PARTIAL AND INTUITIONISTIC LOGIC

Abstract
In the paper we propose a kind of interpretation of partial logic in the intuitionistic logic, or rather its part. We will show that finite binary trees determine this special fragment of intuitionistic logic.

1. Introduction
Formal logic has its ambitious aim, i.e. not only the linguistic representation of the world, but a representation of the knowledge about the world. There are many attempts to characterize adequate the concept of the state of knowledge. The main difference between knowledge and the actual truth is that the latter has to take into account its partiality. In other words – there are sentences that have some logical values, but the observer does not know if the particular proposition is true or false. In the paper we would like to show that partial logic that has been used by Lepage and Lapierre (definition and axiomatization can be found in [7]) can express the same philosophical issues as the intuitionistic one.

It is not the first time that partial logics and intuitionism meet. For example, in [3] the logic considered by Humberstone is some special kind of intuitionistic calculus. However, it coincides with partial logic under the proviso the following condition is assumed (for every propositional variable $p$):

if $w \not\models p$ and $w \not\models \neg p$,

then there exist $v \geq w$ and $u \geq w$ such that $v \models p$ and $u \models \neg p$. 
Let $L = (L, \neg, \land)$ be a propositional language of type $(1, 2)$ and $L' = (L', \neg, \land, \lor, \rightarrow)$ the language of type $(1, 2, 2, 2, 2)$.

Put $Val$ to be the set of partial interpretations, i.e. functions $w : L \rightarrow \{0, \perp, 1\}$ such that
\begin{align*}
\text{if } w(\varphi) = 0, & \text{ then } w(\neg \varphi) = 1 \\
\text{if } w(\varphi) = 1, & \text{ then } w(\neg \varphi) = 0 \\
\text{if } w(\varphi) = \perp, & \text{ then } w(\neg \varphi) = \perp \\
w(\varphi \land \psi) = 1 & \text{ for } w(\varphi) = w(\psi) = 1 \\
w(\varphi \land \psi) = 0 & \text{ if } 0 \in \{w(\varphi), w(\psi)\} \\
w(\varphi \land \psi) = \perp & \text{ in the remaining cases.}
\end{align*}

The set $\{0, \perp, 1\}$ can be partially ordered by the relation $\sqsubseteq$, which is generated by the set $\{(\perp, 0), (\perp, 1)\}$. We put $k \sqsubseteq l$ iff $k \sqsubseteq l$ & $k \neq l$. $\sqsubseteq$ can be naturally extended on the set $Val : w \sqsubseteq v$ iff $\forall p \in Var, w(p) \sqsubseteq v(p)$.

For any $w \in Val$ put $w[p/0]$ for the function that differs from $w$ at most in such a manner that $w[p/0](p) = 0$. And, similarly, for $w[p/1]$.

Put $\vdash_3$ for a consequence relation on the language $L$ that can be defined as follows:
\begin{itemize}
\item for any $L$-formulas $\psi_1, \ldots, \psi_n$ and $\varphi$:
\item $\psi_1, \ldots, \psi_n \vdash_3 \varphi$ iff for every partial interpretation $w$ we have if $w(\psi_1) = w(\psi_2) = \ldots = w(\psi_n) = 1$, then $w(\varphi) = 1$, as well.
\end{itemize}

The above definition is semantical in its nature, but according to [7] it can be characterized in purely syntactical manner (and by the usage of finitary means only).

Let $t : L \rightarrow L$ be the function removing double negations: $t(p) = p$, $t(\varphi \land \psi) = t(\varphi) \land t(\psi)$, $t(\neg p) = \neg t(p)$, $t(\neg(\varphi \land \psi)) = \neg(t(\varphi) \land t(\psi))$, $t(\neg \neg \varphi) = t(\varphi)$.

Now, let $Tr' : L \rightarrow L'$ be characterized as follows:
\begin{itemize}
\item $Tr'(p) = p$ for any propositional variable $p$
\item $Tr'(\neg p) = \neg Tr'(p)$
\end{itemize}
\[ Tr'(\varphi \land \psi) = Tr'(\varphi) \land Tr'(\psi) \]
\[ Tr'(\neg(\varphi \land \psi)) = Tr'(\neg\varphi) \lor Tr'(\neg\psi) \]

Finally, put \( Tr = Tr' \circ t \).

**Definition 2.1.** By Kripke model for intuitionistic logic we shall mean a structure \( M = (W, \leq, V) \), where \( W \) is a non-empty set of worlds, \( \leq \) is quasi order on \( W \), and \( V : \text{Var} \rightarrow \mathcal{P}(W) \) is a valuation, such that \( V(p) \) is \( \leq \)-closed, i.e. for every \( w \in V(p) \) and \( w \leq v \), we have \( v \in V(p) \). Every model defines a relation \( \models_{M} \subseteq W \times L' \) (that will be noted \( \models_{\cdot} \)):

- \( x \models_{p} \) iff \( x \in V(p) \)
- \( x \models_{\neg \varphi \land \psi} \) iff \( x \models_{\neg \varphi} \land x \models_{\neg \psi} \)
- \( x \models_{\varphi \lor \psi} \) iff \( x \models_{\neg \varphi} \lor x \models_{\neg \psi} \)
- \( x \models_{\neg \varphi} \) iff \( \forall_{y \in W}(x \leq y \Rightarrow y \models_{\neg \varphi}) \)
- \( x \models_{\varphi \rightarrow \psi} \) iff \( \forall_{y \in W}(x \leq y \land y \models_{\neg \varphi} \Rightarrow y \models_{\neg \psi}) \).

Let \( W_{3} := \{ x_{w} : w \in \text{Val} \} \), \( \leq_{3} \subseteq W_{3}^{2} \), \( x_{w} \leq_{3} x_{v} \) iff \( w \subseteq v \) and \( x_{w} \in V_{3}(p) \) iff \( w(p) = 1 \).

**Theorem 2.2.** \( M_{3} = (W_{3}, \leq_{3}, V_{3}) \) is a Kripke model, for which the following holds

- \( x_{w} \models_{\neg Tr(\varphi)} \) iff \( w(\varphi) = 1 \) and, consequently \( (a) \) \( x_{w} \models_{\neg Tr(\neg \varphi)} \) iff \( w(\varphi) = 0 \)
- \( x_{w} \models_{Tr(\varphi)} \) iff \( w(\varphi) = 1 \)

**Proof:** (Index 3 will be removed for the convenience.)

At first we have to show that \( M_{3} \) is indeed a Kripke model. It is easy to see that \( \leq \) preserves reflexivity and transivity from \( \sqsubseteq \). Hence \( \leq \) is quasi ordering.

Let \( p \) be any propositional variable \( p \), \( x_{w} \in V(p) \) and \( x_{w} \leq_{3} x_{v} \), so \( w \subseteq v \). Then from \( w(p) = 1 \) and the definitions we obtain \( v(p) = 1 \), so \( x_{v} \in V(p) \).

We are going to prove \((a)\), with induction of the length of formula.

- \( x_{w} \models_{Tr(p)} = p \) iff \( w \in V(p) \) iff \( w(p) = 1 \)
- \( x_{w} \models_{\neg Tr(p)} = \neg Tr(p) \) iff \( \forall_{v \supseteq w} x_{v} \models_{\neg Tr(p)} \) iff \( \forall_{v \supseteq w} v(p) \neq 1 \) iff \( w(p) = 0 \) iff \( w(\neg p) = 1 \).
\[
x_w \vdash Tr(\varphi \land \psi) = Tr(\varphi) \land Tr(\psi) \text{ iff } x_w \vdash Tr(\varphi), Tr(\psi) \text{ iff } w(\varphi) = w(\psi) = 1 \text{ iff } w(\varphi \land \psi) = 1
\]
\[
x_w \vdash Tr(\neg \neg \varphi) = Tr(\varphi) \text{ iff } w(\neg \neg \varphi) = w(\varphi) = 1
\]
\[
x_w \vdash Tr(\neg(\varphi \land \psi)) = Tr(\neg \varphi) \lor Tr(\neg \psi) \text{ iff } x_w \vdash Tr(\neg \varphi) \text{ or } x_w \vdash Tr(\neg \psi) \text{ iff } w(\neg \varphi) = 1 \text{ or } w(\neg \psi) = 1 \text{ iff } w(\neg(\varphi \land \psi)) = 1.
\]

The remaining conditions follow from the definition of partial interpretation. \qed

It is easy to see that every formula \(Tr(\varphi)\) can be generated by \(\land, \lor\) from some set of literals \(\{p_1, \ldots, p_n, \neg q_1, \ldots, \neg q_m\}\), i.e. negation may precede a variable only.

Put \(\vdash_I\) for the relation of consequence for intuitionistic logic. Let \(RMax(\vdash_I)\) be the set of \(\vdash_I\)-relatively maximal sets, in the sense that the set of \(L'\)-formulas \(\Sigma\) is an element of \(RMax(\vdash_I)\) iff there is a non-empty set of formulas \(\Pi\) closed under \(\lor\), fulfilling the conditions:

• for every \(\chi \in \Pi\), \(\Sigma \notmodels_I \chi\)
• if \(\psi \not\in \Sigma\), then there exists \(\chi \in \Pi\) such that \(\Sigma, \psi \models_I \chi\)

For any set of formulas \(\Sigma\), let \(Var(\Sigma)\) be the set of all variables occurring in \(\Sigma\) (or to be more precise – the set of all formulas from \(\Sigma\) that are variables) and \(Var^-(\Sigma) = \{\neg p : p \in Var \& \neg p \in \Sigma\}\). Let us define a relation \(\leq_I \subseteq RMax(\vdash_I) \times RMax(\vdash_I)\) in the following manner:

\[
\Sigma \leq_I \Pi \text{ iff } Var(\Sigma) \subseteq Var(\Pi) \text{ and } Var^-(\Sigma) \subseteq Var^-(\Pi).
\]

Naturally, \(\leq_I\) is a quasi order.

**Lemma 2.3.** For every \(\Sigma \in RMax(\vdash_I)\) and formulas \(\varphi, \psi\):

i) \(\varphi \land \psi \in \Sigma\) iff \(\varphi, \psi \in \Sigma\)

ii) \(\varphi \lor \psi \in \Sigma\) iff \(\varphi \in \Sigma\) or \(\psi \in \Sigma\)

iii) \(\neg p \in \Sigma\) iff \(\forall \Pi \geq_I \Sigma p \not\in \Pi\).
Proof: Ad i) follows from the fact that $\Sigma$ is a theory

Ad ii) $(\Rightarrow)$ let $\varphi \lor \psi \in \Sigma$ and for a contrary $\varphi, \psi \not\in \Sigma$. Let $\Pi$ be an appropriate set that is closed under $\lor$. Thus for some $\chi_1, \chi_2 \in \Pi$ we have $\Sigma, \varphi \vdash \chi_1$ and $\Sigma, \psi \vdash \chi_2$. Consequently, $\Sigma \vdash \varphi \lor \psi \rightarrow \chi_1 \lor \chi_2$. By intuitionistic logic we obtain $\Sigma \vdash \chi_1 \lor \chi_2$, so $\Sigma \vdash \chi_1 \lor \chi_2$ and $\chi_1 \lor \chi_2 \in \Pi$, since $\Pi$ is $\lor$-closed.

A contradiction.

$(\Leftarrow)$ obvious

Ad iii) $(\Rightarrow)$ if $\neg p \not\in \Sigma$ and $\Sigma \subseteq \Pi$, then $\neg p \in Var^-(\Sigma) \subseteq Var^-(\Pi)$. Due to the fact that $\Pi$ is an element of $RMax(\vdash_f)$, there is a set, say $\Gamma$, closed under $\lor$, for which $\Pi$ is maximal. Assume for a contradiction, that $p \in \Pi$. However, $\vdash_f \neg p \rightarrow (\neg p \rightarrow \psi)$ for any formula $\psi$. In particular, since $\Gamma \neq \emptyset$, for some $\varphi \in \Gamma$: $\vdash_f \neg p \rightarrow (\neg p \rightarrow \varphi)$. This immediately implies $\Pi \vdash \varphi$, which is impossible due to the definition of $RMax(\vdash_f)$ (and the fact that $\Pi$ is a theory).

$(\Leftarrow)$ Let $\neg p \not\in \Sigma$, thus by the thesis of intuitionistic logic $(p \rightarrow \neg p) \rightarrow \neg p$ we obtain $p \rightarrow \neg p \not\in \Sigma$. Consequently, $\Sigma, p \not\vdash \neg p$. So, we can extend $\Sigma \cup \{p\}$ to the set, relatively maximal to $\{\neg p, \neg p \lor \neg p, \neg p \lor \neg p \lor \neg p, \ldots\}$

Put $M_1 = (RMax(\vdash_f), \leq_f, V_1)$ where $\Sigma \in V_1(p)$ iff $p \in \Sigma$.

**Lemma 2.4.** For any $L$-formula $\varphi$ we have $\Sigma \models Tr(\varphi)$ iff $Tr(\varphi) \in \Sigma$

**Proof:** Follows from Lemma 2.3 and the remark concerning the general form of formulas from the set $Tr(L)$.

**Definition 2.5.** (See [1] )Let $\mathfrak{M} = (W, \leq, V)$ and $\mathfrak{N} = (W', \leq', V')$ be Kripke models. Any relation $\emptyset \neq Z \subseteq W \times W'$ will be called bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ (in symbols $Z : \mathfrak{M} \leftrightarrow \mathfrak{N}$) iff for every $xZx'$ the following conditions hold:

i) $x \in V(p)$ iff $x' \in V'(p)$ for any propositional variable $V$

ii) if $x \leq y$ then there exists $y' \in W'$ such that $x' \leq' y'$ and $yZy'$

iii) if $x' \leq' y'$ then there can be found $y \in W$ fulfilling $x \leq y$ and $yZy'$

In case $Z : \mathfrak{M} \leftrightarrow \mathfrak{N}$ and $xZx'$ we will write $Z : \mathfrak{M}, x \xrightarrow{L} \mathfrak{N}, x'$.

**Theorem 2.6.** For every models $\mathfrak{M} = (W, \leq, V)$, $\mathfrak{N} = (W', \leq', V')$ and $x \in W, x' \in W'$ such that $Z : \mathfrak{M}, x \xrightarrow{L} \mathfrak{N}, x'$ and every $L'$-formula $\varphi$:

$$x \models_{\mathfrak{M}} \varphi \text{ iff } x' \models_{\mathfrak{N}} \varphi.$$
Proof: Detailed proof concerning modal formulas can be found in [1] Theorem 2.20. Due to the fact that intuitionistic formulas can be expressed by means of the modal ones, our theorem follows.

Put $Z_K \subseteq W_3 \times R\text{Max}(\vdash_I)$ for the binary relation defined as follows: $x_w Z_K \Sigma$ iff for every propositional variable $p$: $w(p) = 1$ iff $p \in \text{Var}(\Sigma)$ and $w(p) = 0$ iff $\neg p \in \text{Var}^- (\Sigma)$

Lemma 2.7. $Z_K : \mathfrak{M}_3 \mathcal{E} \mathfrak{M}_I$. Moreover $\text{Dom}(Z_K) = W_3$ and $\text{Codom}(Z_K) = R\text{Max}(\vdash_I)$.

Proof: The first condition of bisimulation is an easy conclusion of the definitions. The same can be stated about non-emptiness of $Z_K$. Assume that $x_w Z_K \Sigma$ and $x_w \leq_3 x_v$. Put $\text{At}_v^+ := \{p \in \text{Var} : v(p) = 1\}$, $\text{At}_v^- := \{\neg p \in \text{Var} : v(p) = 0\}$ and $R_v := \{p, \neg p : v(p) = \bot\}$. Let $R_v^*$ be the closure $R_v$ under $\lor$. It can be easily shown, that from $\text{At}_v^+ \cup \text{At}_v^-$ it does not $\vdash_I$-follow any formula of $R_v^*$. So, the set $\text{At}_v^+ \cup \text{At}_v^-$ can be extended to the $\vdash_I$-theory, say $\Pi$ maximal w.r.t. $R_v^*$. Thus $x_v Z_K \Pi$ and $\text{Var}(\Sigma) \subseteq \text{Var}(\Pi)$ and $\text{Var}^- (\Sigma) \subseteq \text{Var}^- (\Pi)$. Consequently, $\Sigma \leq_I \Pi$ - we have proved that the second condition of bisimulation holds true.

For the third condition of bisimulation let $\Sigma \leq_I \Pi$ and $x_w Z_K \Sigma$. Then the valuation $v$ defined by

\[ v(p) = \begin{cases} 
1, & \text{when } p \in \Pi; \\
0, & \text{when } \neg p \in \Pi; \\
\bot, & \text{otherwise}
\end{cases} \]

has the property $x_w \leq_3 x_v$ and $x_v Z_K \Pi$.

The above observations lead us to the conclusion about the domain and codomain of $Z_K$.

Theorem 2.8. For any finite set of $L$-formulas $\Sigma_f \cup \{\varphi\}$:

$\Sigma \vdash_3 \varphi$ iff $\vdash_I \bigwedge \text{Tr}(\Sigma_f) \rightarrow \text{Tr}(\varphi)$.

Proof: ($\Rightarrow$) Let $\vdash_I \bigwedge \text{Tr}(\Sigma_f) \rightarrow \text{Tr}(\varphi)$. Then there exists a maximal $\vdash_I$-theory $\Sigma$ such that $\bigwedge \text{Tr}(\Sigma_f) \in \Sigma$ and $\text{Tr}(\varphi) \notin \Sigma$. According to Lemma 2.4, Theorem 2.6 and Lemma 2.7, there exists a point $x_w \in W_3$ such
that $x_w \vdash \bigwedge Tr(\Sigma_f)$ and $x_w \not\vdash Tr(\varphi)$. Due to Theorem 2.2 we obtain
\[ \forall \psi \in \Sigma_f, w(\psi) = 1 \text{ and } w(\varphi) \neq 1. \]
(⇐) straightforward

Put $M_{BI}$ the class of Kripke models based on binary trees, $M_{BI} \supseteq M_{BI}'$ its subclass of models having finite carrier.

**Lemma 2.9.** For any formula of the form $\varphi \rightarrow \psi$ where $\varphi$ and $\psi$ are implication free, the following holds:

$$\vdash_I \varphi \rightarrow \psi \iff \models_{M_{BI}} \varphi \rightarrow \psi.$$  

**Proof:** (⊆) if $\varphi \rightarrow \psi$ is a thesis of intuitionistic logic, then it is valid in all Kripke models, in particular in all models being finite binary trees.

(⊇) Naturally, the logic $\vdash_I$ is determined by finite trees. So, if a formula $\varphi \rightarrow \psi$ is not a thesis, then it is refutable in such a tree. In the sequel, by height of a tree we shall mean the length of the longest branch in this tree. By height of a node $w$ we shall mean the height of the tree generated by $w$.

Let $T = (W, \leq)$ be an arbitrary tree, $V$ a valuation. We are going to show, that for every $i \in \mathbb{N}$ and every $w \in W$: if $w$ has height that equals to $i$ and the number of immediate successors of $w$ is greater than 2, then a subtree generated by $w$ can be replaced by such a binary tree with the root $w'$ and the valuation $V$ by a valuation $V'$, that for every formula $\varphi$ that contains at most one $\rightarrow$ we have

$$\text{if } w' \vDash' \varphi \text{ then } w \vDash \varphi$$  

and for every $\rightarrow$-free formula $\varphi$

$$\text{if } w \vDash \varphi \text{ then } w' \vDash' \varphi$$  

where $\vDash$ corresponds to $(W, \leq, V)$ and $\vDash'$ to the obtained Kripke model equipped with the valuation $V'$.

Let $w \in W$ and $x_1, \ldots, x_n$ (for $n > 2$) be a node with its immediate successors. Then all of subtrees generated by $x_i$'s can be replaced by subtrees generated by some $(x'_i)$'s that fulfill:
for any formula $\rightarrow$-free $x_i \models \varphi \iff x'_i \models' \varphi$ (3)

If height of $x_i$ equals 1 then we just put $x'_i = x_i$, otherwise then the induction hypothesis can be used.

Define new tree with the added nodes $w_1, \ldots, w_{n-1}$ and the new relation $\leq'$:

$$w_k \leq' w_j \text{ iff } j \leq k \text{ and } w_k \leq' x'_j \text{ iff } j \leq k + 1.$$ 

Let $V'$ be a valuation that is the usual set theoretic sum of:

- the valuations associated with the trees generated by $x'_i$’s
- $V$ restricted to all nodes that do not belong to the subtree generated by $w$
- for any $i \in \{1, \ldots, n\}$ and a propositional variable $p$: $w_i \in V'(p)$ iff $w \in V(p)$

Claim (i) $w_i \models' \varphi \iff w \models \varphi$ (where $\varphi$ is $\rightarrow$-free)

and similarly for $\land$ and $\lor$ can be shown in a standard way.

Claim (ii) $w \models \varphi \Rightarrow w_i \models' \varphi$ (where $\varphi$ is $\rightarrow$-free)

$w \models \neg \varphi$ implies $\forall 1 \leq i \leq n x_i \not\models' \varphi$ and $\forall 1 \leq i \leq n x'_i \not\models' \varphi$. Assume for contrary, that $w_i \models' \varphi$ for some $1 \leq i < n$. In this case we would have $x_1 \models \varphi$.

Claim (iii) $w_{n-1} \models' \varphi \Rightarrow w \models \varphi$ (where $\varphi$ is $\rightarrow$-free)

the cases when $\varphi$ is a propositional variable, conjunction or disjunction are omitted.

If $w_{n-1} \models' \neg \varphi$, then $\forall 1 \leq i \leq n x_i \not\models' \neg \varphi$ and $\forall 1 \leq i \leq n x'_i \not\models' \varphi$. Assume that $w \not\models \varphi$, then by Claim (ii) we would obtain $w_{n-1} \models' \varphi$ which is impossible.

Claim (iv) $w_{n-1} \models' \varphi \rightarrow \psi \Rightarrow w \models \varphi \rightarrow \psi$ (where $\varphi$ and $\psi$ are $\rightarrow$-free)

Let $w_{n-1} \models' \varphi \rightarrow \psi$ and (a) $w \models \varphi$ and (b) $w \not\models \psi$. Then by Claim (ii) and (a) we have $w_{n-1} \models' \varphi$ and by Claim (iii) together with (b) we derive $w_{n-1} \not\models' \psi$ which is impossible.

The case when $x_i \models \varphi$ and $x'_i \not\models \psi$ can be also excluded due to the fact that $x_i$ and $x'_i$ are equivalent on $\{\land, \lor, \neg\}$-fragment of the language $L'$. \[\square\]
Theorem 2.10. For every finite set $\Sigma \cup \{\varphi\}$ of $L$:
$$\Sigma \vdash_3 \varphi \iff \models_{M_d} \bigwedge Tr(\Sigma) \rightarrow Tr(\varphi)$$

Proof: Follows from Lemma 2.9 and Theorem 2.8.

3. Conclusions and work for the future

In the paper we have presented some interpretation of propositional partial logic, called also $t$-validity.

The interpretation is not onto function, that is, not all formulas are values of $Tr$. Therefore we have obtained some characterization of the fragment of $L'$. It consists of formulas built up from propositional variables or their negations, by the usage of $\land$ and $\lor$ (maybe containing at most one implication). The image $Tr(L)$ of $Tr(L) \rightarrow Tr(L) = \{Tr(\varphi) \rightarrow Tr(\psi) : \varphi, \psi \in L\}$ is not “a conservative restriction” of classical logic, since $\not\vdash_3 \neg(p \land \neg p)$ and $Tr(\neg(p \land \neg p)) = p \lor \neg p$.

Naturally, we do not need to be restricted to propositional language only. [4] presents first order calculus based on partial logic. There arises a natural question - is it interpretable in the first order intuitionistic logic? On the other hand – the theory of types has a counterpart in partial logic (see [5] and [6]). Is it possible to find an appropriate intuitionistic theory of types?

References


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