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Gemma Robles
Francisco Salto
José M. Méndez

A WEAK LOGIC WITH THE AXIOM MINGLE LACKING THE VARIABLE-SHARING PROPERTY

Abstract

As it is well known, Relevance Logic R plus the axiom mingle (R-Mingle) does not have the variable-sharing property (vsp). The aim of this paper is to improve this result by defining a weak logic with the axiom mingle and not included in minimal logic B_M lacking the vsp.

1. Introduction

As it is well-known (cf. [1]), a necessary property of any relevant logic S is the following:

Definition 1 (Variable-sharing property —vsp). *If $A \to B$ is a theorem of S, then A and B share at least one propositional variable.*

Not less well-known is the fact that the axiom mingle, to wit,

$$\text{M. } A \to (A \to A)$$

is not a theorem of standard relevant logics because if it is added to R, the logic of relevant implication, formulas such as, e.g., the following

$$\neg(A \to A) \to (B \to B)$$

are derivable in the resulting system, R-Mingle (RM) (cf. [1], §29.5).

The aim of this paper is to define a weak relevant logic, labelled Σ, to which M cannot be added if the vsp is to be preserved. This logic Σ is included in the logic of entailment E but does not include Sylvan and Plumwood's minimal logic B_M (see [9]). In Section 4 of the paper, an alternative to Σ, Σ', with essentially the same properties predicable of Σ, is introduced. We end the paper with some brief notes on the relationship between relevance and mingle.

2. The logic Σ

Routley and Meyer's basic positive logic B_+ can be axiomatized as follows (cf. [8]):

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules:

Modus ponens (MP): $(\vdash A \to B \ \& \vdash A) \Rightarrow \vdash B$

Adjunction (Adj): $(\vdash A \ \& \vdash B) \Rightarrow \vdash A \wedge B$

Suffixing (Suf): $\vdash (A \to B) \Rightarrow \vdash (B \to C) \to (A \to C)$

Prefixing (Pref): $\vdash (B \to C) \Rightarrow \vdash (A \to B) \to (A \to C)$

Notice that the rule transitivity

Transitivity (Trans): $\vdash (A \to B) \ \ \& \ \vdash (B \to C) \Rightarrow \vdash (A \to C)$

is immediate given Suf or Pref.

Then, the logic Σ is axiomatized by adding the following axioms and rule to B_+:

A7. $\{[(A \to A) \wedge (B \to B)] \to C\} \to C$

A8. $A \to \neg\neg A$

A9. $(A \to B) \to (\neg B \to \neg A)$

rdn. $\vdash A \Rightarrow \vdash \neg\neg A \to A$

Remark 1. *The label rdn stands for "Rule of elimination of double negation". The De Morgan law*

$$dm.\ (\neg A \vee \neg B) \to \neg(A \wedge B)$$

is immediate in Σ *by A2, A5, A9 and Adj. Finally, A7 is one of the characteristic axioms of the logic of entailment E (see [1] or [2]).*

Now, Sylvan and Plumwood's logic B_M (cf. [9]) is the result of adding to B_+ the axioms

$$\text{A10. } \neg(A \wedge B) \to (\neg A \vee \neg B)$$
$$\text{A11. } (\neg A \wedge \neg B) \to \neg(A \vee B)$$

and the rule

$$\text{Contraposition (con). } \vdash A \to B \Rightarrow \vdash \neg B \to \neg A$$

We note that B_M is, when negation is present, the minimal logic in Routley and Meyer's ternary relational semantics.

Then, it is proved:

Proposition 1. *(a)* Σ *is included in the logic of entailment E. (b)* B_M *is not included in* Σ*.*

Proof: (a) Cf., e.g., [2]. (b) A10 is not derivable in Σ: cf. MSI in §5. □

So, it seems that Σ is not a strong logic. Nevertheless, let use the label ΣM to refer to the result of adding the axiom mingle (M) to Σ. In the next section, it is proved that ΣM lacks the vsp.

3. ΣM does not have the vsp

Notice that, by Proposition 1(a), Σ has the vsp. But, similarly, as it is the case in R-Mingle (RM), we have the following:

Proposition 2. *The wff* $\neg(A \to A) \to (B \to B)$ *is provable in* Σ*M.*

PROOF:

1. $\neg A \to (\neg A \to \neg A)$ M
2. $\neg A \to (\neg\neg A \to \neg\neg A)$ 1, A9, Suf

3. $(\neg\neg A \to \neg\neg A) \to (A \to \neg\neg A)$ A8, Suf.
4. $\neg A \to (A \to \neg\neg A)$ Trans, 2, 3
5. $\neg[(A \to A) \wedge (B \to B)] \to \{[(A \to A) \wedge (B \to B)] \to \neg\neg[(A \to A) \wedge (B \to B)]\}$ Substitution in 4
6. $\{[(A \to A) \wedge (B \to B)] \to \neg\neg[(A \to A) \wedge (B \to B)]\} \to \neg\neg[(A \to A) \wedge (B \to B)]$ A7
7. $\neg[(A \to A) \wedge (B \to B)] \to \neg\neg[(A \to A) \wedge (B \to B)]$ Trans, 5, 6
8. $(A \to A) \wedge (B \to B)$ A1 and Adj.
9. $\neg\neg[(A \to A) \wedge (B \to B)] \to [(A \to A) \wedge (B \to B)]$ rdn, 8
10. $\neg[(A \to A) \wedge (B \to B)] \to [(A \to A) \wedge (B \to B)]$ Trans, 7, 9
11. $\neg[(A \to A) \wedge (B \to B)] \to (B \to B)$ Trans, A2, 10
12. $[\neg(A \to A) \vee \neg(B \to B)] \to (B \to B)$ dm (cf. Remark 1), 11. Trans.
13. $\neg(A \to A) \to [\neg(A \to A) \vee \neg(B \to B)]$ A4
14. $\neg(A \to A) \to (B \to B)$ Trans, 12, 13

$\square$

Consequently, we have the following corollary of Proposition 2:

COROLLARY 1. *The logic ΣM does not have the vsp.*

Given the axioms and rules used in the proof displayed above, it is possible, obviously, to build up logics lacking the vsp independent of ΣM. But of all these logics, there is one that in our opinion merits consideration. It is the logic Σ' introduced in the following section.

4. The logic Σ'

The logic Σ' is the result of changing A7 in Σ for the rule of "demodalization":

$$\text{rdem. } \vdash A \Rightarrow \vdash (A \to A) \to A$$

REMARK 2. *The rule rdem is named after the thesis*

$$A \to [(A \to A) \to A]$$

called "demodalizer" by relevant logicians (see e.g. [3], p.126)

We note that the following rule

$$\text{rdem}'. \vdash A \Rightarrow \vdash (A \to \neg\neg A) \to \neg\neg A$$

is derivable in Σ' by rdem, A8, rdn, Pref and Suf.

On the other hand, by Σ'M we refer to the result of adding the axiom M to Σ'. Then, we have:

PROPOSITION 3. *Σ and Σ' are independent logics. In fact, ΣM and $\Sigma' M$ are independent logics.*

PROOF: (a) The rule rdem is not derivable in ΣM: cf. MSI in §5. (b) A7 is not derivable in Σ'M: cf. MSII in §5. □

PROPOSITION 4. *(a) B_M is not included in Σ'. (b) The logic of entailment E does not include Σ'.*

PROOF: (a) A10 is not derivable in Σ': cf. MSII in §5. (b) The rule rdem is not derivable in E: cf. MSIII in §5. □

We note that, as Σ' is clearly included in R, Σ' has the vsp. Nevertheless, it is proved:

PROPOSITION 5. *The formula $\neg(A \to A) \to (B \to B)$ is provable in $\Sigma' M$.*

PROOF: Exactly as that of Proposition 2 but justifying now step 6 by rdem$'$. □

5. Sets of matrices

The following set of matrices are used in some of the preceding proofs. Designated values are starred.

Matrix set I (MSI):

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	0
*2	0	0	3	3	2
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

MSI verifies ΣM (that is, it satisfies the axioms and rules of ΣM) but falsifies rdem ($v(A) = 2$) and A10 ($v(A) = 2$, $v(B) = 1$).

Matrix set II (MSII):

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	1	3	0
2	1	1	3	3	0
*3	1	1	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
*3	3	3	3	3

MSII verifies Σ'M but falsifies A7 ($v(A) = v(B) = v(C) = 0$) and A10 ($v(A) = 2$, $v(B) = 1$).

Matrix set III (MSIII):

$\to$	0	1	2	3	$\neg$
0	2	2	2	2	3
*1	0	2	2	2	2
*2	0	0	2	2	1
*3	0	0	0	2	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

MSIII verifies the logic of entailment E as axiomatized in [1], §27.1.1 or [2], §R2, but falsifies rdem ($v(A) = 1$).

REMARK 3. *MSI and MSIII show that rdem is derivable neither in ΣM nor in E, respectively. That is, it is possible that the premises of the rule are assigned a designated value while the conclusion is assigned a non-designated one. But it will be easy to show that this rule is admissible in both systems (cf. [7]. On the concepts "derivable rule" and "admissible rule", see [1], pp. 53-54).*

6. Conclusions

We have introduced two weak relevant logics (independent of each other), Σ and Σ' incompatible with mingle in the sense that if the axiom mingle is added to any one of them, the resulting logic lacks the vsp. Does this fact entail that, as Anderson and Belnap affirm (cf. [1], §29.5), relevance and mingle are incompatible? By no means. There are at least two paths in which relevance and mingle converge:

1. The first is the one that results when the De Morgan negation characteristic of relevance logics is restricted (cf. [4], [5]).
2. The second one is to drop A7 (or rdem). In this sense, we note that Ticket Entailment Logic plus the mingle axiom has the vsp (cf. [6]).

We end the paper by posing a problem: is there a Routley-Meyer type semantics for Σ and Σ'? In other words, which are the corresponding semantical postulates for rdn and rdem?

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Dpto. de Psicología, Sociología y Filosofía, Universidad de León
Campus de Vegazana, s/n, 24071, León
e-mail: gemmarobles@gmail.com
e-mail: francisco.salto@unileon.es

Universidad de Salamanca
Campus Unamuno, Edificio FES, 37007, Salamanca, Spain
e-mail: sefus@usal.es