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M. Spinks and R. Veroff*

SLANEY'S LOGIC $\mathbf{F}^{**}$ IS CONSTRUCTIVE LOGIC WITH STRONG NEGATION

Abstract

In [19] Slaney *et al.* introduced a little known deductive system $\mathbf{F}^{**}$ in connection with the problem of the indeterminacy of future contingents. The main result of this paper shows that, up to definitional equivalence, $\mathbf{F}^{**}$ has a familiar description: it is precisely Nelson's constructive logic with strong negation [25].

1. Introduction

Let $\Sigma[\mathbf{IPC}]$ denote the Hilbert-style presentation of Blok and Pigozzi [2, Example 2.2.2] of the intuitionistic propositional calculus **IPC** over the language $\Lambda[\mathbf{IPC}] := \{\wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1}\}$, where $\wedge, \vee, \to$ are binary logical connectives, $\neg$ is a unary logical connective, and $\mathbf{0}$ and $\mathbf{1}$ are nullary logical connectives respectively. *Constructive logic with strong negation*, denoted **N**, is the deductive system over the language $\Lambda[\mathbf{N}] := \Lambda[\mathbf{IPC}] \cup \{\sim\}$, where $\sim$ is a unary logical connective (called the *strong negation*), determined by the axioms and inference rules of $\Sigma[\mathbf{IPC}]$ together with the axioms [25]:[1]

$$\begin{array}{cc} \sim p \to (p \to q) & \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q) \\ \sim(p \to q) \leftrightarrow (p \wedge \sim q) & \sim(\neg p) \leftrightarrow p \\ \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q) & \sim(\sim p) \leftrightarrow p. \end{array}$$

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[1] Constructive logic with strong negation originates with the work of David Nelson [13, 14]. The presentation of **N** given here is taken from Vakarelov [25].

(Here $p \leftrightarrow q$ abbreviates $(p \to q) \wedge (q \to p)$, etc.)

Let $\mathbf{FL}_{ew}$ denote the full Lambek calculus with exchange and weakening, over the language $\Lambda[\mathbf{FL}_{ew}] := \{\wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1}\}$, where $\wedge$, $\vee$, $*$, and $\Rightarrow$ are binary logical connectives and $\mathbf{0}$ and $\mathbf{1}$ are nullary logical connectives respectively. For an explicit axiomatisation of $\mathbf{FL}_{ew}$ in the signature $\Lambda[\mathbf{FL}_{ew}]$, see [22, Section 5]. *Nelson* $\mathbf{FL}_{ew}$*-logic*, in symbols $\mathbf{NFL}_{ew}$, is the axiomatic extension of $\mathbf{FL}_{ew}$ by the axioms

$$\sim\sim p \Rightarrow p \qquad \text{(Double Neg.)}$$

$$\big(p \wedge (q \vee r)\big) \Rightarrow \big((p \wedge q) \vee (p \wedge r)\big) \qquad \text{(Distributivity)}$$

$$\big(p \Rightarrow (p \Rightarrow (p \Rightarrow q))\big) \Rightarrow \big(p \Rightarrow (p \Rightarrow q)\big) \qquad \text{(3-potency)}$$

$$\big((p \Rightarrow (p \Rightarrow q)) \wedge (\sim q \Rightarrow (\sim q \Rightarrow \sim p))\big) \Rightarrow (p \Rightarrow q) \qquad \text{(Nelson).}$$

(Here $\sim p$ abbreviates $p \Rightarrow \mathbf{0}$, etc.)

In [21, 22] the current authors showed that, to within definitional equivalence, constructive logic with strong negation may be presented as a substructural logic, to wit, $\mathbf{NFL}_{ew}$. A detailed (algebraic) analysis of constructive logic with strong negation, considered as a substructural logic, can be found in the paper [4] of Busaniche and Cignoli.

In response to the well known philosophical problems surrounding the indeterminacy of future contingents, in [19] Slaney *et al.* introduced a certain little known logic $\mathbf{F}^{**}$. The deductive system $\mathbf{F}^{**}$, which has language $\Lambda[\mathbf{F}^{**}] := \{\wedge, \vee, \Rightarrow, \sim\}$ where $\wedge$, $\vee$, and $\Rightarrow$ are binary logical connectives and $\sim$ is a unary logical connective, is presented by the following collection of axioms and inference rules:

$$p \Rightarrow \big((p \Rightarrow q) \Rightarrow q\big) \qquad \text{(A1)}$$

$$(p \Rightarrow q) \Rightarrow \big((q \Rightarrow r) \Rightarrow (p \Rightarrow r)\big) \qquad \text{(A2)}$$

$$(p \wedge q) \Rightarrow p \qquad \text{(A3)}$$

$$(p \wedge q) \Rightarrow q \qquad \text{(A4)}$$

$$\big((p \Rightarrow q) \wedge (p \Rightarrow r)\big) \Rightarrow \big(p \Rightarrow (q \wedge r)\big) \qquad \text{(A5)}$$

$$p \Rightarrow (p \vee q) \qquad \text{(A6)}$$

$$q \Rightarrow (p \vee q) \qquad \text{(A7)}$$

$$\big((p \Rightarrow r) \wedge (q \Rightarrow r)\big) \Rightarrow \big((p \vee q) \Rightarrow r\big) \qquad \text{(A8)}$$

$$\big(p \wedge (q \vee r)\big) \Rightarrow \big((p \wedge q) \vee r\big) \qquad \text{(A9)}$$

$$\sim\sim p \Rightarrow p \qquad \text{(A10)}$$

$$(p \Rightarrow {\sim} q) \Rightarrow (q \Rightarrow {\sim} p) \tag{A11}$$

$$p \Rightarrow (q \Rightarrow p) \tag{A12}$$

$$\big((p \Rightarrow (p \Rightarrow q)) \wedge ({\sim} q \Rightarrow (p \Rightarrow q))\big) \Rightarrow (p \Rightarrow q) \tag{A13}$$

$$p, p \Rightarrow q \vdash_{\mathbf{F}^{**}} q \tag{MP}$$

$$p, q \vdash_{\mathbf{F}^{**}} p \wedge q. \tag{ADJ}$$

Restall further studies $\mathbf{F}^{**}$ and its connections with the problem of future contingents in [17].

As the formulas $p \Rightarrow q$ and ${\sim} q \Rightarrow {\sim} p$ are synonymous (in the sense of Smiley [20]) over each of $\mathbf{NFL}_{ew}$ and $\mathbf{F}^{**}$, the pivotal axiom (A13) of $\mathbf{F}^{**}$ is a theorem of $\mathbf{NFL}_{ew}$, and conversely, the crucial axiom (Nelson) of $\mathbf{NFL}_{ew}$ is a theorem of $\mathbf{F}^{**}$. In light of the definitional equivalence of $\mathbf{NFL}_{ew}$ and $\mathbf{N}$, it is therefore natural to enquire as to the precise connection (if any) between the deductive systems $\mathbf{N}$ and $\mathbf{F}^{**}$.

This query is particularly germane inasmuch as Slaney *et al.* have shown that the logic $\mathbf{F}^{**}$ has many desirable properties, including: a simple and intuitive frame semantics; an elegant natural deduction presentation much in the style of Lemmon [9]; metacompleteness; the disjunction property; and the finite model property.

The aim of this paper is thus to establish the following theorem:

THEOREM 1.1.

1. *The map* $\delta : \Lambda[\mathbf{F}^{**}] \to \mathrm{Fm}_{\Lambda[\mathbf{N}]}$ *defined by*

$$p \wedge q \mapsto p \wedge q$$
$$p \vee q \mapsto p \vee q$$
$$p \Rightarrow q \mapsto (p \to q) \wedge ({\sim} q \to {\sim} p)$$
$${\sim} p \mapsto {\sim} p$$

   is an interpretation of $\mathbf{F}^{**}$ *in* $\mathbf{N}$.

2. *The map* $\varepsilon : \Lambda[\mathbf{N}] \to \mathrm{Fm}_{\Lambda[\mathbf{F}^{**}]}$ *defined by*

$$p \wedge q \mapsto p \wedge q$$
$$p \vee q \mapsto p \vee q$$
$$p \to q \mapsto p \Rightarrow (p \Rightarrow q)$$
$$\neg p \mapsto p \Rightarrow \big(p \Rightarrow {\sim}(p \Rightarrow p)\big)$$
$${\sim} p \mapsto p \Rightarrow {\sim}(p \Rightarrow p)$$

$$\mathbf{0} \mapsto \sim(p \Rightarrow p)$$
$$\mathbf{1} \mapsto p \Rightarrow p$$

is an interpretation of $\mathbf{N}$ *in* $\mathbf{F}^{**}$.

3. *The interpretations* δ *and* ε *are mutually inverse.*

Hence the deductive systems $\mathbf{N}$ *and* $\mathbf{F}^{**}$ *are definitionally equivalent.*

(Here and elsewhere in this paper the notion of definitional equivalence used is that of [22].)

The proof of Theorem 1.1 proceeds via a series of lemmas, several of which were obtained with the assistance of the automated reasoning program Prover9 [10], using the method of proof sketches [28]. In the sequel, results having machine-oriented proofs obtained from first principles are flagged with '*' for easy identification. For the complete set of automated proofs supporting this paper, see the companion Web site [23].

2. Proof of Theorem 1.1

Throughout this section we assume familiarity with the theory of regularly algebraisable logics, as presented in [6] or [7].

Let **RW** denote the deductive system presented by the axioms (A1)–(A11) and the rules of inference (MP) and (ADJ). The following lemma is essentially well known.

Lemma 2.1. *The deductive system* $\mathbf{F}^{**}$ *is regularly algebraisable with finite system of equivalence formulas* $\{p \Rightarrow q, q \Rightarrow p\}$.

Proof: It is well known that **RW** is finitely equivalential with finite system of equivalence formulas $\{p \Rightarrow q, q \Rightarrow p\}$. As finite equivalentiality of a deductive system is preserved on passage to axiomatic extensions (*cf.* [1, Corollary 4.9]), we have that $\mathbf{F}^{**}$ is finitely equivalential.

Since $\mathbf{F}^{**}$ is finitely equivalential, to see $\mathbf{F}^{**}$ is regularly algebraisable it suffices by [1, Corollary 4.8] to show

$$p, q \vdash_{\mathbf{F}^{**}} p \Rightarrow q \tag{1}$$
$$p, q \vdash_{\mathbf{F}^{**}} q \Rightarrow p. \tag{2}$$

Now the derivation

1. $\vdash_{\mathbf{F}^{**}} q \Rightarrow (p \Rightarrow q)$ (A12)

2. $p, q \vdash_{\mathbf{F}^{**}} q$ (Hyp.)
3. $p, q \vdash_{\mathbf{F}^{**}} p \Rightarrow q$ 1., 2., (MP)

establishes (1), and the derivation of (2) is similar. □

Let $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ denote the equivalent quasivariety semantics of $\mathbf{F}^{**}$. By [26, Theorem 3.2.4, p. 182] $\mathbf{1}^f := x \Rightarrow x$ is a constant term of $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$; moreover, the regular algebraisability of $\mathbf{F}^{**}$ guarantees that $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ satisfies an identity of the form $\varphi \approx \mathbf{1}^f$ for each axiom φ of the presentation of $\mathbf{F}^{**}$ given in Section 1. Denote any identity so obtained by $\varphi[\approx \mathbf{1}^f]$. By [7, Theorem 30], $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ is axiomatised by the identities (A1)$[\approx \mathbf{1}^f]$–(A13)$[\approx \mathbf{1}^f]$ together with the quasi-identities:

$$x \approx \mathbf{1}^f \ \text{ and } \ x \Rightarrow y \approx \mathbf{1}^f \ \text{ implies } \ y \approx \mathbf{1}^f \tag{3}$$

$$x \approx \mathbf{1}^f \ \text{ and } \ y \approx \mathbf{1}^f \ \text{ implies } \ x \wedge y \approx \mathbf{1}^f \tag{4}$$

$$x \Rightarrow y \approx \mathbf{1}^f \ \text{ and } \ y \Rightarrow x \approx \mathbf{1}^f \ \text{ implies } \ x \approx y. \tag{5}$$

Recall next that the deductive system $\mathbf{NFL}_{ew}$ is regularly algebraisable [22, Section 5, p. 420], and further, that its equivalent quasivariety semantics is the variety NFL_{ew} of all Nelson FL_{ew}-algebras [22, Corollary 5.6]. Here, an *FL_{ew}-algebra* $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a commutative integral residuated lattice with distinguished least element $0 \in A$; note that any FL_{ew}-algebra satisfies the identity $x \Rightarrow x \approx \mathbf{1}$. A *Nelson FL_{ew}-algebra* is a 3-potent, distributive, involutive FL_{ew}-algebra satisfying the *Nelson identity*

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim^n} y \Rightarrow ({\sim^n} y \Rightarrow {\sim^n} x)\big) \approx x \Rightarrow y. \tag{N}$$

(Here ${\sim^n} x$ abbreviates $x \Rightarrow \mathbf{0}$, etc.) For details, see Spinks and Veroff [21, Section 2.4].

Now we have to hand all the ingredients needed to establish:

LEMMA 2.2 (*). *The map* $\delta_1 : \Lambda[\mathbf{F}^{**}] \to \mathrm{Fm}_{\Lambda[\mathbf{FL}_{ew}]}$ *defined by*

$$x \wedge y \mapsto x \wedge y \qquad x \Rightarrow y \mapsto x \Rightarrow y$$
$$x \vee y \mapsto x \vee y \qquad {\sim} x \mapsto x \Rightarrow \mathbf{0}$$

is an interpretation of $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ *in* NFL_{ew}.

PROOF: Let $\mathbf{A} \in \mathsf{NFL}_{ew}$. Since $\mathbf{A}$ is an FL_{ew}-algebra, we certainly have that $\mathbf{A}^{\delta_1} \models$ (A1)$[\approx \mathbf{1}^f]$–(A12)$[\approx \mathbf{1}^f]$, and further, that $\mathbf{A}^{\delta_1} \models$ (3)–(5). To see $\mathbf{A}^{\delta_1} \in \mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$, therefore, it remains only to show that $\mathbf{A}^{\delta_1} \models$ (A13)[$\approx \mathbf{1}^f$]. Since $\mathbf{A}$ satisfies

$$x \Rightarrow y \approx {\sim}^n \, y \Rightarrow {\sim}^n \, x \tag{6}$$

we have also that $\mathbf{A}$ satisfies

$$\begin{aligned} \mathbf{1} &\approx \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}^n \, y \Rightarrow ({\sim}^n \, y \Rightarrow {\sim}^n \, x))\big) \Rightarrow (x \Rightarrow y) && \text{by (N)} \\ &\approx \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}^n \, y \Rightarrow (x \Rightarrow y))\big) \Rightarrow (x \Rightarrow y) && \text{by (6)} \end{aligned}$$

It follows that $\mathbf{A}^{\delta_1} \models$ (A13)[$\approx \mathbf{1}^f$], as desired. □

The proof of the next lemma is an easy computation.

Lemma 2.3 (*). $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ *satisfies the identities*

$$x \Rightarrow x \approx y \Rightarrow y \tag{7}$$

$${\sim}\, x \approx x \Rightarrow {\sim}(y \Rightarrow y) \tag{8}$$

$$x \Rightarrow y \approx {\sim}\, y \Rightarrow {\sim}\, x. \tag{9}$$

Lemma 2.4 (*). *The map* $\varepsilon_1 : \Lambda[\mathbf{FL}_{ew}] \to \mathrm{Fm}_{\Lambda[\mathbf{F}^{**}]}$ *defined by*

$$\begin{array}{rcl} x \wedge y \mapsto x \wedge y & \qquad & x \Rightarrow y \mapsto x \Rightarrow y \\ x \vee y \mapsto x \vee y & \qquad & \mathbf{0} \mapsto {\sim}(x \Rightarrow x) \\ x * y \mapsto {\sim}(x \Rightarrow {\sim}\, y) & \qquad & \mathbf{1} \mapsto x \Rightarrow x \end{array}$$

is an interpretation of NFL_{ew} *in* $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$.

Proof: Let $\mathbf{A} \in \mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$. Since $\mathbf{F}^{**}$ is an axiomatic extension of the extension of **RW** by the weakening axiom (A12), we can infer that $\mathbf{A}^{\varepsilon_1}$ is a distributive involutive FL_{ew}-algebra (*cf.* [27, Sections 3–4]). In view of [4, Remark 2.1, Theorem 2.2], to see that $\mathbf{A}^{\varepsilon_1}$ is a Nelson FL_{ew}-algebra it therefore suffices to show $\mathbf{A}^{\varepsilon_1}$ satisfies the identity

$$\big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}^n \, y \Rightarrow ({\sim}^n \, y \Rightarrow {\sim}^n \, x))\big) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}. \tag{10}$$

Since $\mathbf{A} \models$ (A13)[$\approx \mathbf{1}^f$], we have also that $\mathbf{A}$ satisfies

$$\begin{aligned} \mathbf{1}^f &\approx \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}\, y \Rightarrow (x \Rightarrow y))\big) \Rightarrow (x \Rightarrow y) \\ &\approx \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}\, y \Rightarrow ({\sim}\, y \Rightarrow {\sim}\, x))\big) \Rightarrow (x \Rightarrow y) && \text{by (9).} \end{aligned}$$

By several applications of (8), it follows that $\mathbf{A}$ satisfies the identity

$$\mathbf{1}^f \approx \big((x \Rightarrow (x \Rightarrow y)) \wedge ((y \Rightarrow \mathbf{0}^{\varepsilon_1}) \Rightarrow ((y \Rightarrow \mathbf{0}^{\varepsilon_1}) \Rightarrow (x \Rightarrow \mathbf{0}^{\varepsilon_1})))\big) \Rightarrow (x \Rightarrow y).$$

It follows that $\mathbf{A}^{\varepsilon_1} \models (10)$, as desired. □

THEOREM 2.5.

1. *The map* $\delta_1 : \Lambda[\mathbf{F}^{**}] \to \mathrm{Fm}_{\Lambda[\mathbf{FL}_{ew}]}$ *of Lemma 2.2 is an interpretation of* $\mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$ *in* NFL_{ew}.
2. *The map* $\varepsilon_1 : \Lambda[\mathbf{FL}_{ew}] \to \mathrm{Fm}_{\Lambda[\mathbf{F}^{**}]}$ *of Lemma 2.4 is an interpretation of* NFL_{ew} *in* $\mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$.
3. *The interpretations* δ_1 *and* ε_1 *are mutually inverse.*

Hence the variety NFL_{ew} *and the quasivariety* $\mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$ *are term equivalent.*

PROOF: It remains only to establish Item (3).

Suppose $\mathbf{A} \in \mathsf{NFL}_{ew}$ and $a, b \in A$. Then with $\mathbf{A}^{\delta_1} \in \mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$ and $\mathbf{A}^{\delta_1\varepsilon_1} \in \mathsf{NFL}_{ew}$, we have:

(i) $a \wedge^{\mathbf{A}^{\delta_1\varepsilon_1}} b = a \wedge^{\mathbf{A}} b$ and $a \vee^{\mathbf{A}^{\delta_1\varepsilon_1}} b = a \vee^{\mathbf{A}} b$ and $a \Rightarrow^{\mathbf{A}^{\delta_1\varepsilon_1}} b = a \Rightarrow^{\mathbf{A}} b$.

(ii) $a *^{\mathbf{A}^{\delta_1\varepsilon_1}} b = {\sim}^{\mathbf{A}^{\delta_1}}(a \Rightarrow^{\mathbf{A}^{\delta_1}} {\sim}^{\mathbf{A}^{\delta_1}} b)$
$= \big(a \Rightarrow^{\mathbf{A}} (b \Rightarrow^{\mathbf{A}} \mathbf{0}^{\mathbf{A}})\big) \Rightarrow^{\mathbf{A}} \mathbf{0}^{\mathbf{A}}$
$= a *^{\mathbf{A}} b$ by [15, Lemma 3.1.(2)].

(iii) $\mathbf{0}^{\mathbf{A}^{\delta_1\varepsilon_1}} = {\sim}^{\mathbf{A}^{\delta_1}}(a \Rightarrow^{\mathbf{A}^{\delta_1}} a) = (a \Rightarrow^{\mathbf{A}} a) \Rightarrow^{\mathbf{A}} \mathbf{0}^{\mathbf{A}} = \mathbf{1}^{\mathbf{A}} \Rightarrow^{\mathbf{A}} \mathbf{0}^{\mathbf{A}} = \mathbf{0}^{\mathbf{A}}$.

(iv) $\mathbf{1}^{\mathbf{A}^{\delta_1\varepsilon_1}} = a \Rightarrow^{\mathbf{A}^{\delta_1}} a = a \Rightarrow^{\mathbf{A}} a = \mathbf{1}^{\mathbf{A}}$.

Thus $\mathbf{A}^{\delta_1\varepsilon_1} = \mathbf{A}$.

Suppose $\mathbf{A} \in \mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$ and $a, b \in A$. Then with $\mathbf{A}^{\varepsilon_1} \in \mathsf{NFL}_{ew}$ and $\mathbf{A}^{\varepsilon_1\delta_1} \in \mathsf{Alg\,Mod}^*\,\mathbf{F}^{**}$, we have:

(i) $a \wedge^{\mathbf{A}^{\varepsilon_1\delta_1}} b = a \wedge^{\mathbf{A}} b$ and $a \vee^{\mathbf{A}^{\varepsilon_1\delta_1}} b = a \vee^{\mathbf{A}} b$ and $a \Rightarrow^{\mathbf{A}^{\varepsilon_1\delta_1}} b = a \Rightarrow^{\mathbf{A}} b$.

(ii) ${\sim}\, a^{\mathbf{A}^{\varepsilon_1\delta_1}} = a \Rightarrow^{\mathbf{A}^{\varepsilon_1}} \mathbf{0}^{\mathbf{A}^{\varepsilon_1}}$
$= a \Rightarrow^{\mathbf{A}} {\sim}^{\mathbf{A}}(b \Rightarrow^{\mathbf{A}} b)$
$= {\sim}^{\mathbf{A}} a$ by (8).

Thus $\mathbf{A}^{\varepsilon_1\delta_1} = \mathbf{A}$. □

Recall next that constructive logic with strong negation **N** is regularly algebraisable [16, Chapter XII], and moreover, that its equivalent quasivariety semantics is the variety N of all Nelson algebras [16, Chapter V]. Here, a *Nelson algebra* is an algebra $\langle A; \wedge, \vee, \rightarrow, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 1, 0, 0\rangle$ such that $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and moreover the following identities are satisfied [3, Definition 5.1]:

$$
\begin{aligned}
(x \wedge \sim x) \wedge (y \vee \sim y) &\approx x \wedge \sim x & x \rightarrow x &\approx \mathbf{1} \\
(x \rightarrow y) \wedge (x \rightarrow z) &\approx x \rightarrow (y \wedge z) & \neg x &\approx x \rightarrow \mathbf{0} \\
x \wedge (\sim x \vee y) &\approx x \wedge (x \rightarrow y) & (x \wedge y) \rightarrow z &\approx x \rightarrow (y \rightarrow z). \\
(x \rightarrow y) \wedge (\sim x \vee y) &\approx \sim x \vee y
\end{aligned}
$$

The following theorem is the main result of [21].

Theorem 2.6. *[21, Theorem 1.1]*

1. *The map* $\delta_2 : \Lambda[\mathbf{FL}_{ew}] \rightarrow \mathrm{Fm}_{\Lambda[\mathbf{N}]}$ *defined by*

$$
\begin{aligned}
x \wedge y &\mapsto x \wedge y \\
x \vee y &\mapsto x \vee y \\
x * y &\mapsto \sim(x \rightarrow \sim y) \vee \sim(y \rightarrow \sim x) \\
x \Rightarrow y &\mapsto (x \rightarrow y) \wedge (\sim y \rightarrow \sim x) \\
\mathbf{0} &\mapsto \mathbf{0} \\
\mathbf{1} &\mapsto \mathbf{1}
\end{aligned}
$$

   is an interpretation of NFL_{ew} *in* N.

2. *The map* $\varepsilon_2 : \Lambda[\mathbf{N}] \rightarrow \mathrm{Fm}_{\Lambda[\mathbf{FL}_{ew}]}$ *defined by*

$$
\begin{aligned}
x \wedge y &\mapsto x \wedge y \\
x \vee y &\mapsto x \vee y \\
x \rightarrow y &\mapsto x \Rightarrow (x \Rightarrow y) \\
\neg x &\mapsto x \Rightarrow (x \Rightarrow \mathbf{0}) \\
\sim x &\mapsto x \Rightarrow \mathbf{0} \\
\mathbf{0} &\mapsto \mathbf{0} \\
\mathbf{1} &\mapsto \mathbf{1}
\end{aligned}
$$

   is an interpretation of N *in* NFL_{ew}.

3. *The interpretations* δ_2 *and* ε_2 *are mutually inverse.*

Hence the varieties N *and* NFL_{ew} *are term equivalent.*

Since term equivalence is an equivalence relation on quasivarieties (*cf.* [11, Section 4.12, p. 246]), on combining Theorem 2.5 with Theorem 2.6 and simplifying the resulting interpretations, we have:

THEOREM 2.7.

1. *The map* $\delta : \Lambda[\mathbf{F}^{**}] \to \mathrm{Fm}_{\Lambda[\mathbf{N}]}$ *defined by*

$$\begin{aligned} x \wedge y &\mapsto x \wedge y \\ x \vee y &\mapsto x \vee y \\ x \Rightarrow y &\mapsto (x \to y) \wedge ({\sim}\, y \to {\sim}\, x) \\ {\sim}\, x &\mapsto {\sim}\, x \end{aligned}$$

is an interpretation of $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ *in* N.

2. *The map* $\varepsilon : \Lambda[\mathbf{N}] \to \mathrm{Fm}_{\Lambda[\mathbf{F}^{**}]}$ *defined by*

$$\begin{aligned} x \wedge y &\mapsto x \wedge y \\ x \vee y &\mapsto x \vee y \\ x \to y &\mapsto x \Rightarrow (x \Rightarrow y) \\ \neg x &\mapsto x \Rightarrow \big(x \Rightarrow ({\sim}(x \Rightarrow x))\big) \\ {\sim}\, x &\mapsto x \Rightarrow {\sim}(x \Rightarrow x) \\ \mathbf{0} &\mapsto {\sim}(x \Rightarrow x) \\ \mathbf{1} &\mapsto x \Rightarrow x \end{aligned}$$

is an interpretation of N *in* $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$.

3. *The interpretations* δ *and* ε *are mutually inverse.*

Hence the variety N *and the quasivariety* $\mathsf{Alg\,Mod}^* \, \mathbf{F}^{**}$ *are term equivalent.*

Recall from general algebra that a quasivariety K with a constant term $\mathbf{1}$ is *relatively* $\mathbf{1}$*-regular* if, whenever $\mathbf{A} \in \mathsf{K}$ and $\theta, \phi \in \mathrm{Con}_{\mathsf{K}}\, \mathbf{A}$ with $\mathbf{1}^{\mathbf{A}}/\theta = \mathbf{1}^{\mathbf{A}}/\phi$, we have that $\theta = \phi$. (Here $\mathrm{Con}_{\mathsf{K}}\, \mathbf{A}$ denotes the set of all congruences θ on $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathsf{K}$). By van Alten [26, Theorem 3.2.4, p. 182], the equivalent quasivariety semantics of any regularly algebraisable deductive system $\mathbf{S}$ is a relatively $\mathbf{1}$-regular quasivariety K (for some constant term $\mathbf{1}$ of K).

Now we have all the machinery in place to state the following result, which gives a sufficient condition for lifting the term equivalence of well behaved 'quasivarieties of logic' directly to the setting of definitional equivalence for the associated deductive systems.

THEOREM 2.8. *[22, Theorem 4.6] Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be two regularly algebraisable deductive systems over language types* Λ_1 *and* Λ_2*. Let* K_1 *and* K_2 *be the relatively* $\mathbf{1}^{\mathsf{K}_1}$*-regular and relatively* $\mathbf{1}^{\mathsf{K}_2}$*-regular quasivarieties comprising the equivalent quasivariety semantics of* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *respectively. Suppose* K_1 *and* K_2 *are term equivalent with interpretations* $\alpha : \Lambda_1 \to \mathrm{Fm}_{\Lambda_2}$ *and* $\beta : \Lambda_2 \to \mathrm{Fm}_{\Lambda_1}$ *such that* $(\mathbf{1}^{\mathsf{K}_1})^\alpha = \mathbf{1}^{\mathsf{K}_2}$ *and* $(\mathbf{1}^{\mathsf{K}_2})^\beta = \mathbf{1}^{\mathsf{K}_1}$*. Then* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are definitionally equivalent with the same mutually inverse interpretations.*

On examining the content of Theorem 2.7, it is clear that the conditions stipulated by Theorem 2.8 are met. The main result of this paper, Theorem 1.1, thus follows directly from Theorem 2.7 and Theorem 2.8.

3. Concluding Remarks

The natural deduction presentation of $\mathbf{F}^{**}$ given in [19] suggests that the structural rule

$$\frac{\Gamma \vdash \varphi, \psi, \psi \qquad \Gamma \vdash \varphi, \varphi, \psi}{\Gamma \vdash \varphi, \psi}$$

should be derivable in any sequent calculus formulation of $\mathbf{NFL}_{ew}$. See [19, Section II, p. 9].

On the other hand, in [18, Section 4, p. 289] Slaney implicitly observes that the structural rule

$$\frac{\Gamma, \Gamma, \Pi \vdash \varphi \qquad \Gamma, \Pi, \Pi \vdash \varphi}{\Gamma, \Pi \vdash \varphi}$$

should be derivable in any sequent calculus formulation of $\mathbf{NFL}_{ew}$.

Collectively, (11) and (12) hint that a (cut-free) sequent calculus formulation of $\mathbf{NFL}_{ew}$ may be obtained upon adjoining the structural rule

$$\frac{\Gamma, \Gamma, \Pi \vdash \Sigma, \Delta, \Delta \qquad \Gamma, \Pi, \Pi \vdash \Sigma, \Sigma, \Delta}{\Gamma, \Pi \vdash \Sigma, \Delta}$$

to a sequent calculus formulation of the involutive full Lambek calculus with exchange and weakening. This has been established recently in [12]; in this connection, see also [5].

Added in Proof

The North American Collecting Editor J.M. Dunn has pointed out to the authors that Thomason in [24] has provided a Kripke semantics for **N** and that Slaney, Girle, and Surendonk in [19] have provided a Kripke semantics for $\mathbf{F}^{**}$ that essentially differ only in that the semantics for $\mathbf{F}^{**}$ has "contraposition" built into it by requiring falsity preservation backwards as well as truth preservation forwards. These two semantics can be used to show the translatability of $\mathbf{F}^{**}$ into **N** (and also the converse, though this is not as transparent). Dunn [8] contains the appropriate results and further references.

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Mathematical Institute
University of Bern
CH-3012 Bern, Switzerland
e-mail: mspinksau@yahoo.com.au

Department of Computer Science
University of New Mexico
Albuquerque, NM 87131, U.S.A.
e-mail: veroff@cs.unm.edu