1. Introduction

Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. In abstract algebraic logic terms, the implicative fragment of Intuitionistic Propositional Logic is regularly and strongly algebraizable and its equivalent algebraic semantic is the class of Hilbert algebras.

In [1], Diego proves that the class of Hilbert algebras is equational, proving that an algebra $\langle A, \rightarrow \rangle$ of type $(2)$ is a Hilbert algebra if and only if it satisfies the following equations:

\[
\begin{align*}
(x \rightarrow x) \rightarrow x & \approx x, \\
x \rightarrow x & \approx y \rightarrow y, \\
x \rightarrow (y \rightarrow z) & \approx (x \rightarrow y) \rightarrow (x \rightarrow z), \\
(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) & \approx (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y).
\end{align*}
\]

In [6] Padmanabhan and Rudeanu proved that Hilbert algebras and some subvarieties of Hilbert algebras are 2-based, i.e., they admit an equational base with two independent equations. They also posed the following conjecture:

“All finitely-based equational classes emanating from Boolean algebras and having the congruence distributive property are 2-based.”
Motivated by this conjecture and the results in their paper, in this short note we will prove the following:

**Theorem 1.** *Every non trivial finitely-based subvariety of Hilbert algebras is 2-based.*

The *equational spectrum* of an equational class $C$ is the set of natural numbers $n$ such that $C$ admits an independent base of $n$ equations.

Using Theorem 1 and a theorem of Tarski [7], we easily obtain the following corollary (see also [6, Theorems 2, 3 and 4]).

**Corollary 2.** *The equational spectrum of every non trivial finitely-based subvariety of Hilbert algebras is $[2, \omega)$*

2. **Proof of Theorem 1**

In order to prove Theorem 1 we first recall the necessary results and fix some notation.

We will denote by $\mathfrak{m}$ the *term algebra* in the language $\{\to\}$, i.e., the absolutely free algebra with countably many generators (called propositional variables). Given an algebra $\langle A, \to \rangle$ a *valuation* on $A$ is a homomorphism $v: \mathfrak{m} \to A$. As usual $A$ satisfies an equation $t \approx s$ if and only if $v(t) = v(s)$ for every valuation $v$ on $A$, in symbols $A \models t \approx s$.

In [6] Padmanabhan and Rudeanu present the following base of equations for the class of Hilbert algebras.

**Lemma 3.** [6, Lemma 4] *The class of Hilbert algebras admits the following equational base:

\[
(x \to x) \to y \approx y \quad (1)
\]

\[
((x \to y) \to (x \to z)) \to ((u \to w) \to ((w \to u) \to u)) \approx ((x \to (y \to z)) \to ((w \to u) \to ((u \to w) \to w)) \quad (2)
\]

In the following lemmas we recall some necessary results on Hilbert algebras.
LEMMA 4. [6, Theorem 1] The unique one-based subvariety of Hilbert algebras is the trivial one.¹

LEMMA 5. [6, Lemma 3] Let $A$ be a Hilbert algebra. If $A \models t \to u \approx s \to u$, where $u$ is a propositional variable that does not occur in $t$ nor in $s$, then $A \models s \approx t$.

Given $\langle A, \to \rangle$ an algebra of type (2) and $a, a_1, \ldots, a_n \in A$, we define:

$$
(a_n, \ldots, a_1; a) = \begin{cases} 
a_1 \to a & \text{if } n = 1, \\
 a_n \to (a_{n-1}, \ldots, a_1; a) & \text{if } n > 1.
\end{cases}
$$

The following lemma is a trivial observation but will be crucial in the proof of our main theorem.

LEMMA 6. Let $\langle A, \to \rangle$ be an algebra of type (2). If $A$ satisfies equation (1), then the following hold:

(i) $\{a \to a\} \subseteq A$ is a subalgebra of $\langle A, \to \rangle$, i.e., for every term $t(x_1, \ldots, x_n)$ and every element $a \in A$, $t^A(a \to a, a \to a, \ldots, a \to a) = a \to a$.

(ii) For every $a, b, c \in A$,

$$(a \to a, \ldots a \to a, b, a \to a, \ldots, a \to a; c) = b \to c$$

We are now ready to prove the main theorem of this note.

PROOF OF THEOREM 1

We first fix some notation:

$$L = ((x \to y) \to (x \to z)) \to ((u \to w) \to ((u \to u) \to u))$$

and

$$R = (x \to (y \to z)) \to ((u \to u) \to ((u \to w) \to w)).$$

Now let $\mathcal{K}$ be a non trivial finitely-based subvariety of Hilbert algebras, and let

$$t_1 \approx s_1$$

$$\vdots$$

$$t_n \approx s_n$$

¹The trivial subvariety is the class formed by singleton algebras and the empty algebra. See the remark about Hilbert algebras considered with a constant “1” at the end of this note.
be a base of equations for $K$. Without loss of generality we may assume that the variables of $\text{var}\{t_i, s_i\} \cap \text{var}\{t_j, s_j\} = \emptyset$ whenever $i \neq j$, and that $\text{var}\{t_i, s_i\} \cap \text{var}\{L, R\} = \emptyset$ for every $i = 1, \ldots, n$.

We claim that
\[
\begin{align*}
\{ (x \to x) \to y \} & \approx y \quad \text{(a)} \\
(t_1, \ldots, t_n; L) & \approx (s_1, \ldots, s_n; R) \quad \text{(b)}
\end{align*}
\]
is a base for $K$.

Since every algebra in $K$ is a Hilbert algebra that satisfies $t \approx s$, by Lemma 3 we have that every algebra in $K$ satisfies (a) and (b).

Now let us fix an algebra $A$ satisfying (a) and (b).

**Claim 1:** $A$ is a Hilbert algebra.

If $A$ is the empty algebra, the proof is trivial. If $A$ is not empty, by Lemma 3 it is enough to prove that $A \models L \approx R$. Let $v: \mathfrak{Fm} \to A$ be a valuation. Let us fix $a \in A$ and let $v': \mathfrak{Fm} \to A$ be the valuation uniquely determined by:
\[
v'(p) = \begin{cases} 
v(p) & \text{if } p \in \text{var}\{L, R\}, \\
 a \to a & \text{otherwise}, \end{cases}
\]
where $p$ denotes a propositional variable.

Since $A$ satisfies (a), by Lemma 6 $v'(t_i) = v'(s_i) = a \to a$. By the definition of $v'$, $v(L) = v'(L)$ and $v(R) = v'(R)$. Thus
\[
v(L) = v'(L) = (v'(t_1), \ldots, v'(t_n); v'(L)) = v'(t_1, \ldots, t_n; L) \equiv v'(s_1, \ldots, s_n; R) = (v'(s_1), \ldots, v'(s_n); v'(R)) = v'(R) = v(R),
\]
proving that $A$ is a Hilbert algebra.

**Claim 2:** For every $i = 1, \ldots, n$, $A \models t_i \to u \approx s_i \to u$.

Let $v: \mathfrak{Fm} \to A$ be a valuation. Let us fix $a \in A$ and let $v': \mathfrak{Fm} \to A$ be the valuation defined by:
\[
v'(p) = \begin{cases} 
v(p) & \text{if } p \in \text{var}\{t_i, s_i\}, \\
 v(u) & \text{if } p \in \{u, w\} \\
 a \to a & \text{otherwise}, \end{cases}
\]
where $p$ denotes a propositional variable.
Observe first that $v'(L) = v'((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow ((u \rightarrow w) \rightarrow ((w \rightarrow u) \rightarrow u)) = v'(u) = v(u)$ and similarly $v'(R) = v(u)$.

Now by Lemma 6 and the fact that $A$ satisfies (a) and (b),

$$
v(t_i \rightarrow u) = v'(t_i) \rightarrow v'(u)
= (a \rightarrow a, \ldots a \rightarrow a, v'(t_i), a \rightarrow a, \ldots, a \rightarrow a; v'(L))
= (v'(t_1), \ldots, v'(t_i), \ldots, v'(t_n); v'(L))
= v'(t_1, \ldots, t_n; L)
= v'(s_1, \ldots, s_n; R)
= (v'(s_1), \ldots, v'(s_i), \ldots, v'(s_n); v'(R))
= (a \rightarrow a, \ldots a \rightarrow a, v'(s_i), a \rightarrow a, \ldots, a \rightarrow a; v'(R))
= v'(s_i) \rightarrow v'(u)
= v(s_i \rightarrow u)
$$

which proves the claim.

By Claim 1, we have that $A$ is a Hilbert algebra, so we are able to use Lemma 5 which combined with Claim 2 proves that $A \models t_i \approx s_i$ for each $i \in \{1, \ldots, n\}$. Thus $A \in \mathcal{K}$, proving that (a) and (b) form an equational base for $\mathcal{K}$.

The fact that (a) and (b) are independent follows from Lemma 4 and the assumption that $\mathcal{K}$ is not trivial. \qed

A remark

In the literature Hilbert algebras are usually presented as algebras $\langle A, \rightarrow, 1 \rangle$ (see for example [1],[4]) where the constant 1 is the top element of the algebra in the natural order and denotes the value naturally associated with “truth”. Since for every Hilbert Algebra $\langle A, \rightarrow, 1 \rangle$, $1 = a \rightarrow a$ for each $a \in A$, the presentation for Hilbert algebras chosen here only differs in the fact that the empty algebra is allowed to be a Hilbert algebra. This is not a big change when we study the algebraic structure of Hilbert algebras, since adding or not adding the constant only differs in one trivial case. The choice has some consequences on the categorical properties of the variety since the algebra with only one element is initial and a terminal object if we consider the constant in the language, while if we allow the empty algebra clearly the algebra with one element is no longer a terminal object. Nevertheless, the use of the constant allows us to simplify many definitions, proofs, statements and it allows us to make natural connections.
with other algebraic structures related to logics, which justify the use of the constant.

However, in this note we have decided not to include the constant in the language for two reasons. First, because the constant itself does not appear in the implication fragment of Intuitionistic Logic. But apart from the philosophical and historical reasons, the most important motivation was to show that the use of the constant, which certainly increases the structure of the algebra of term $3m$, and therefore provides more equations to describe subvarieties, does not have any impact on the results described in this note.

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References


Dep. de Matemáticas – Facultad de Ciencias Exactas
Universidad Nacional del Centro de la Provincia de Buenos Aires
Pinto 399 – Tandil (7000), Argentina
e-mail: lcabrer@exa.unicen.edu.ar