A NEW VERSION OF AN OLD MODAL INCOMPLETENESS THEOREM

Abstract
Thomason [5] showed that a certain modal logic $L \subset S4$ is incomplete with respect to Kripke semantics. Later Gerson [3] showed that $L$ is also incomplete with respect to neighborhood semantics. In this paper we show that $L$ is in fact incomplete with respect to any class of complete Boolean algebras with operators, i.e. that it is completely incomplete.

1. Introduction
In 1974, two modal incompleteness theorems were published in the same issue of the same journal. Fine [2] presented a logic above $S4$ and Thomason [5] presented one between $T$ and $S4$, and both showed that their logics were incomplete with respect to Kripke semantics. In 1975, a paper by Gerson [3] followed in which he showed that both logics were both also incomplete with respect to neighborhood semantics. Then, in 2003 Litak [4] showed that Fine’s logic is in fact as he calls it completely incomplete, i.e. it is incomplete with respect to any class of complete Boolean algebras with operators (or BAOs for short). It is known that Kripke frames correspond to the class of complete, atomic and completely additive BAOs, and that neighborhood frames for normal logics such as the ones we are considering correspond to the class of complete, atomic BAOs (see Thomason [6] and Došen [1]). In the present paper, we show what one might call a complement to Litak’s result, viz. that Thomason’s logic is also completely incomplete.
2. An incompleteness theorem

2.1. Algebraic preliminaries

When considering an arbitrary complete BAO \( A \) below, we will always assume there is some Kripke frame \( \langle W, R \rangle \) such that \( A = \langle A, \&^c, -, 0, \diamond \rangle \) is a subalgebra of \( \langle \wp(W), \cap, \setminus, 0, m_R \rangle \), where \( \setminus \) is set-theoretic complementation with respect to \( W \) and for \( U \subseteq W \), \( m_R(U) := \{ w \in W \mid \exists v \in U (wRv) \} \); the Jónsson-Tarski theorem tells us that any BAO is such a subalgebra up to isomorphism. We will make use of a few observations about suprema in a complete BAO \( A \). Let \( X, Y \subseteq A \). By \( \bigcup X \) and \( \bigvee X \) we denote the supremum of \( X \) in \( \wp(W) \) and in \( A \), respectively. Because \( A \) is not a regular subalgebra of \( \wp(W) \), it may be the case that \( \bigcup X \neq \bigvee X \).

Firstly, note however that it will always be the case that \( \bigcup X \leq \bigvee X \).

Secondly, \( \bigcup X \leq \bigcup Y \) implies \( \bigvee X \leq \bigvee Y \), as for all \( a \in X \), \( a \leq \bigcup X \leq \bigcup Y \leq \bigvee Y \), so \( \bigvee Y \) is an upper bound for \( X \) in \( \mathfrak{A} \). Now since \( \bigvee X \) is the least upper bound of \( X \) in \( \mathfrak{A} \), it follows that \( \bigvee X \leq \bigvee Y \). Thirdly, \( \bigcup X \cap \bigcup Y = \emptyset \) implies \( \bigvee X \land \bigvee Y = 0 \),

for if \( \bigcup X \cap \bigcup Y = \emptyset \) but \( \bigvee X \land \bigvee Y > 0 \) then \( \bigvee X \cap \bigvee Y > 0 \). If this were not the case, then we would get \( \bigcup Y \leq \bigvee Y \setminus \bigvee X \in \mathfrak{A} \), contradicting the fact that \( \bigvee Y \) is the least upper bound of \( Y \) in \( \mathfrak{A} \). So, there must be some \( b \in Y \) such that \( b \cap \bigvee X > 0 \), and now we know that \( \bigcup X \setminus \bigvee X \setminus b \), for otherwise \( \bigvee Y \) would not be the least upper bound of \( X \) in \( \mathfrak{A} \). It follows that there must be some \( a \in X \) such that \( a \land b > 0 \); however this contradicts our assumption that \( \bigcup X \cap \bigcup Y = \emptyset \). It follows that \( (2) \) is true. Finally,

\begin{equation}
\bigvee_{a \in X} \diamond a \leq \diamond \bigvee_{a \in X} a.
\end{equation}

Take any \( a \in X \), then \( a \leq \bigvee X \). Since \( \diamond \) is order-preserving, it follows that \( \diamond a \leq \diamond \bigvee X \). Since \( a \) was arbitrary, we see that \( \diamond \bigvee X \) is an upper bound for \( \{ \diamond a \mid a \in X \} \). It follows that \( (3) \) holds.
2.2. A case of complete incompleteness

Consider the axioms

\[ A_i := □(q_i \to r) \quad (i = 1, 2), \]
\[ B_i := □(r \to ≻q_i) \quad (i = 1, 2), \]
\[ C_1 := □¬(q_1 \land q_2), \]
\[ A := r \land □p \land □²p \land A_1 \land A_2 \land B_1 \land B_2 \land C_1 \]
\[ \to ≻(r \land □(r \to q_1 \lor q_2)), \]
\[ B := □(p \to q) \to (□p \to □q), \]
\[ C := □p \to p, \]
\[ D := (p \land □²q) \to (◊q \lor □²(q \land ◊p)), \]
\[ E := (□p \land ¬□²p) \to ◻(□²p \land ¬□³p), \]
\[ F := □p \to □²p. \]

Let \( L \) be the logic containing all propositional tautologies, \( A, B, C, D \) and \( E \) and closed under modus ponens, substitution and necessitation. This is the same logic as found in [5]; note that \( B, C \) and \( F \) are perhaps better known as the \( K \)-axiom, the \( T \)-axiom and the 4-axiom respectively. It is therefore not hard to see that \( T \subseteq L \subseteq S4 \). We will see below that the latter inclusion is strict, because \( S4 \not\subseteq F \subseteq L \).

**Lemma 1.** Let \( \mathfrak{A} \) be a complete BAO. If \( \mathfrak{A} \models L \), then \( \mathfrak{A} \not\models F \).

**Proof:** Let \( \mathfrak{A} \) be a complete BAO on which \( B, C, D \) and \( E \) are valid, but \( F \) is not. We will show that \( \mathfrak{A} \not\models A \), proving the statement of the lemma.

The fact that \( \mathfrak{A} \not\models F \) must be witnessed by some \( a \in \mathfrak{A} \) such that \( □a \not\subseteq □²a \). Since by \( C \), \( □²a \leq □a \), it follows that \( □²a \not\leq □a \). For \( n \geq 1 \) we define

\[ b_n := □^n a \setminus □^{n+1} a, \]

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1 Incidentally, axiom \( B \) plays no role in the proof of our result, nor does the fact that the modalities of BAOs are additive. We only use the facts that \( □ \) (and hence also \( ◻ \)) is monotone and that \( □1 = 1 \). By assuming \( B \) we are following Thomason and ensuring that \( T \subseteq L \).

2 Using the algebraizability of modal logic, we view modal formulas as equations, e.g. the formula \( A \) corresponds to the equation \( A = 1 \).
where \( c \setminus d := c \land -d \). By the above, we already know that \( b_1 > 0 \). To inductively show that all \( b_n > 0 \), suppose that \( b_n > 0 \), but \( b_{n+1} = 0 \). Then replace \(^3 p\) with \( \Box^{n-1} a \) in \( E \), so we get

\[
b_n = \sqcap \Box^{n-1} a \land -((\Box^2 \Box^{n-1} a \land -((\Box^3 \Box^{n-1} a))) = \sqcap b_{n+1} = \Box 0 = 0,
\]

which is a contradiction, so it must be that \( b_{n+1} > 0 \). This completes our induction. Note that

\[
\text{for all } 1 \leq i < j, \quad b_i \land b_j = 0, \quad (4)
\]

since \( b_j \leq \Box^j a \leq \Box^{i+1} a \leq -b_i \). Next, we will show by induction on \( n \) that

\[
\text{for all } n \geq 2 \text{ and for all } 1 \leq i < n, \quad b_i \leq \Box b_n. \quad (5)
\]

The base case \( n = 2 \) follows immediately from \( E \). We will show that if \( (5) \) holds for \( n \), it must also hold for \( n+1 \). We only consider \( i < n \) (for if \( i = n \), we can immediately apply \( E \)). By our induction hypothesis, \( b_i \leq \Box b_n \) and by \( E \), \( b_n \leq \Box b_{n+1} \), so we have \( b_i \leq \Box^2 b_{n+1} \), i.e. \( b_i = b_i \land \Box^2 b_{n+1} \). Reverting to definitions, we find that \( \Box b_i = \Box (\Box a \setminus \Box^{i+1} a) \). As \( \Box^i a \setminus \Box^{i+1} a \leq -\Box^{i+1} a \), we get that \( \Box (\Box^i a \setminus \Box^{i+1} a) \leq \Box -\Box^{i+1} a = -\Box^{i+2} a \), so \( \Box^{i+2} a \land \Box b_i = 0 \).

Since also \( b_{n+1} \leq \Box^{n+1} a \leq \Box^{i+2} a \) (as \( i < n \)), it follows that \( b_{n+1} \land \Box b_i = 0 \), so replacing \( p \) with \( b_i \) and \( q \) with \( b_{n+1} \) in \( D \), we find that

\[
b_i = b_i \land \Box^2 b_{n+1} \leq \Box b_{n+1} \lor \Box^2 (b_{n+1} \land \Box b_i) = \Box b_{n+1} \lor \Box^2 0 = \Box b_{n+1}.
\]

It follows that \( (5) \) holds for \( n+1 \), so by induction \( (5) \) is true for all \( n \geq 2 \).

Now we define the following elements of \( \mathfrak{A} \):

\[
p := a, \quad q_i := \bigvee_{n \geq 0} b_{3n+i} \quad (i = 1, 2, 3), \quad r := \bigvee_{n \geq 1} b_n.
\]

(Note that this is where we use the assumption that \( \mathfrak{A} \) is complete.) We will use these elements to show that \( A \) is not valid. First of all, as \( \bigcup_{n \geq 0} b_{3n+i} \leq \bigcup_{n \geq 1} b_n \), it follows by \( (1) \) that \( q_i \leq r \) for \( i = 1, 2, 3 \), so \( q_i \rightarrow r = 1 \), whence \( A_1 = A_2 = \Box 1 = 1 \). Secondly, by \( (5) \), for any \( n \geq 1 \) there must exist a

\[^3 \Box^0 a := a.\]
\( k \geq 2 \) such that \( b_n \leq \Diamond b_{3k+i} \), whence \( \bigcup_{n \geq 1} b_n \leq \bigcup_{n \geq 0} \Diamond b_{3n+i} \). By (1), this means that

\[
\text{for all } 1 \leq i \leq 3, \quad r = \bigvee_{n \geq 1} b_n \leq \bigvee_{n \geq 0} \Diamond b_{3n+i} \leq \Diamond \bigvee_{n \geq 0} b_{3n+i} = \Diamond q_i, \quad (6)
\]

where the latter inequality follows from (3). Therefore, \( r \to \Diamond q_i = 1 \), so \( B_1 = B_2 = \Box 1 = 1 \). Finally, using (4), we see that \( \bigcup_{n \geq 0} b_{3n+i} \cap \bigcup_{n \geq 0} b_{3n+j} = 0 \) for all \( 1 \leq i < j \leq 3 \). It follows by (2) that

\[
\text{for all } 1 \leq i < j \leq 3, \quad q_i \land q_j = 0 \quad (7)
\]

whence \( C_1 = \Box - 0 = 1 \). Combining all this, we find that

\[
r \land \Box p \land -\Box^2 p \land A_1 \land A_2 \land B_1 \land B_2 \land C_1 = r \land (\Box a \setminus \Box^2 a) = b_1 > 0,
\]

i.e. the left hand side of \( A \) is non-zero. However, we have \( r = q_1 \lor q_2 \lor q_3 \). Since the \( q_i \) are disjoint by (7), this means that \( r \land -q_1 \land -q_2 = q_3 \). By (6), \( r \leq \Diamond q_3 \), so

\[
0 = r \land -\Diamond q_3 = r \land \Box - q_3 = r \land \Box - (r \land -q_1 \land -q_2) = r \land \Box (r \to q_1 \lor q_2).
\]

It follows that \( \Diamond (r \land \Box (r \to q_1 \lor q_2)) = 0 \), i.e. the right hand side of \( A \) is zero. We conclude that \( \mathfrak{A} \not\models A \).

For \( C \) some class of BAOs, we define \( \Delta \models_C \Gamma \) if for every \( \mathfrak{A} \in C \), \( \mathfrak{A} \models \Delta \) only if \( \mathfrak{A} \models \Gamma \).

**Corollary 2.** Let \( C \) be any class of complete BAOs. Then \( \{A, B, C, D, E\} \models_C F \).

**Lemma 3.** \( F \not\in \mathbf{L} \).

**Proof:** See the proof of the Theorem in [5]. Thomason proves our lemma by showing that the veiled recession frame, which is a general frame equivalent to an incomplete BAO, validates \( \mathbf{L} \) while \( \neg F \) can be satisfied on it.

The lemmas give us the following:

**Theorem 4.** \( \mathbf{L} \) is completely incomplete.
3. Acknowledgements

This paper is the result of Eric Pacuit asking me to write a paper about neighborhood semantics for modal logic for a class of his in January of 2006 at the Institute for Logic, Language and Computation. He was very helpful although the result strayed somewhat from the original assignment. I would also like to thank fellow students Gaëlle Fontaine and Christian Kissig for discussions. Finally, I would like to express my gratitude to the anonymous referee for their comments and suggestions.

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