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SIMPLE EXPANSIONS OF CLASSES SATISFYING FRAENKEL-CARNAP PROPERTIES

Abstract

In the 1920's Fraenkel and Carnap raised the question of whether or not every finitely axiomatizable semantically complete theory formulated in the theory of types is categorical. The question remains open. Recent papers have provided partial answers to this and a related question for theories formulated in second-order languages. These papers singled out subclasses of interpretations and showed that the second-order theories of members of these classes are categorical, if the theories are finitely axiomatizable. This paper continues the search for partial answers. It focuses on simple expansions of classes previously studied.

Keywords: second-order logic, categoricity, finite axiomatizability, homogeneous models

1. Introduction

In the late 20's Fraenkel and Carnap raised the question of whether or not all finitely axiomatizable semantically complete theories are categorical. Carnap explicitly restricted attention to theories formulated in the simple theory of types. He announced a positive answer, but his proof was flawed. It appears that the question was forgotten until the beginning of this century ([1], [2], and [3]). Related results were established in the late 60's: that for each $n \geq 2$, a semantically complete, finitely axiomatizable and satisfiable theory formulated in an n^{th}-order language has models in a single cardinality and this cardinal is characterizable in the language in the sense that there is a sentence in the language true on all and only interpretations of that cardinality. It follows that if the non-logical vocabulary of the language is empty, every finitely axiomatizable, semantically complete theory is categorical. A proof of this consequence, by Dana Scott, is outlined in [3]. Similar results have been shown to hold when the non-logical vocabulary is non-empty but contains neither functional constants nor predicate constants of degree greater than 1 ([16]).

For the most part, attempts to find a positive answer to the Fraenkel-Carnap question have been restricted to theories formulated in second-order languages. Even in this case the Fraenkel-Carnap question remains open. It has been shown that the Fraenkel-Carnap question has a positive answer when restricted to certain classes of interpretations ([14], [15], [17]). The purpose of this paper is to add to the classes for which the Fraenkel-Carnap question has a positive answer.

K is a finite type. Members of K are either individual constants, n-ary predicate constants or n-ary functional constants. L_K is the second-order language whose non-logical vocabulary is K. Interpretations for L_K (*interpretations of type* K) are ordered pairs $\mathfrak{A}$=(A, $f_{\mathfrak{A}}$), where A is a non-empty set (*the domain of* $\mathfrak{A}$) and $f_{\mathfrak{A}}$ is a function on K defined in the usual way. Sentences in L_K are interpreted in the "standard" way on interpretations of type K. L_K is *homogeneous* in the sense that for each member of K there is a variable in the logical vocabulary of L_K of the same grammatical category.

For each $n \geq 1$, $\{k_1,...,k_n\}$ is a set of individual constants not in K. $K[n]$ is the union of K and $\{k_1,..., k_n\}$. Interpretations of type $K[n]$ are of the form $(\mathfrak{A}a_1...a_n)$, where $\mathfrak{A}$ is an interpretation of type K, $a_1,...,a_n$ are in A, and a_i is the denotation of k_i in $(\mathfrak{A}a_1...a_n)$. $(\mathfrak{A}a_1...a_n)$ is *a simple expansion of* $\mathfrak{A}$.

Δ is a class of interpretations of type K. For each $n \geq 1$, Δ_n=$\{(\mathfrak{A}a_1...a_n)|$ $\mathfrak{A}\in \Delta$ and $a_1,..., a_n$ in $A\}$. Δ_n is *a simple expansion of* Δ. Δ is *a class in* L_K iff there is S, a set of sentences in L_K, such that Δ is the class of models of S. Δ is *a finitary class in* L_K iff Δ is the class of models of a finite set of sentences in L_K. If Δ is a class in L_K, then Δ_n is a class in $L_{K[n]}$; and, if Δ is a finitary class in L_K, then Δ_n is a finitary class in $L_{K[n]}$.

Isomorphisms between interpretations of the same type are defined as usual. *An automorphism on* $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$. $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* iff there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. $Th(\mathfrak{A})$ is set of all sentences in L_K true on $\mathfrak{A}$. This set is *the theory of* $\mathfrak{A}$ *(in* L_K*)*. A *pure sentence (in* L_K*)* is a sentence in L_K in which no member of K occurs.

The pure theory of $\mathfrak{A}$ is the set of pure sentences true on $\mathfrak{A}$. Interpretations of type K are *equivalent (in L_K)* iff their theories are identical. A set of sentences, T, is a *basis for $Th(\mathfrak{A})$* provided the logical consequences of T in L_K are exactly the members of $Th(\mathfrak{A})$. $Th(\mathfrak{A})$ is *finitely axiomatizable* iff there is a finite set of sentences that is a basis for $Th(\mathfrak{A})$. $Th(\mathfrak{A})$ is *quasi-finitely axiomatizable* iff there is a finite set of sentences such that the union of that set and the pure theory of $\mathfrak{A}$ is a basis for $Th(\mathfrak{A})$. $\mathfrak{A}$ is *finitely characterizable* iff there is a finite subset of $Th(\mathfrak{A})$ all of whose models are isomorphic. $\mathfrak{A}$ is *quasi-finitely characterizable* iff there is T, a finite subset of $Th(\mathfrak{A})$, such that all models of T of the same cardinality are isomorphic. Since isomorphic interpretations are equivalent, if $\mathfrak{A}$ is finitely characterizable, then $Th(\mathfrak{A})$ is finitely axiomatizable. If $\mathfrak{A}$ is quasi-finitely characterizable, then $Th(\mathfrak{A})$ is quasi-finitely axiomatizable (Lemma 4.1, [15] page 287). If $\mathfrak{A}$ is finite, then the first-order theory of $\mathfrak{A}$ is finitely axiomatizable and categorical ([5] page 106). If $Th(\mathfrak{A})$ is finitely axiomatizable, and β is the cardinality of A, then all models of $Th(\mathfrak{A})$ are of cardinality β and β is *characterizable in L_K* in the sense that there is a pure sentence in L_K true on all and only interpretations of cardinality β (Corollary 2.2, [15] page 256).

Δ *satisfies the Fraenkel-Carnap property* iff every member of Δ is finitely characterizable, if its theory is finitely axiomatizable. Δ *satisfies the quasi Fraenkel-Carnap property* iff every member of Δ is quasi-finitely characterizable, if its theory is quasi-finitely axiomatizable. Since L_K is homogeneous, if Δ satisfies the quasi Fraenkel-Carnap property, then Δ satisfies the Fraenkel-Carnap property.

Assume that Δ satisfies the quasi Fraenkel-Carnap property. The purpose of this paper is to explore the question of whether or not Δ_n satisfies the quasi Fraenkel-Carnap property. If this question has a positive answer, it follows that Δ_n satisfies the Fraenkel-Carnap property. It is shown in [17] that if Δ is the class of all infinite binary relational systems (A, ρ) were ρ is an equivalence relation on A, then Δ and Δ_n satisfy the quasi Fraenkel-Carnap property. In the following attention is restricted to simple expansions of classes previously shown to satisfy the quasi Fraenfel-Carnap property. All of the classes considered below are classes in a second-order language. Some are finitary classes.

2. The General Setting

This section provides the general setting for sections 3 and 5. Suppose that Δ is a finitary class in L_K all of whose members are infinite and that Δ satisfies the quasi Fraenkel-Carnap property. Δ_n is a finitary class in $L_{K[n]}$. The following lemmas, in essence, reduce the problem of showing that Δ_n satisfies the quasi Fraenkel-Carnap property to the study of the automorphisms on the members of Δ. The material from this section draws heavily from [15] and [17]. The first lemma was established in [15] (Theorem 5.1, page 288).

LEMMA 2.1: *Assume that Δ is a finitary class in L_K. Then, the following are equivalent:*

1. *Δ satisfies the quasi Fraenkel-Carnap property; and*
2. *for all ϕ, a sentence in L_K all of whose models are in Δ, and all infinite cardinals β, if all models of ϕ of cardinality β are equivalent in L_K, then all models of ϕ of cardinality β are isomorphic.*

LEMMA 2.2: *Assume that Δ is a finitary class in L_K satisfying the quasi Fraenkel-Carnap property, that ϕ is a sentence in $L_{K[n]}$ all of whose models are in Δ_n, and that β is an infinite cardinal. Then, if all models of ϕ of cardinality β are equivalent in $L_{K[n]}$, then there is ψ, a sentence in L_K, such that*

1. *for all $\mathfrak{A}$ of type K, $\mathfrak{A}$ is a model of ψ iff there are a_1,..., a_n in A such that $(\mathfrak{A}a_1...a_n)$ is a model of ϕ;*
2. *all models of ψ are in Δ;*
3. *all models of ψ of cardinality β are equivalent in L_K; and*
4. *all models of ψ of cardinality β are isomorphic.*

The reasoning here follows that for Lemma 3.1 of [17]. ψ is obtained from ϕ by removing each of k_1,..., k_n occurring in ϕ, replacing them by different individual variables not occurring in ϕ and existentially quantifying over these new variables. Conditions 1, 2 and 3 are immediate. Condition 4 follows from condition 3, the assumption that Δ is a finitary class satisfying the quasi Fraenkel-Carnap property and Lemma 2.1.

LEMMA 2.3: *Assume Δ is a finitary class of infinite interpretations that satisfies the quasi Fraenkel-Carnap property, $n \geq 1$, $\mathfrak{A} \in \Delta$, a_1,..., a_n, b_1,...,*

$b_n \in A$, and if ($\mathfrak{A}a_1...a_n$) is equivalent to ($\mathfrak{A}b_1...b_n$) in $L_{K[n]}$, then there is an automorphism on $\mathfrak{A}$ that takes a_i to b_i, for all i. Then, Δ_n satisfies the quasi Fraenkel-Carnap property.

PROOF: Suppose that ϕ is a sentence in $L_{K[n]}$ all of whose models are in Δ_n and that all models of ϕ of cardinality β are equivalent in $L_{K[n]}$. Suppose ($\mathfrak{A}a_1...a_n$) and ($\mathfrak{B}b_1...b_n$) are models of ϕ of cardinality β. Let ψ be the sentence constructed from ϕ as per Lemma 2.2. $\mathfrak{A}$ and $\mathfrak{B}$ are models of ψ of cardinality β. Thus, there is f, an isomophism from $\mathfrak{A}$ to $\mathfrak{B}$. f is also an isomorphism from ($\mathfrak{A}a_1...a_n$) to ($\mathfrak{B}f(a_1)...f(a_n)$). Therefore, ($\mathfrak{B}f(a_1)...f(a_n)$) and ($\mathfrak{B}b_1...b_n$) are equivalent in $L_{K[n]}$. Since $\mathfrak{B}$ is in Δ, there is g, an automorphism on $\mathfrak{B}$ taking $f(a_i)$ to b_i, for all i. The composition of f and g is an isomorphism between ($\mathfrak{A}a_1...a_n$) and ($\mathfrak{B}b_1...b_n$).

It is natural to ask whether or not the converse of Lemma 2.3 holds: if Δ_n satisfies the quasi Fraenkel-Carnap property, then for all $\mathfrak{A}$ in Δ, all a_1,..., a_n, b_1,..., b_n in A, if ($\mathfrak{A}a_1...a_n$) and ($\mathfrak{A}b_1...b_n$) are equivalent, then there is an automorphism on $\mathfrak{A}$ taking a_i to b_i. It is shown in section 4 that this converse fails.

A weaker form of Lemma 2.1 was used in [17] to establish that certain classes in L_K satisfy Fraenkel-Carnap properties. This weaker result, together with weaker forms of Lemma 2.2 and Lemma 2.3 are used in section five below.

LEMMA 2.4: *Assume that Δ is a class in L_K, that ϕ is a sentence in L_K, that β is an infinite cardinal, and that if all models of ϕ in Δ of cardinality β are equivalent in L_K, then all models of ϕ in Δ are isomorphic. Then, Δ satisfies the quasi Fraenkel-Carnap property.*

Lemma 2.4 is essentially Theorem 5.2 [15] (page 289). The condition in Lemma 2.2 and Lemma 2.3 that Δ is a finitary class in L_K can be replaced by the condition that Δ is a class in L_K, if the condition that Δ satisfies the quasi Fraenkel-Carnap property is replaced by the condition that if all models of ϕ in Δ of cardinality β are equivalent in L_K, then they are isomorphic. Let ϕ be a sentence in L_K. ϕ is *well behaved in* Δ iff for all infinite cardinals β, if all models of ϕ in Δ of cardinality β are equivalent in L_K, then all models of ϕ in Δ of cardinality β are isomorphic. By Lemma 2.4, if Δ is a class in L_K and every sentence in L_K is well behaved in Δ, then Δ satisfies the quasi Fraenkel-Carnap property. For completeness, the weaker versions of Lemma 2.2 and of Lemma 2.3 are stated below. The reasoning for these results essentially follows the reasoning above.

LEMMA 2.5: *Assume that* Δ *is a class in* L_K, *that every sentence in* L_K *is well behaved in* Δ, *that* ϕ *is a sentence in* $L_{K[n]}$ *all of whose models are in* Δ_n *and that* β *is an infinite cardinal. Then, if all models of* ϕ *of cardinality* β *are equivalent in* $L_{K[n]}$, *then there is* ψ, *a sentence in* L_K, *such that*

1. *for all* $\mathfrak{A}$, *of type* K, $\mathfrak{A}$ *is a model of* ψ *iff there are* $a_1,\ldots, a_n$ *in* A *such that* $(\mathfrak{A}a_1\ldots a_n)$ *is a model of* ϕ;
2. *all models of* ψ *are in* Δ;
3. *all models of* ψ *of cardinality* β *are equivalent in* L_K; *and*
4. *all models of* ψ *of cardinality* β *are isomorphic.*

LEMMA 2.6: *Assume that* Δ *is a class in* L_K *all of whose members are infinite and that every sentence in* L_K *is well behaved in* Δ. *Then, for all* $n \geq 1$, Δ_n *satisfies the quasi Fraenkel-Carnap property, if for all* $\mathfrak{A} \in \Delta$, *all* $a_1,\ldots, a_n, b_1,\ldots, b_n$ *in* A, *if* $(\mathfrak{A}a_1\ldots a_n)$ *is equivalent to* $(\mathfrak{A}b_1\ldots b_n)$, *then there is an automorphism on* $\mathfrak{A}$ *taking* a_i *to* b_i, *for all* i.

3. Dedekind Algebras

K=$\{s\}$ and s is a unary functional constant. $\mathfrak{A}$, an interpretation of type K, is a *Dedekind algebra* provided $f_{\mathfrak{A}}(s)$ is an injection on A. $\mathbb{D}$ is the class of infinite Dedekind algebras. $\mathbb{D}$ is a finitary class in L_K. It follows from Lemma 2 (page 94) of [14] that $\mathbb{D}$ satisfies the quasi Fraenkel-Carnap property. Reasoning proceeds by applying Lemma 2.3 above. In this application extensive use is made of the model theory of the first-order theories of Dedekind algebras ([12]).

LEMMA 3.1: *Assume that* $\mathfrak{A}$ *is an infinite Dedekind algebra, that* $a_1,\ldots, a_n, b_1,\ldots, b_n$ *are in* A, *and that* $(\mathfrak{A}a_1\ldots a_n)$ *and* $(\mathfrak{A}b_1\ldots b_n)$ *are equivalent in* $L_{K[n]}$. *Then there is an automorphism on* $\mathfrak{A}$ *taking* a_i *to* b_i, *for all* i.

PROOF: Assume that $\mathfrak{A}$ is an infinite Dedekind algebra and that $a_1,\ldots, a_n, b_1,\ldots, b_n$ are members of A. Suppose that $(\mathfrak{A}a_1\ldots a_n)$ and $(\mathfrak{A}b_1\ldots b_n)$ are equivalent in $L_{K[n]}$. There are two cases to consider.

Suppose that A is of cardinality $\aleph_0$. By Corollary 10 of [12] (page 360) all countably infinite Dedekind algebras are homogeneous prime. By supposition, $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are elementarily equivalent. Thus, as $\mathfrak{A}$ is homogeneous prime, there is an automorphism on $\mathfrak{A}$ taking a_i to b_i, for all i.

Suppose that $\mathfrak{A}$ is uncountable. By the Tarski-Vaught theorem, there is $\mathfrak{B}$, a countably infinite elementary subalgebra of $\mathfrak{A}$, whose domain includes $\{a_1,..., a_n, b_1,...,b_n\}$. Thus, $(\mathfrak{B}a_1...a_n)$ is an elementary subalgebra of $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{B}b_1...b_n)$ is an elementary subalgebra of $(\mathfrak{A}b_1...b_n)$. $\mathfrak{B}$ is an infinite Dedekind algebra and $(\mathfrak{B}a_1...a_n)$ is elementarily equivalent to $(\mathfrak{B}b_1...b_n)$. By reasoning as in the first case, there is an automorphism f on $\mathfrak{B}$ taking a_i to b_i, for all i. It suffices to find an automorphism on $\mathfrak{A}$ that extends f. By Theorem 3 of Weaver [2002] (page 351) B is closed under both $f_{\mathfrak{A}}(s)$ and its inverse. Let $\mathfrak{A}[A-B]$ be that subalgebra of $\mathfrak{A}$ generated by $A-B$. Since B is closed under $f_{\mathfrak{A}}(s)$ and its inverse, $A-B$ is the domain of $\mathfrak{A}[A-B]$. Further, $f_{\mathfrak{A}}(s)$=$f_{\mathfrak{B}}(s)\cup f_{\mathfrak{A}[A-B]}(s)$. Let h be any automorphism on $\mathfrak{A}[A-B]$. Since B and $A-B$ are disjoint, $f\cup h$ is a bijection on A. It is easily verified that $f\cup h$ is an automorphism on $\mathfrak{A}$.

By Theorem 8 of [12] (page 359) in the uncountable case $\mathfrak{A}$ need not be homogeneous prime. The above reasoning introduces a type of homogeneous model that, to the authors' knowledge, has not previously been studied in the model theory of first-order languages. The following theorem is immediate from Lemma 3.1 and Lemma 2.3.

THEOREM 3.1: *For all $n \geq 1$, $\mathbb{D}_n$ satisfies both the quasi Fraenkel-Carnap property and the Fraenkel-Carnap property.*

4. Strict Well Orders

K=$\{Q\}$ and Q is a binary relational constant. $\mathbb{W}$ is the class of infinite strict well orders. Members of $\mathbb{W}$ are of the form $\mathfrak{A}$=(A, ρ) where ρ is a strict well order on the set A. Members of $\mathbb{W}_n$ are of the form $(\mathfrak{A}a_1...a_n)$. $\mathbb{W}$ is a finitary class in L_K. It follows from Lemma 3.1 and Theorem 5.1 of [15] that $\mathbb{W}$ satisfies the quasi Fraenkel-Carnap property. Recall that isomorphisms between strict well orders are unique. Hence, all well orders are *rigid* in the sense that they have but one automorphism: the identity map on their domain. It follows that Lemma 2.3 cannot be applied here.

To understand this, let $\mathfrak{A}$ be a well order of cardinality greater than $2^{\aleph_0}$. A simple cardinality argument shows that there are distinct a and b in A such that $(\mathfrak{A}a)$ and $(\mathfrak{A}b)$ are equivalent in $L_{K[1]}$.

LEMMA 4.1: *Assume that ϕ is a sentence in $L_{K[n]}$ all of whose models are in $\mathbb{W}_n$, that β is an infinite cardinal, and that all models of ϕ of cardinality β are equivalent in $L_{K[n]}$. Then, all models of ϕ of cardinality β are isomorphic.*

PROOF: Let $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{B}b_1...b_n)$ be models of ϕ of cardinality β. Since $\mathbb{W}$ satisfies the quasi Fraenkel-Carnap property, by Lemma 2.2, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Let f be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. $(\mathfrak{A}a_1...a_n)$ is isomorphic to $(\mathfrak{B}f(a_1)...f(a_n))$. It suffices to show that $f(a_i)$=b_i.

Suppose that $f(a_i)\neq b_i$. Let $\mathfrak{B}$=(B, ρ). By supposition, either ($f(a_i)$, b_i)$\in \rho$ or (b_i, $f(a_i)$)$\in \rho$. Suppose that (b_i, $f(a_i)$)$\in \rho$. Let x_1,..., x_n be distinct individual variables not occurring in ϕ. ϕ' is the result of replacing every occurrence of k_j in ϕ by an occurrence of x_j, for all $j \neq i$. ϕ_i is the sentence obtained from ϕ' by existential quantification over all the individual variables above but x_i. All models of ϕ_i of cardinality β are equivalent in $L_{K\cup\{k_i\}}$. $(\mathfrak{B}f(a_i))$ and $(\mathfrak{B}b_i)$ are models of ϕ_i. Let e be the least member of B such that $(\mathfrak{B}e)$ is a model of ϕ_i and (e, $f(a_i)$)$\in \rho$. $(\mathfrak{B}e)$ and $(\mathfrak{B}f(a_i))$ are models of ϕ_i; hence, equivalent in $L_{K\cup\{k_i\}}$. Let ψ be the result of replacing every occurrence of k_i in ϕ_i by x_i. The following sentence is true on $(\mathfrak{B}f(a_i))$ but false on $(\mathfrak{B}e)$:

$$\exists x_i(Q(x_i, k_i)\&\psi). \tag{1}$$

The reasoning in the second case is analogous.

THEOREM 4.1: For all $n \geq 1$, $\mathbb{W}_n$ satisfies both the quasi Fraenkel-Carnap property and the Fraenkel-Carnap property.

It follows from Theorem 4.1 and above remarks that the following conditions are not equivalent when Δ is a finitary class satisfying the quasi Fraenkel-Carnap property:

1. for all $\mathfrak{A} \in \Delta$, all a_1,..., a_n, b_1..., $b_n \in A$, if $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are equivalent in $L_{K[n]}$, then there is an automorphism on $\mathfrak{A}$ taking a_i to b_i; and
2. Δ_n satisfies the quasi Fraenkel-Carnap property.

5. Groups, Fields and Vector Spaces

Several classes of algebras were shown in [15] (page 289) to satisfy both the quasi Fraenkel-Carnap property and the Fraenkel-Carnap property. It is shown in this section that the simple expansions of these classes also satisfy these properties. The results of this section use Lemma 2.3 or Lemma 2.6. Suppose that $\mathfrak{A}$ is one of these algebras, and that $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are equivalent in $L_{K[n]}$. Thus, $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are elementarily equivalent. Reasoning proceeds by showing that $\mathfrak{A}$ is saturated; hence, homogeneous in the sense of [4] (page 226) or homogeneous prime in the sense of [6] (page 69). It follows that there is an automorphism on $\mathfrak{A}$ taking a_i to b_i. The details are outlined in the proof of Lemma 5.1.

LEMMA 5.1: *Assume that Δ is a finitary class all of whose members are infinite and that either*

1. *Δ contains countable members and the first-order theory of any member of Δ is categorical in all infinite powers; or*
2. *Δ contains only uncountable members and the first-order theory of any member of Δ is categorical in all uncountable powers.*

Then, for all $\mathfrak{A} \in \Delta$, $n \geq 1$, and $a_1,\ldots, a_n$, $b_1,\ldots, b_n$ in A, if $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are equivalent in $L_{K[n]}$, there is an automorphism on $\mathfrak{A}$ taking a_i to b_i, for all i.

PROOF: Suppose that Δ contains members that are countably infinite. Let $\mathfrak{A}$ be a member of Δ. Suppose that $\mathfrak{A}$ is countably infinite and that $(\mathfrak{A}a_1...a_n)$ is equivalent to $(\mathfrak{A}b_1...b_n)$ in $L_{K[n]}$. By hypothesis, all countably infinite models of the first-order theory of $\mathfrak{A}$ are isomorphic, By Theorem 2.3.13 of [7] (page 101) $\mathfrak{A}$ is saturated. Hence, $\mathfrak{A}$ is homogeneous prime (see [6], page 69). By supposition, $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are elementarily equivalent and, as $\mathfrak{A}$ is homogeneous prime, there is an automorphism on $\mathfrak{A}$ taking a_i to b_i. Suppose that $\mathfrak{A}$ is uncountable and that $(\mathfrak{A}a_1...a_n)$ is equivalent to $(\mathfrak{A}b_1...b_n)$ in $L_{K[n]}$. The first-order theory of $\mathfrak{A}$ is categorical in all infinite powers. By Theorem 37.4 of [9] (page 257) every uncountable model of the first-order theory of $\mathfrak{A}$ is saturated. Hence, as above, $\mathfrak{A}$ is homogeneous prime. Since $(\mathfrak{A}a_1...a_n)$ and $(\mathfrak{A}b_1...b_n)$ are elementarily equivalent, $\mathfrak{A}$ has the desired automorphism. When Δ contains only uncountable members the reasoning proceeds as in the second case above.

Suppose that Δ is the class of all infinite Abelian groups all of whose elements are of prime order p. By remarks on page 289 of [15], Δ is a finitary class. By Proposition 1.4.7 of [7] (page 40), the first-order theory of any member of Δ is categorical in all infinite powers. Hence, by Lemma 2.3 and Lemma 5.1, Δ satisfies the quasi Fraenkel-Carnap property. Suppose that Δ is the class of all uncountable divisible torsion-free Abelian groups. As above, Δ is a finitary class. By Proposition 1.4.8 of [7] (page 40), the first-order theory of any member of Δ is categorical in all uncountable powers. Reasoning as above, Δ satisfies the quasi Fraenkel-Carnap property. This reasoning establishes the following.

THEOREM 5.1: *Assume that Δ is either*

1. *the class of all infinite Abelian groups all of whose elements are of prime order p; or*
2. *the class of all uncountable divisible torsion-free Abelian groups.*

Then, Δ_n satisfies both the quasi Fraenkel-Carnap property and the Fraenkel-Carnap property, for all n.

It was shown in [15] that each of the following satisfy the quasi Fraenkel-Carnap property:

1. the class of all uncountable algebraically closed fields of characteristic p, where p is a prime;
2. the class of all uncountable algebraically closed fields of characteristic 0; and
3. the class of all uncountable n-dimensional vector spaces over an algebraically closed field of characteristic p, where p is prime.

Each of these classes is a class in a second-order language. Further, the reasoning in [15] showed that each sentence in this language is well behaved in the class. Hence, Lemma 2.6 can be applied to show that the simple expansions of these classes satisfy the quasi Fraenkel-Carnap property. By Proposition 1.4.10 of [7] and Corollary 1.17 in [4] the first-order theory of any member of one of the above classes is categorical in all uncountable powers. Thus by reasoning as above, all members of these classes are saturated, hence, homogeneous prime. Therefore, if $(\mathfrak{A}a_1...a_n)$ is equivalent to $(\mathfrak{A}b_1...b_n)$ in $L_{K[n]}$, then there is an automorphism on $\mathfrak{A}$ taking a_i to b_i. This reasoning establishes the following.

THEOREM 5.2: *The simple expansions of each of the following classes satisfy both the quasi Fraenkel-Carnap property and the Fraenkel-Carnap property:*

1. *the class of all uncountable algebraically closed fields of characteristic p, where p is a prime;*
2. *the class of all uncountable algebraically closed fields of characteristic 0; and*
3. *the class of all uncountable n-dimensional vector spaces over an algebraically closed field of characteristic p, where p is prime.*

6. Open Questions

The question of whether or not the simple expansions of the class Δ satisfy the quasi Freankel-Carnap property, if Δ does, remains open. Similar remarks hold for the case in which Δ satisfies the Fraenkel-Carnap property. The reasoning for Lemma 2.2 can be modified to show that if $Th((\mathfrak{A}a_1...a_n))$ is finitely axiomatizable, then $Th(\mathfrak{A})$ is also finitely axiomatizable. Hence, isomorphisms between models of the first theory can be constructed by composing isomorphisms between models of the second theory and automorphisms on $\mathfrak{A}$. In all of the examples considered above, save the well orders, the proofs used Lemma 2.3 and proceeded by showing that members of Δ are "homogeneous" in some sense. Classes of infinite rigid interpretations (e.g. the rigid linear orderings of [8], page 133) look like a interesting area for further exploration, precisely because Lemma 2.3 cannot be applied in this case either.

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