normal.

In this paper we investigate some weak modal logics. For these logics
we are using the following lemmas.
Lemma 1.6. For any modal system $\Sigma$ which includes the following set

$$E_{PL} := \{ \Box \varphi \equiv \Box \psi \mid \varphi, \psi \in PL \} ,$$

1. $\Box \top \in \Sigma$ iff $\Box PL \subseteq \Sigma$.
2. If $\text{sub}(X) \subseteq \Sigma$, then $\text{sub}(K) \subseteq \Sigma$.

Proof: 1. For any $\tau \in PL$, $\Box \top \equiv \top \in PL$ and $\Box \tau \equiv \Box \top \in \Sigma$, since $E_{PL} \subseteq \Sigma$. Hence, by $PL$, also $\Box \tau \in \Sigma$, since $\Box \top \in \Sigma$.

2. For any $\varphi, \psi \in \text{For}$, $\Box \varphi \equiv (\top \supset \varphi) \in PL$ and $\Box \psi \equiv (\top \supset \psi) \in PL$. So if $E_{PL} \subseteq \Sigma$, then $\Box \varphi \equiv \Box(\top \supset \varphi)$ and $\Box \psi \equiv \Box(\top \supset \psi)$ belong to $\Sigma$. Moreover, if $\langle X \rangle \in \Sigma$, then $\Box(\Box(\top \supset \varphi) \land \Box(\top \supset \psi)) \supset \Box(\top \supset \psi) \in \Sigma$. Hence $\Box(\Box(\top \supset \varphi) \supset (\Box \varphi \supset \Box \psi)) \in \Sigma$, by $PL$.

Lemma 1.7 ([6]). For any modal system $\Sigma$: $\Sigma$ includes the following set

$$M_{PL} := \{ \Box \Box \varphi \supset \Box \psi \mid \varphi, \psi \in PL \}$$

iff $E_{PL} \subseteq \Sigma$ and $\text{sub}(\top \mid \top) \subseteq \Sigma$.

Lemma 1.8. For any modal system $\Sigma$ which includes $M_{PL}$:

$$\Box PL \subseteq \Sigma$$

iff $\Box \top \in \Sigma$ iff $\Sigma$ has some formula of the form $\Box \Box \varphi$.

Lemma 1.9 ([6]). For any modal system $\Sigma$ the following conditions are equivalent:

(a) $\Sigma$ includes the following set

$$R_{PL} := \{ \Box(\Box \varphi \land \Box \psi) \supset \Box \chi \mid (\varphi \land \psi) \supset \chi \in PL \} ,$$

(b) $M_{PL} \subseteq \Sigma$ and $\text{sub}(K) \subseteq \Sigma$,

(c) $M_{PL} \subseteq \Sigma$ and $\text{sub}(X) \subseteq \Sigma$,

(d) $M_{PL} \subseteq \Sigma$ and $\text{sub}(C) \subseteq \Sigma$,

(e) $E_{PL} \subseteq \Sigma$ and $\text{sub}(R) \subseteq \Sigma$.

Lemma 1.10. Fix any system $\Sigma$:

1. If $E_{PL} \subseteq \Sigma$, then $\Sigma$ contains all instances of the following formula

$$\Diamond p \equiv \neg \Box \neg p$$

(df $\Diamond$)
2. If $R_{PL} \subseteq \Sigma$, then $\Sigma$ contains all instances of the following formulae
\[
\diamond (p \lor q) \equiv (\diamond p \lor \diamond q) \quad \text{(R)}
\]
\[
\diamond (p \supset q) \equiv (\Box p \supset \diamond q) \quad \text{(R^{cc})}
\]

**Lemma 1.11.** For any modal system $\Sigma$:
1. If $E_{PL} \subseteq \Sigma$, then $\Sigma$ contains all instances of the following formula
\[
(p \prec q) \equiv \neg \diamond (p \land \neg q) \quad \text{(df'' \prec)}
\]
2. If $R_{PL} \subseteq \Sigma$, then $\Sigma$ contains all instances of
\[
(p \prec q) \equiv \Box (p \equiv q) \quad \text{(df'' \prec)}
\]

**Lemma 1.12.** For any modal system $\Sigma$:
1. If $\Sigma$ contains all instances of the following formula
\[
\Box p \supset p \quad \text{(T)}
\]
then $\Sigma$ is closed under the following rule
\[
\Box \varphi \supset \varphi \quad \text{if } \Box \varphi \in \Sigma, \text{ then } \varphi \in \Sigma. \quad \text{(RN,)}
\]
2. If $\Sigma$ is closed under (RN,), then $\Sigma$ is closed under the following rule of detachment for ‘$\prec$’ (strict version of modus ponens)
\[
\Box \varphi \prec \psi \in \Sigma \text{ and } \varphi \in \Sigma, \text{ then } \psi \in \Sigma. \quad \text{(SMP)}
\]
3. If $E_{PL} \subseteq \Sigma$ and $\Sigma$ is closed under (SMP), then $\Sigma$ is closed under (RN,).

**Proof:** For 3. Let $\Box \varphi \in \Sigma$. Since $E_{PL} \subseteq \Sigma$, we have that $\Box \varphi \equiv \Box (\top \supset \varphi) \in PL$, we have that $\Box \varphi \equiv \Box (\top \supset \varphi) \in PL$. Hence, by PL, $\Box (\top \supset \varphi) \in \Sigma$. So $\varphi \in \Sigma$, by (SMP) and PL.

\[\square\]

### 1.2. t-regular modal systems
In [6] a modal system is called $t$-regular iff it includes the set $R_{PL}$. Thus, the set $R_{PL}$ replaces the rule (RR) in the formulation of regular systems.
By definition, any modal system which includes some t-regular system, is also t-regular. So, if $\mathcal{A}$ is a t-regular logic, then $\mathcal{A}[\Phi]$ is. Moreover, every regular system is t-regular.

**Fact 1.13.** For any t-regular modal system $\Sigma$ the following conditions are equivalent:

(a) $\Box \top \in \Sigma$,

(b) $\Sigma$ contains all instances of the following formula

\[
\Box p \supset \Diamond p
\]

**Fact 1.14.** For any t-regular modal system $\Sigma$, if $\Sigma$ contains one of the following formula, then $\Sigma$ contains all the following formulae:\(^1\)

\[
\begin{align*}
\Box p & \supset (p \lor \Box q) \\
\Diamond q & \supset (\Box p \supset p) \\
\Diamond (q \supset q) & \supset (\Box p \supset p) \\
\neg \Box (q \land \neg q) & \supset (\Box p \supset p)
\end{align*}
\]

The logic $\mathcal{C}_1$ from [7] is the smallest t-regular system. $\mathcal{C}_1$ is a logic and $\mathcal{C}_1 := \text{PL}[R_{\text{PL}}]$. The logics $\mathcal{D}_1$ and $\mathcal{E}_1$ from [4] are respectively the smallest t-regular logics which contain $\Box \top$ and $\Box \text{PL}$, i.e. $\mathcal{D}_1 := \text{PL}[R_{\text{PL}}, \Box] = \mathcal{C}_1[\Box \top]$ and $\mathcal{E}_1 := \text{PL}[R_{\text{PL}}, \Box] = \mathcal{C}_1[\Box \top]$. We have that $\mathcal{C}_1 \subset \mathcal{D}_1 \subset \mathcal{E}_1$ and $\mathcal{C}_1 \subset \mathcal{C}_1[\Box \top] \subset \mathcal{E}_1$ (see [6]).

Notice that $\mathcal{E}_1 = \mathcal{C}_1[\Box \top, \Box \top]$. Indeed, from $\mathcal{C}_1$ and $\Box \top$ we obtain ‘$\Diamond (q \supset q)$’, and hence (T), by ($\Box \top$) and (MP).

### 1.3. t-normal modal systems

In [6] a modal system is called t-normal iff it contains all instances of (K) and includes the set $\Box \text{PL}$. Thus, the set $\Box \text{PL}$ replaces the rule (RN) in the formulation of normal systems. By definitions, any modal system which includes some t-normal system, is also t-normal. So, if $\mathcal{A}$ is a t-normal logic, then $\mathcal{A}[\Phi]$ is. Moreover, every normal system is t-normal.

\(^1\)The name ‘$\Box \top$’ is an abbreviation for ‘quasi-T’, because for normal logics with (T) (resp. ($\Box \top$)) we use reflexive (resp. quasi-reflexive) standard Kripke models.
By lemmas 1.6–1.9 we obtain:

**Lemma 1.15.** For any system the following conditions are equivalent:

(a) it is t-normal,

(b) it is t-regular and contains $\Box \top$,

(c) it is t-regular and contains some formula of the form $⌜\Box \varphi⌝$.

In [4] the logic $S_{0.5}$ is the smallest modal logic which includes $\Box \text{Taut}$, and contains $(K)$ and $(T)$. The logic $S_{0.5}^\circ$ is associated with Lemmon’s $S_{0.5}$. It is the smallest logic which includes $\Box \text{Taut}$ and contains $(K)$. The logic $S_{0.5}^\circ$ is the smallest modal logic which includes $\Box \text{Taut}$ and contains $(K)$. Of course, by uniform substitution, $S_{0.5}$ and $S_{0.5}^\circ$ include the set $\Box \text{PL}$; so $S_{0.5}^\circ$ is the smallest t-normal system, and $S_{0.5}$ is the smallest t-normal system which includes sub$(T)$. So we have that $S_{0.5}^\circ := PL[\Box \text{Taut}, K] = C1[\Box \top]$ and $S_{0.5} := PL[\Box \text{Taut}, K,T] = S_{0.5}^\circ[\top] = E1[\Box \top]$. It is the case that $S_{0.5}^\circ \subseteq S_{0.5}^\circ[D,T]$ and $S_{0.5} \subseteq S_{0.5}^\circ[D,T]$ (see e.g. [6] and Corollary 3.5 in the second part). Notice that $S_{0.5} = S_{0.5}^\circ[D,T]$.

By Lemma 1.12, the logic $S_{0.5}$ is closed under $(\text{RN}^*)$ and $(\text{SMP})$. However for any $\varphi \in \text{For}$: $⌜\Box \varphi⌝ \in S_{0.5}^\circ$ iff $\varphi \in PL$ iff $⌜\Box \varphi⌝ \in S_{0.5}$ (see Fact 3.8 in the second part). So $S_{0.5}^\circ$, $S_{0.5}^\circ[D]$ and $S_{0.5}^\circ[T]$ are also closed under $(\text{RN}_*)$ and $(\text{SMP})$.

### 1.4. Replacement for tautologous equivalents

We say that a modal system $\Sigma$ is an $\text{rte-system}$ iff $\Sigma$ is closed under replacement for tautological equivalents, i.e.:

$$\forall \varphi, \psi, \chi \in \text{For}: \text{ if } \varphi \equiv \psi \in \text{PL and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad \text{(rte)}$$

We consider the following sets of formulae:

---

2Notice that the rules $(\text{RN}_*)$ and $(\text{SMP})$ are not derivable in $S_{0.5}^\circ$, $S_{0.5}^\circ[D]$ and $S_{0.5}^\circ[T]$ in the following sense. We can consider $S_{0.5}^\circ$ (resp. $S_{0.5}^\circ[D]$; $S_{0.5}^\circ[T]$; $S_{0.5}$) as being axiomatized by axioms $\text{PL}$, sub$(K)$ (resp. plus sub$(D)$; sub$(T)$; sub$(T)$) and the sole rule $(\text{MP})$. Of course, in such axiomatic system of $S_{0.5}^\circ$ (resp. $S_{0.5}^\circ[D]$; $S_{0.5}^\circ[T]$), if $\varphi \notin \text{PL}$, then from $⌜\Box \varphi⌝$ we do not obtain $\varphi$, since $\text{PL}$, sub$(K)$, sub$(D)$, sub$(T)$ do not include $\Box \varphi \vdash \varphi$. 

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\[ \text{REP}_{PL} := \{ \gamma \chi \equiv \chi[\varphi/\psi] \subseteq \chi \in \text{For} \land \gamma \varphi \equiv \psi \subseteq \text{PL} \}, \]

\[ \text{PL}_{rte} := \{ \tau[\varphi/\psi_1, \ldots, \varphi_k/\psi_k] \subseteq \tau \in \text{PL} \land \gamma \varphi_1 \equiv \psi_1 \subseteq \text{PL}, \ldots, \gamma \varphi_k \equiv \psi_k \subseteq \text{PL} \}, \]

where \( \tau[\varphi/\psi_1, \ldots, \varphi_k/\psi_k] \) is any formula that results from \( \tau \) by replacing zero, one or more occurrences of \( \varphi_i \), in \( \tau \), by \( \psi_i \). Since \( \gamma \chi \equiv \chi \subseteq \text{PL} \), we have that: \( \text{REP}_{PL} \subseteq \text{PL}_{rte} \) and \( \Box \text{REP}_{PL} \subseteq \Box \text{PL}_{rte} \).

We will now focus on general properties of rte-systems.

**Lemma 1.16.** For any system \( \Sigma \) the following conditions are equivalent:

- (a) \( \Sigma \) is an rte-system,
- (b) \( \text{PL}_{rte} \subseteq \Sigma \),
- (c) \( \text{REP}_{PL} \subseteq \Sigma \).

1. \( \Sigma \) is closed under the following replacement

\[ \forall \varphi, \psi, \chi \in \text{For} : \text{if } \gamma \varphi \equiv \psi \subseteq \text{PL}, \text{ then } \gamma \chi \equiv \chi[\varphi/\psi] \subseteq \Sigma. \]

**Proof:**
- \((a) \Rightarrow (b)\) If \( \gamma \varphi \equiv \psi \subseteq \text{PL}, i = 1, \ldots, k, \) and \( \tau \in \text{PL} \subseteq \Sigma \) then \( \tau[\varphi/\psi_i] \subseteq \Sigma, \ldots, \tau[\varphi/\psi_1, \ldots, \varphi_k/\psi_k] \subseteq \Sigma \), by (rte). Thus, \( \text{PL}_{rte} \subseteq \Sigma \).
- \((b) \Rightarrow (c)\) By the fact that \( \text{REP}_{PL} \subseteq \text{PL}_{rte} \).
- \((c) \Rightarrow (a)\) If \( \gamma \varphi \equiv \psi \subseteq \text{PL}, \) then \( \gamma \chi \equiv \chi[\varphi/\psi] \subseteq \text{REP}_{PL} \subseteq \Sigma \).

Moreover, if \( \chi \in \Sigma \), then \( \chi[\varphi/\psi] \subseteq \Sigma \), by (PL).

- \((c) \Rightarrow (d)\) Obvious.

- \((d) \Rightarrow (e)\) Suppose that \( \gamma \varphi \equiv \psi \subseteq \text{PL} \). First we consider the possibility that \( \chi = \varphi \). Then \( \chi[\varphi/\psi] = \varphi \) or \( \chi[\varphi/\psi] = \psi \).

Thus we may assume henceforth that \( \chi \neq \varphi \). The proof proceeds by induction on the complexity of \( \chi \). We give it for the cases in which \( \chi \) is \((*)\) atomic; \((***)\) \( \neg \chi_1 \equiv \chi_2 \equiv \chi \), for \( \circ = \lor, \land, \supset, \equiv \); and \((***)\) a necessitation, \( \Box \chi_1 \).

For \((*)\): There is no replacement in this case. For \((***)\): by the assumption.

For the inductive case \((***)\) we assume, for induction, that the result holds for all sentences shorter than \( \chi \). So \( \gamma \chi_1 \equiv \chi_1[\varphi/\psi] \subseteq \text{A} \) and \( \gamma \chi_2 \equiv \chi_2[\varphi/\psi] \subseteq \text{A} \). It follows (by (PL)) that \( \gamma \chi_1 \equiv \gamma \chi_1[\varphi/\psi] \subseteq \text{A} \) and \( \gamma (\chi_1 \circ \chi_2) \equiv (\chi_1 \circ \chi_2)[\varphi/\psi] \subseteq \text{A} \), for \( \circ = \lor, \land, \supset, \equiv \).

By lemmas 1.16, 1.6, 1.9 and 1.15 we obtain:
Corollary 1.17. For any rte-system $\Sigma$:

1. $E_{PL} \subseteq \Sigma$.
2. $\Box \top \in \Sigma$ iff $\Box PL \subseteq \Sigma$.
3. If $\Box \top \in \Sigma$ and $\text{sub}(X) \subseteq \Sigma$, then $\Sigma$ is t-normal; consequently $\text{R}_{PL} \subseteq \Sigma$, $\text{sub}(X) \subseteq \Sigma$ and $\text{sub}(R) \subseteq \Sigma$.
4. If $\text{sub}(X) \subseteq \Sigma$, then $\text{sub}(K) \subseteq \Sigma$.

Of course, any modal system which includes some rte-system, is also an rte-system. So if $\Lambda$ is an rte-logic, then $\Lambda[\Phi]$ is.

Fact 1.18. The set $\text{PL}_{rte}$ is the smallest rte-system and rte-logic.

Proof: Of course, $\text{PL} \subseteq \text{PL}_{rte}$. Let $\tau_1 \supset \chi_2 \in \text{PL}_{rte}$ and $\chi_1 \in \text{PL}_{rte}$, i.e., for some $\tau_0 \in \text{PL}$, $\psi_0 \in \text{PL}$, we have that: $\chi_1 = \tau_0[\psi_1, \ldots, \psi_k]$, $\tau_0 \supset \psi_0 \in \text{PL}$, $\chi_2 = \psi_0[\psi_{k+1}, \ldots, \psi_{k+m}]$ and $\tau_1 \equiv \chi_1 \in \text{PL}$, $\ldots, \tau_1 \equiv \chi_{k+m} \in \text{PL}$. Hence $\psi_0 \in \text{PL}$, so $\chi_2 \in \text{PL}_{rte}$. Thus, $\text{PL}_{rte}$ is a modal system. From Lemma 1.16, $\text{PL}_{rte}$ is the smallest rte-system.

For any uniform substitution $s$ of formulae for propositional letters, $s(\tau[\psi_1, \ldots, \psi_k]) = s(\tau)[s(\psi_1), \ldots, s(\psi_k)]$ and $s(\tau) \in \text{PL}$. ⊣

Notice that $S0.5^5$ (and so also $S0.5^5[D]$ and $S0.5^5[T_q]$) is not closed under (rte). For example, the formulae:

a) $\Box \Box p \supset \Box \Box \neg \neg p$

b) $\Box \Box \neg \neg p \supset \Box \Box p$

do not belong to these logics (see e.g. Fact 3.6 in the second part).

Corollary 1.19. For any rte-system $\Sigma$ which includes $M_{PL}$ and has some formula of the form $\tau \varphi \supset \chi \in \text{PL}$, by Lemma 1.8:

1. $\Box \text{REP}_{rte} \subseteq \Box \text{PL}_{rte} \subseteq \Sigma$.
2. $\Sigma$ is closed under the following replacement

\[
\forall \varphi, \psi, \chi \in \text{For}: \text{ if } \tau \varphi \equiv \psi \in \text{PL}, \text{ then } \tau \chi \supset \chi[\tau/\psi] \in \Sigma. \quad \text{(srte)}
\]

Proof: 1. Let $\tau \in \text{PL}$. By Corollary 1.17, $\Box \tau \in \Sigma$. So if $\tau \varphi \equiv \psi \in \text{PL}$, then $\Box \tau[\tau/\psi] \in \Sigma$, by (rte).

2. By 1, $\Box(\chi \supset \chi[\tau/\psi])$ and $\tau(\chi[\tau/\psi] \supset \chi) \in \Sigma$. ⊣
Moreover, we obtain:

**Lemma 1.20.** For any rte-system $\Sigma$:

\[
\text{if } \text{sub}(\square(\chi)) \subseteq \Sigma, \text{ then } \text{sub}(\square(\chi)) \subseteq \Sigma.
\]

**Proof:** If \( \square((\square(T \supset \varphi) \land \square(\varphi \supset \psi)) \supset \square(T \supset \psi)) \uparrow \in \Sigma \), then \( \square((\square(\varphi \supset \psi) \supset (\square\varphi \supset \square\psi)) \uparrow \in \Sigma \), by PL and two applications of (rte), since \( \square\varphi \equiv (T \supset \varphi) \in \Sigma \). 

Let $S0.5_{\mathrm{rte}}$, $S0.5_{\mathrm{rte}}[D]$, $S0.5_{\mathrm{rte}}[T_q]$ and $S0.5_{\mathrm{rte}}$ be, respectively, such versions of the logics $S0.5^\circ$, $S0.5^\circ[D]$, $S0.5_{\mathrm{rte}}[T_q]$ and $S0.5$ that are closed under (rte). Thus, $S0.5_{\mathrm{rte}}$ is the smallest t-normal rte-system; so $S0.5_{\mathrm{rte}} = PL[REP_{PL}, K, \square T]$. The logics $S0.5_{\mathrm{rte}}[D]$, $S0.5_{\mathrm{rte}}[T_q]$ and $S0.5_{\mathrm{rte}}$ are the smallest t-normal rte-logics which contain $(D)$, $(T_q)$ and $(T)$, respectively. Thus, $S0.5_{\mathrm{rte}} = S0.5_{\mathrm{rte}}[T] = PL[REP_{PL}, K, T, \square T]$ and $S0.5_{\mathrm{rte}}[D] = PL[REP_{PL}, K, D, \square T]$. We have that $S0.5_{\mathrm{rte}} \subseteq S0.5_{\mathrm{rte}}[D] \subseteq S0.5_{\mathrm{rte}}$, because $(D) \notin S0.5_{\mathrm{rte}}$ and $(T) \notin S0.5_{\mathrm{rte}}[D]$. Moreover, we have that $S0.5_{\mathrm{rte}} \subseteq S0.5_{\mathrm{rte}}[T_q] \subseteq S0.5_{\mathrm{rte}}$, because $(T_q) \notin S0.5_{\mathrm{rte}}$ and $(T) \notin S0.5_{\mathrm{rte}}[T_q]$ (see [6]).

By Lemma 1.12, the logic $S0.5_{\mathrm{rte}}$ is closed under $(RN_\ast)$ and (SMP). However for any $\varphi \in \text{For}$: $\square\varphi \equiv S0.5_{\mathrm{rte}}$ if $\varphi \in PL_{\mathrm{rte}}$ if $\square\varphi \equiv S0.5_{\mathrm{rte}}$ (see Fact 4.5 in the second part). So, by Lemma 1.16, $S0.5_{\mathrm{rte}}$ is also closed under $(RN_\ast)$ and (SMP).

Let $C1_{\mathrm{rte}}$, $D1_{\mathrm{rte}}$, $C1_{\mathrm{rte}}[T_q]$ and $E1_{\mathrm{rte}}$ be, respectively, such versions of the logics $C1$, $D1$, $C1[T_q]$ and $E1$ that are closed under (rte). The logic $C1_{\mathrm{rte}}$ is the smallest t-regular rte-system; so $C1_{\mathrm{rte}} = PL[R_{\mathrm{rte}}, REP_{PL}]$. $D1_{\mathrm{rte}}$, $C1_{\mathrm{rte}}[T_q]$ and $E1_{\mathrm{rte}}$ are smallest t-regular rte-logics which contain $(D)$, $(T_q)$ and $(T)$, respectively. We have that $C1_{\mathrm{rte}} \subseteq D1_{\mathrm{rte}} \subseteq E1_{\mathrm{rte}}$ and $C1_{\mathrm{rte}} \subseteq C1_{\mathrm{rte}}[T_q] \subseteq E1_{\mathrm{rte}}$ (see [6]).

Finally notice that for the smallest rte-logic $PL_{\mathrm{rte}}$ we have “valuation semantics”. Let $Val^{\circ}_{\mathrm{rte}}$ be the set of all valuations $V$: For $\rightarrow \{0, 1\}$ from $Val^{\circ}$ satisfying the following condition:

\[
\forall \varphi, \psi, \chi \in \text{For}: \text{if } \square\varphi \equiv \psi \uparrow \in PL, \text{ then } V(\chi) = V(\chi[\varphi/\psi]).
\]

For the set $Val^{\circ}_{\mathrm{rte}}$ we have a fact analogous to Lemma 1.1 for $Val^{\circ}$. 

---

*Semantical Investigations on Some Weak Modal Logics. Part I*
LEMMA 1.21. 1. $V \in \text{Val}_{\text{ite}}^d$ iff for some $v : \text{PA}t \to \{0,1\}$ such that

$$\forall \varphi, \psi, \chi \in \text{For} : \text{if } \varphi \equiv \psi \in \text{PL}, \text{ then } v(\square \chi) = v(\square \chi[\psi/\varphi]), \quad (\star_{\text{PA}t})$$

$V$ is the unique extension of $v$ by classical truth conditions for truth-value operators.

2. For any $\varphi \in \text{For} : \varphi \in \text{PL}_{\text{ite}}$ iff for any $v : \text{PA}t \to \{0,1\}$ satisfying $(\star_{\text{PA}t})$ we have that $V(\varphi) = 1$, where $V$ is the unique extension of $v$ by classical truth conditions for truth-value operators.

3. For any $\varphi \in \text{For} : \varphi \in \text{PL}_{\text{ite}}$ iff for any $V \in \text{Val}_{\text{ite}}^d$, $V(\varphi) = 1$.

PROOF: 1. “$\Leftarrow$” Let $\chi, \varphi, \psi \in \text{For}$ such that $\varphi \equiv \psi \in \text{PL}$. By Lemma 1.1, $V \in \text{Val}^d$ and $V(\varphi) = V(\psi)$.

First we consider the possibility that $\chi = \varphi$. Then $\chi[\psi/\varphi] = \psi$ (when there is no replacement) or $\chi[\psi/\varphi] = \varphi$ (when $\varphi$ is replaced by $\psi$). So $V(\chi) = V(\chi[\psi/\varphi])$, by the assumption.

Thus we may assume henceforth that $\chi \neq \varphi$. The proof proceeds by induction on the complexity of $\chi$. We give it for the cases in which $\chi$ is ($\ast$) atomic; ($\ast\ast$) $\neg \chi_1 \gamma$ or $\chi_1 \circ \chi_2 \gamma$, for $\circ = \lor, \land, \supset, \equiv$; and ($\ast\ast\ast$) a necessitation, $\square \chi_1 \gamma$.

For ($\ast$): There is no replacement. For ($\ast\ast$): For any $\chi_1 \in \text{For}$ we have that $V(\square \chi_1) = v(\square \chi_1)$. So we use the assumption ($\star_{\text{PA}t}$).

For the inductive case ($\ast\ast$) we assume that the result holds for all sentences shorter than $\chi$. So $V(\chi_1) = V(\chi_1[\psi/\varphi])$ and $V(\chi_2) = V(\chi_2[\psi/\varphi])$. We have: $V(\neg \chi_1) = V(\neg \chi_1[\psi/\varphi])$ and $V(\chi_1 \circ \chi_2) = V((\chi_1 \circ \chi_2)[\psi/\varphi])$, since $V \in \text{Val}^d$.

“$\Rightarrow$” We put $v := V_{\text{PA}t}$. By the part “$\Leftarrow$”, the unique extension of $v$ by classical truth conditions for truth-value operators belongs to $\text{Val}_{\text{ite}}^d$, and it is equal to $V$.

2. “$\Leftarrow$” Suppose that $\varphi$ is built by means of truth-value operators, different propositional letters $\alpha_1, \ldots, \alpha_n$ and different necessitations $\square \chi_1 \gamma$, $\ldots$, $\square \chi_m \gamma$ $(n + m \geq 0)$.

If $m = 0$, i.e. $\varphi$ is a classical formula, then $\varphi \in \text{Taut}$. Moreover, $\varphi \in \text{PL}$, if $m > 0$ but there is no $i, j = 1, \ldots, m$ such that $\chi_i = \chi_j[\psi/\varphi]$, for some $\psi, \psi'$ in For such that $\varphi \equiv \psi \in \text{PL}$. Indeed, in none of both cases condition ($\star_{\text{PA}t}$) is connected with $\varphi$, so this formula is true for an arbitrary valuation $v : \text{PA}t \to \{0,1\}$.

Let us the assume that $m > 0$. We define the following equivalence relation in $\{\square \chi_1, \ldots, \square \chi_m\}$:
\(\Box \chi_i \Box \chi_j \iff \chi_i = \chi_j[\psi/\psi']\),

for some \(\psi, \psi' \in \text{For}\) such that \(\psi \equiv \psi' \in \text{PL}\).

If it is the identity relation in \(\{\Box \chi_1, \ldots, \Box \chi_m\}\), then the second considered case holds.

Let \(\|\varrho_1\|_R, \ldots, \|\varrho_k\|_R\) be different equivalence classes from \(\{\Box \chi_1, \ldots, \Box \chi_m\}\). For different formulae \(\varrho_1, \ldots, \varrho_k\) we assign different propositional letters \(\beta_1, \ldots, \beta_k\) (these letters are to be different as well from \(\alpha_1, \ldots, \alpha_n\)). All formulae from \(\|\varrho_i\|_R\) are replaced by \(\beta_i\). In this way we obtain the classical formula \(\varphi^*\). By the assumption we have that \(\varphi^* \in \text{PL}\). Replacing \(\varrho_i\) for \(\beta_i\) in \(\varphi^*\) we obtain \(\varphi^* \in \text{PL}\). The latest formula can be transformed into \(\varphi\) by suitable replacements (reverting to the initial ones) of formula \(\varrho_i\). Thus \(\varphi \in \text{PL}_{\text{rte}}\).

1.5. Strict classical logics. The logics \(\text{S1}, \text{S0.9}, \text{S1}^\circ\) and \(\text{S0.9}^\circ\)

After [1], we say that a logic \(\Lambda\) is strictly classical ("traditionally strict classical") iff \(\Box \text{PL} \subseteq \Lambda\) and \(\Lambda\) is closed under "traditional replacement rule for strict equivalents":

\[
\text{if } \Gamma \vdash \psi \rightarrow \chi \in \Lambda \text{ and } \chi \in \Lambda, \text{ then } \chi[\psi/\psi] \in \Lambda. \quad (\text{RRSE}_\Gamma)
\]

Moreover, a logic \(\Lambda\) is called strict classical iff \(\Box \text{PL} \subseteq \Lambda\) and \(\Lambda\) is closed under the following replacement rule:

\[
\text{if } \Gamma \vdash (\psi \equiv \psi') \in \Lambda \text{ and } \chi \in \Lambda, \text{ then } \chi[\psi/\psi'] \in \Lambda. \quad (\text{RRSE})
\]

We obtain that for modal logics which contain (K) and/or (X), the above notions are equivalent. Firstly we notice that:
Lemma 1.22 ([1]). Every strict-$_T$ or strict classical logic is an rte-system.

Secondly, by lemmas 1.11 and 1.22, and Corollary 1.17 we have that:

Lemma 1.23 ([1]). For every logic $A$ which contains (K) or (X): $A$ is strict-$_T$ classical iff $A$ is strict classical.

The logic $S0.9$ (resp. $S1$) is the smallest strict classical logic which contains the formulae ($T$, $\Box(T)$ and $\Box(K)$ (resp. $\Box(X)$). For these logics see e.g. [1, 4, 6]. By lemmas 1.20 and 1.22, $S0.9 \subseteq S1$. In [3] it was proved that $S0.9 \neq S1$, since $\Box(X) \notin S0.9$ (see also e.g. [1, pp. 15–16]).

In [1] the Feys' logic $S1^\circ$ from [2] is described as the smallest strict-$_T$ classical logic which contains the formulae ($X$) and $\Box(X)$, and is closed under (SMP). In [8] the logic $S1^\circ$ is described as the smallest strict-$_T$ classical logic which contains the formulae ($X$) and $\Box(X)$, and is closed under (RN$_*$). By lemmas 1.12 and 1.22 both characterizations are equivalent.

Again by lemmas 1.20 and 1.22, and Corollary 1.17, ($X$), $\Box(\Box)$ $\in S1^\circ$. Since ($X$) $\in S1$ and $S1$ is closed under (SMP), so $S1^\circ \subseteq S1$. Because ($T$), $\Box(T)$ $\notin S1^\circ$, so $S1^\circ \neq S1$ (see e.g. [1]).

Moreover, in [1] the logic $S0.9^\circ$ is described as the smallest strict-$_T$ classical logic which contains the formulae ($K$) and $\Box(K)$, and is closed under (SMP). We have $S0.9^\circ \subseteq S1^\circ$, because ($K$), $\Box(K) \in S1^\circ$.

Since ($T$) $\notin S0.9^\circ$, $\Box(X) \notin S0.9$, ($K$) $\in S0.9$ and $S0.9$ is closed under (SMP), so $S0.9^\circ \subseteq S0.9$ and $S0.9^\circ \subseteq S1^\circ$.

Notice that, by lemmas 1.12 and 1.22, the logics $S0.9^\circ$, $S0.9$, $S1^\circ$ and $S1$ are also closed under (RN$_*$). We can describe the logic $S0.9^\circ$ (resp. $S0.9$, $S1^\circ$; $S1$) as the smallest logic which includes $\Box$Taut, is closed under (RN$_*$) and (RRSE$_T$), and contains $\Box(K)$ (resp. $\Box(K)$ and $\Box(T)$; $\Box(X)$; $\Box(X)$) and $\Box(T)$).

In the second part of this paper we shall prove that $\Box(K), \Box(T) \notin S0.5_{rte}$, so $S0.5_{rte} \subseteq S0.9^\circ$ and $S0.5_{rte} \subseteq S0.9$.

In [1] the Lewis version $\text{Lew}(A)$ of a logic $A$ is understood as the smallest logic which includes $A$ and contains the formula $\Box \top$, i.e. $\text{Lew}(A) := A[\Box \top] = \text{PL}[A, \Box \top]$.

In [1] a logic is called prenormal iff it is congruential and contains the formula $\Box \top \sdash V (K)^\circ$. Of course, every prenormal logic which contains $\Box \top$ is normal. In [1] were considered the logics PK, PX, PKT and PXT which are the smallest congruential logics respectively containing:
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(K); (K) and (T); (X); (X) and (T). By Lemma 1.6, these logics contain (K), so also \( \square \top \supset (K)'\). Hence they are prenormal and we have that \( PK \subseteq PX \subseteq PXT \) and \( PK \subseteq PKT \subseteq PXT \). In [1] it was proved that \( S0.9^o = \text{Lew}(PK) := PK[\square \top], S0.9 = \text{Lew}(PKT) := PXT[\square \top], S1^o = \text{Lew}(PX) := PX[\square \top] \) and \( S1 = \text{Lew}(PXT) := PXT[\square \top] \).

Finally, notice that the logics \( S1, S0.9, S1^o \) and \( S0.9^o \) are not congruential and that the formulae \( \square (M) \), \( \square (C) \), \( \square p \supset \square (p \lor q) \) (1.1) and \( \Diamond (p \land q) \supset \Diamond p \) (1.2) are not members of \( S1 \), while the formulae \( (M), (C), \square p \supset \square (p \lor q) ' \) and \( ' \Diamond (p \land q) \supset \Diamond p ' \) belong to \( C1 \).

1.6. The logics \( S2^o \) and \( S2 \)

We say the a logic \( A \) is closed under Becker’s rule iff for any \( \varphi, \psi \in \text{For} \):

\[
\varphi \vdash \psi \in A, \text{then } \Box \varphi \vdash \Box \psi \in A. \quad \text{(RB)}
\]

In [4] the logic \( S2 \) is described as the smallest modal logic which includes \( \Box \text{Taut} \), contains the formulae (T), (X)(T), and is closed under (RB). Of course, \( S2 \) includes \( \Box \text{PL} \), contains (K) and, by Lemma 1.12, it is closed under \( \text{(RN\_s)} \) and \( \text{(SMP)} \).

Moreover, the logic \( S2^o \) is described in [8] as the smallest logic which includes \( \Box \text{Taut} \), contains \( \Box (K) \), and is closed under (RB) and \( \text{(RN\_s)} \). Of course, \( S2^o \) includes \( \Box \text{PL} \), contains (K) and, by Lemma 1.12, it is closed under \( \text{(SMP)} \). So \( S2^o \subseteq S2 \). For example \( (T), \Box (T) \notin S2^o \).

In [4] Lemmon proved that \( \Box (X) \in S2 \) and \( S2 \) is closed under \( \text{(RRSE} \_T \text{)} \). His proof shows that also \( \Box (X) \in S2^o \) and \( S2^o \) is closed under \( \text{(RRSE} \_T \text{)} \).

So we have that \( S1^o \subseteq S2^o \) and \( S1 \subseteq S2 \). Thus, \( S2 \) and \( S2^o \) are strict classical, but they are not congruential.

In [1] it was proved that \( S2^o = \text{Lew}(C2) := C2[\square \top] \) and \( S2 = \text{Lew}(E2) := E2[\square \top] \).

Moreover, for every \( \varphi \in \text{For} \):

\[
\Box \varphi ^o \in S2^o \text{ iff } \varphi \in C2, \quad (1.3)
\]
\[
\Box \varphi ^o \in S2 \text{ iff } \varphi \in E2. \quad (1.4)
\]

Hence, the formulae \( \Box (M), \Box (C), (1.1) \) and \( (1.2) \) belong to \( S2^o \), because \( (M), (C), \Box p \supset \Box (p \lor q) ' \) and \( ' \Diamond (p \land q) \supset \Diamond p ' \) belong to \( C1 \).
2. Some new weak t-normal logics and t-normal rte-logics

In the present paper we examine some logics which are not strict classical, but these logics have the form $A[\Box \Phi]$, where $\Phi \subseteq S0.5$ and $A = S0.5^\circ$, $S0.5^\circ[D]$, $S0.5^\circ[T_q]$, $S0.5$, $S0.5^\circ_{rte}$, $S0.5^\circ_{rte}[D]$, $S0.5^\circ_{rte}[T_q]$, $S0.5_{rte}$.

\textbf{Remark 2.1.} By Lemma 1.15, if a logic $A$ is t-regular (resp. a t-regular rte-system) and $\Phi \neq \emptyset$, then $A[\Box \Phi]$ is t-normal (resp. a t-normal rte-system).

For example, $C1[\Box \Phi] = S0.5^\circ[\Box \Phi]$, where $\Phi \neq \emptyset$. Similarly for t-regular logics $D1$, $C1[T_q]$, $E1$, $C1_{rte}$, $D1_{rte}$, $C1_{rte}[T_q]$. $E1_{rte}$ and suitable t-normal logics $S0.5^\circ[D]$, $S0.5^\circ[T_q]$, $S0.5$, $S0.5^\circ_{rte}$, $S0.5^\circ_{rte}[D]$, $S0.5^\circ_{rte}[T_q]$, $S0.5_{rte}$.

\textbf{Remark 2.2.} As we remember (see p. 42) the formulae (†) do not belong to $S0.5$. The formula (†) belongs to $S0.5[\Box K, \Box(\Box p \lor \Box \neg \neg p)]$, where ‘$\Box p \lor \Box \neg \neg p$’ $\in C1$. But $\Box(\Box)$ and

\begin{align*}
a) & \quad \Box \Box \Box p \lor \Box \Box \Box \neg \neg p \\
b) & \quad \Box \Box \Box \neg \neg p \lor \Box \Box \Box p
\end{align*}

do not belong to $S0.5[\Box S0.5]$; so this logic is not an rte-system (see the second part).

In Section 3 for logics $A[\Box \Phi]$, where $A = S0.5^\circ$, $S0.5^\circ[D]$, $S0.5^\circ[T_q]$, $S0.5$, we give simplified semantics formulated by means of some Kripke-style models. In Section 4 we give similar semantics for logics $A[\Box \Phi]$, where $A = S0.5_{rte}$, $S0.5^\circ_{rte}[D]$, $S0.5^\circ_{rte}[T_q]$, $S0.5_{rte}$. In Section 5 we prove that considered logics are determined by some classes of these models.

Firstly notice that by Lemma 1.20 we obtain:

\textbf{Corollary 2.1.} For any rte-logic $A$: $A[\Box \Phi, \Box X] = A[\Box \Phi, \Box K, \Box X]$.

By facts from Section 1 and Corollary 2.1 we obtain:

\textbf{Fact 2.2.} 1. $S0.5^\circ[\Box K] \subseteq S0.5^\circ_{rte}[\Box K] \subseteq S0.9^\circ$.
2. $S0.5[\Box K, \Box T] \subseteq S0.5^\circ_{rte}[\Box K, \Box T] \subseteq S0.9$.
3. $S0.5^\circ[\Box K, \Box X] \subseteq S0.5^\circ_{rte}[\Box X] \subseteq S1^\circ$.
4. $S0.5[\Box T, \Box K, \Box X] \subseteq S0.5^\circ_{rte}[\Box X, \Box T] \subseteq S1$. 
Moreover, we have:

**Lemma 2.3.** For any t-regular logic $\Lambda$ and $\Phi, \Psi \subseteq \text{For}$, if $\Psi \models_{\text{PL}} \Phi$, then $\Lambda[\Box \Phi] \subseteq \Lambda[\Box \Psi]$.

**Proof:** Suppose that $\Psi \models_{\text{PL}} \Phi$, i.e., for every $\varphi \in \Phi$ there is a subset $\{\psi_1, \ldots, \psi_n\}$ of $\Psi$, $n \geq 0$, such that $\Gamma(\psi_1 \land \ldots \land \psi_n) \supset \varphi \in \text{PL}$. Since $\Lambda$ is t-regular, $\Gamma(\Box \psi_1 \land \ldots \land \Box \psi_n) \supset \Box \varphi \in \Lambda$. Hence, $\Box \varphi \in \Lambda[\Box \Psi]$, since $\Box \psi_1, \ldots, \Box \psi_n \in \Lambda[\Box \Psi]$.

By the above lemma we obtain:

**Corollary 2.4.** For any r-regular logic $\Lambda$: $\Lambda[\Box \Phi, \Box C] \subseteq \Lambda[\Box \Phi, \Box R]$, $\Lambda[\Box \Phi, \Box N] \subseteq \Lambda[\Box \Phi, \Box R]$ and $\Lambda[\Box \Phi, \Box C, \Box N] = \Lambda[\Box \Phi, \Box R]$.

From the facts (1.3) and (1.4) we have:

**Fact 2.5.**
1. If $\Phi \subseteq C_2$, then $S_0.5^\circ \text{rte}[\Box \Phi] \subseteq S_2$.
2. If $\Phi \subseteq E_2$, then $S_0.5^\circ \text{rte}[\Box \Phi] \subseteq S_2$.

However in the present paper we are only interested in such a set $\Box \Phi$, as a set of new axioms, which satisfies condition $\Phi \subseteq S_0.5$. Notice that we have the following facts:

$$C_1 = C_2 \cap S_0.5^\circ,$$
$$C_1 \subseteq C_2 \cap S_0.5 \not\subseteq S_0.5^\circ,$$
$$E_1 = E_2 \cap S_0.5.$$

We have: $C_1 \not\subseteq C_2$, $C_1 \not\subseteq S_0.5^\circ \subseteq S_0.5$, $E_1 \subseteq E_2$ and $E_1 \not\subseteq S_0.5$. The remaining facts we will obtain from the semantics presented in [6] (see Fact 3.12 in the second part of this paper).

Therefore the following corollary will be of crucial importance:

**Corollary 2.6.**
1. If $\Phi \subseteq C_2 \cap S_0.5$, then $S_0.5^\circ \text{rte}[\Box \Phi] \subseteq S_2$.
2. If $\Phi \subseteq E_1$, then $S_0.5^\circ \text{rte}[\Box \Phi] \subseteq S_2$.

In Section 6 (see Corollary 6.3 in the second part) we prove that in the subsequent in the above corollary the symbol ‘$\subseteq$’ can be replaced by ‘$\not\subseteq$’. 
References


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