Rodolfo C. Ertola Biraben

ON SOME EXTENSIONS OF INTUITIONISTIC LOGIC

Abstract

We prove that many extensions of Intuitionistic Sentential Calculus ISC with new intuitionistic connectives that are known to be conservative extensions of ISC are not conservative extensions of Intuitionistic Predicate Calculus because formulas such as Kuroda’s are derivable. We thus solve a problem posed by López-Escobar in 1985 and answer a question posed by Humberstone in 2001 regarding a connective called the strongest anticipator.

In 1960 (see [12] and [13]) Smetanich introduced a new propositional intuitionistic constant connective we call γ and proved that the resulting extension ISC + γ of the Intuitionistic Sentential Calculus ISC is conservative. The logic ISC + γ may be given adding to any set of axiom schemas for ISC the following:

\[(γA) \neg\neg\varphi \leftrightarrow (γ \rightarrow \varphi)\]

Note that it is equivalent to add to ISC the following two axiom schemas:

\[(γI) \neg\neg γ,\]

\[(γE) \neg\neg\varphi \rightarrow (γ \rightarrow \varphi)\]

To see that ISC + γ is a conservative extension of ISC use the finite model property of ISC noting previously that in a finite Heyting algebra H, taking Γ = \{y ∈ H : \neg\neg y = 1\}, the corresponding constant minΓ exists being equal to the meet \bigwedge Γ as \neg\neg\bigwedge Γ belongs to Γ, because, for y₁ and y₂ ∈ H, if y₁ and y₂ ∈ Γ, then also y₁ ∧ y₂ ∈ Γ.

In 1977 (see [6] and [7]) Gabbay introduced a new intuitionistic propositional unary connective we call G and also proved that the resulting ex-
tension $ISC + G$ is conservative. The logic $ISC + G$ may be given adding to any set of axiom schemas for intuitionistic logic the following:

$$(GA) \ ((\neg \neg \phi \land (\psi \rightarrow \phi)) \rightarrow \psi) \leftrightarrow (G\phi \rightarrow \psi).$$

Note that it is equivalent to add to $ISC$ the following two axiom schemas:

$$(GI) \ (\neg \neg \phi \land (G\phi \rightarrow \phi)) \rightarrow G\phi,$n

$$(GE) \ ((\neg \neg \phi \land (\psi \rightarrow \phi)) \rightarrow \psi) \rightarrow (G\phi \rightarrow \psi).$$

To see that $ISC + G$ is a conservative extension of $ISC$ proceed as in the previous case considering, in a given Heyting algebra $H$, the unary operation $Gx = \min \{y \in H : \neg \neg x \land (y \rightarrow x) \leq y\}$.

Circa 1978 (see [10]) Kuznetsov introduced another new intuitionistic propositional unary connective we call $S$ (called “successor” in [1], where it is noted that in Heyting chains where the corresponding algebraic operation exists, it is the successor with respect to the order) and also proved that the resulting extension $ISC + S$ is conservative. The logic $ISC + S$, that may be seen as a generalization of $ISC + \gamma$, appears in the literature with different axiomatizations and names, e.g. in [4] is called $HC \Box$ and in [5] is called $KM$. It may be given adding to any set of axiom schemas for $ISC$ the following:

$$(SA) \ ((\psi \rightarrow \phi) \rightarrow \psi) \leftrightarrow (S\phi \rightarrow \psi).$$

Note that it is equivalent to add to $ISC$ the following two axiom schemas:

$$(SI) \ (S\phi \rightarrow \phi) \rightarrow S\phi,$n

$$(SE) \ ((\psi \rightarrow \phi) \rightarrow \psi) \rightarrow (S\phi \rightarrow \psi).$$

To see that $ISC + S$ is a conservative extension of $ISC$ proceed as in the previous cases considering, in a given Heyting algebra $H$, the unary operation $Sx = \min \{y \in H : y \rightarrow x \leq y\}$.

Note that the axiom schemas for the three given connectives, that the author has not seen in the literature, allow to define the corresponding algebraic operations with just one equation. The reader may see [15] and the references therein for additional information concerning the mentioned connectives or, in general, regarding the issue of new intuitionistic connectives.

In the three extensions considered the question naturally arises whether the corresponding extensions of Intuitionistic First Order Predicate Calculus $IPC$ are also conservative.
In 1985 (see [11, p. 118]), López-Escobar posed the following problem: “Suppose that $S$ is a schema, essentially involving quantifiers, such that $S$ is provable in the Classical Predicate Calculus, $\text{CPC}$, but not in the $\text{IPC}$. Then, is there a sentential connective $\oplus$ (with associated rules) so that $\text{ISC} + \oplus$ is a conservative extension of $\text{ISC}$ and $\text{IPC} + \oplus \vdash S$?” Moreover, he mentioned, as a possible example for $S$, the formula $\forall x \sim \neg Px \rightarrow \neg \neg \forall x Px$. This formula, very well known not to be derivable in $\text{IPC}$, is sometimes known as the Kuroda formula.

In 2001 (see [8, p. 433]), Humberstone asked the following question, where he writes $\text{ILa}$ for $\text{ISC}$ enriched with a connective he called “the strongest anticipator”: “is quantified $\vdash_{\text{ILa}}$, presumed formulated with the standard intuitionistic rules for the quantifiers, a conservative extension of intuitionistic predicate logic, or is there some untoward interaction of the kind discussed in [20] for the case of dual intuitionistic implication?” where [20] refers to the article by López-Escobar cited in the previous paragraph and “untoward interaction” refers to the fact proven by López-Escobar in the same paper, i.e. [11], that the extension of $\text{IPC}$ with the dual of the conditional is not conservative. Moreover, the logic $\text{ILa}$ is the extension of $\text{ISC}$ given by the unary connective $a$, the axiom $(a\varphi \rightarrow \varphi) \rightarrow \varphi$ and the rule: if $\psi \rightarrow \varphi \vdash \varphi$ then $a\varphi \vdash \psi$.

Is this paper we simultaneously answer the question whether the mentioned extensions of $\text{IPC}$ by adding the connectives by Smetanich, Kuznetsov and Gabbay are conservative, solve López-Escobar’s problem for the case of the Kuroda formula and answer Humberstone’s question whether quantified $\text{ILa}$ is a conservative extension of $\text{IPC}$. Our solution was inspired by the papers [3] and [4], in particular, by the not explicitly proven Statement 3 of paper [4], for which we provide a variant better suited to our needs.

Our solution follows from the following

**Theorem 1.** $\vdash_{\text{IPC} + S} \forall x((Px \rightarrow \psi) \rightarrow Px) \rightarrow ((\forall x Px \rightarrow \psi) \rightarrow \forall x Px)$, for $x$ not free in $\psi$. 
Proof. Just consider the following derivation:

1. $\forall x((P_x \rightarrow \psi) \rightarrow P_x)$ Sup
2. $\forall xP_x \rightarrow \psi$ Sup
3. $(P_u \rightarrow \psi) \rightarrow Pa$ 1, IPC
4. $S\psi \rightarrow Pa$ 3, SE
5. $S\psi \rightarrow \forall xPx$ 4, IPC
6. $S\psi \rightarrow \psi$ 2, 5, ISC
7. $S\psi$ 6, SI
8. $\forall xPx$ 5,7, ISC □

Note that the formulas of the schema in Theorem 1, for $\psi$ belonging to the language of $IPC$, belong to the language of $IPC$, that is, the connective $S$ does not appear in them. Note, also, that the formula that results from the schema in the theorem substituting $\bot$ for $\psi$ is equivalent to the Kuroda formula. So, it follows that $IPC + S$ is not a conservative extension of $IPC$.

Now, it also follows that we have solved López-Escobar’s problem in the case of the Kuroda formula. In fact, we may give two solutions for $\oplus$ in López-Escobar’s question: not only the unary connective $S$ but also the constant connective $\gamma$. This is because if in the given derivation we first substitute every occurrence of $S\psi$ for $\gamma$ and then every remaining occurrence of $\psi$ for $\bot$, then the resulting sequence of formulas is a derivation of the Kuroda formula in the logic $IPC + \gamma$ (in the justifications for lines 4 and 7 just add $IPC$ and substitute $\gamma$ for $S$). Moreover, in the logic $ISC + \gamma + S$ it may be seen that $S\bot = \gamma$. It follows that also $IPC + \gamma$ is not a conservative extension of $IPC$.

Moreover, it is known and easily seen that in the logic $ILA$ the constant $\gamma$ may be defined as $a\bot$ (see [8, p. 399]). Consequently, it follows that quantified $ILA$ is not a conservative extension of $IPC$, thus answering Humberstone’s question. After completing the first draft of this paper, an anonymous referee called our attention to the fact that the answer to Humberstone’s question had already been obtained and may be found in [9] as Observation 4.38.10 on p. 626, where it is credited to Brian Weatherson.

It is also the case that the extensions of $IPC$ with the unary connectives connectives $h$, $\gamma$ or $T$ of [2], or with any connective resulting in a logic where $\gamma$ may be defined, will not be conservative.
Finally, in the case of the extension $IPC + G$, using the following notation:

\[(GS) \ (\neg\neg\psi \land \forall x((P x \rightarrow \psi) \rightarrow P x)) \rightarrow ((\forall x P x \rightarrow \psi) \rightarrow \forall x P x),\]

for $x$ not free in $\psi$, we have the following

**Theorem 2.** $\vdash_{IPC+G} GS$.

**Proof.** Similar to Theorem 1. □

It follows that $IPC + G$ is not a conservative extension of $IPC$, because $GS$ is not derivable in $IPC$. In order to see this, note that e.g. for $\psi = q$, where $q$ is any propositional letter of $IPC$, the resulting formula is not derivable in $IPC$ using, for instance, the Kripke model with universe given by the set of infinite nodes $k_i$, for $i$ belonging to the set of natural numbers $N = \{0, 1, \ldots\}$, with the natural ordering and adding an extra node $l_i$ (that will be maximal) to the left (or to the right) of each $k_i$, and defining the universes of the nodes and the forcing relation $\vdash$ as follows: for the $k_i$ and $l_i$, define the universes $U(k_i) = U(l_i) = \{0, 1, \ldots, i\}$ and stipulate that $k_i \vdash P a$ iff $a \in \{0, 1, \ldots, i - 1\}$, but $l_i \vdash P a$ iff $a \in \{0, 1, \ldots, i\}$ and also $l_i \vdash q$, for all $i \in N$. Now, both $\neg\neg q \land \forall x ((P x \rightarrow q) \rightarrow P x)$ and $\forall x P x \rightarrow q$ are forced at node $k_0$, but $\forall x P x$ is not. The given Kripke model is a variation of the model given in [14, p.82].

**Acknowledgement.** The author thanks an anonymous referee for his remarks regarding Humberstone’s question. The author was partially supported by CONICET Project PIP 112-200801-02543.

**References**


CLE
Unicamp
Campinas
Brasil

UNIVERSIDADE ESTADUAL DE CAMPINAS
Centro de Lógica, Epistemologia e História da Ciência-CLE
Rodolfo C. Ertola Biraben
Rua Sérgio Buarque de Holanda, 251
Cidade Universitária Zefirino Vaz-Bairro Barão Geraldo
13083-859-Campinas (SP)-Brasil