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Gemma Robles
José M. Méndez

AXIOMATIZING $S4_+$ AND J_+ WITHOUT THE SUFFIXING, PREFIXING AND SELF-DISTRIBUTION OF THE CONDITIONAL AXIOMS

Abstract

In this note, we axiomatize the positive fragments of Lewis' S4, $S4_+$, and of intuitionistic logic, J_+, by extending Routley and Meyer's basic positive logic B_+ with the contraction axiom and, respectively, the restricted K axiom and the K axiom.

1. Introduction

The suffixing, prefixing and self-distribution of the conditional axioms are, respectively,

$$\text{suf. } (A \to B) \to [(B \to C) \to (A \to C)]$$
$$\text{pref. } (B \to C) \to [(A \to B) \to (A \to C)]$$
$$\text{sdtr. } [A \to (B \to C)] \to [(A \to B) \to (A \to C)]$$

The sdtr axiom can also be rendered in the form

$$\text{sdtr'. } (A \to B) \to [[A \to (B \to C)] \to (A \to C)]$$

Standard axiomatizations of relevant, entailment or strict implication logics always include at least one of these axioms. Thus, for example, suf and pref are two of the axioms of Ackermann's Π' (cf. [1]). Or, to take another example, at least one of the aforementioned axioms is used in each one of the alternative formulations of E and R provided by Anderson and Belnap (cf. [2]. Consult the head "Axiom chopping"). Or, to take a last example, one at least of these axioms is present in each one of the several formulations provided for the implicational fragments of E, R, S3 and S4 in [4] and [5].

The aim of this note is to prove that the positive fragment of Lewis' S4, $S4_+$, and the positive fragment of intuitionistic logic J, J_+, can be axiomatized without any of the aforementioned axioms. This result has some interest in the sense that allows us to define and isolate weak logics of strict implication or subsystems of intuitionistic logic with certain properties. Thus, for example, it has been used in defining some weak logics of strict implication which are paraconsistent (cf. [7]).

The proof to be here developed depends on the K axiom or the S4-version of it (cf. A8, A9 below). So, the question whether a similar result can be obtained for relevant or entailment logics is left open.

In particular, we shall proceed as follows.

As is known, Routley and Meyer's basic positive logic B_+ can be axiomatized as follows (cf. [8], [9]):

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules:

Modus ponens (MP): $(\vdash A \to B \ \& \vdash A) \Rightarrow \ \vdash B$

Adjunction (Adj): $(\vdash A \ \& \vdash B) \Rightarrow \ \vdash A \wedge B$

Suffixing (Suf): $\vdash (A \to B) \Rightarrow \ \vdash (B \to C) \to (A \to C)$

Prefixing (Pref): $\vdash (B \to C) \Rightarrow \ \vdash (A \to B) \to (A \to C)$

The rule Transitivity, which is immediate from MP and Suf or Pref, and the Law of commutativity of the conjunction, which follows easily from A2, A3 and Adj, shall be used in the sequel.

$$\textit{Transitivity}\ \text{(Trans)}:\ (\vdash A \to B\ \ \&\ \ \vdash B \to C) \Rightarrow\ \vdash A \to C$$
$$\text{c}\wedge:\ \vdash (A \wedge B) \to (B \wedge A)$$

The aim of this note is to prove that the positive fragment of Lewis S4, $S4_+$, can be axiomatized by adding to B_+ the contraction and the restricted K axiom, respectively.

$$\text{A7.}\ [A \to (A \to B)] \to (A \to B)$$
$$\text{A8.}\ (A \to B) \to [C \to (A \to B)]$$

An easy consequence of this result is that the positive fragment of intuitionistic logic J, J_+, can be axiomatized by adding to B_+ A7 and the K axiom

$$\text{A9.}\ A \to (B \to A)$$

The logic B_+ plus A7 and A8 shall be labelled $S4B_+$.

Some results in [5] and [6] shall be presupposed.

2. $S4B_+$ is deductively included in $S4_+$

As is known, $S4_+$ can be axiomatized as follows (cf. [3]): A1, A2, A4, A8, and

$$\text{A10.}\ (A \to B) \to [(A \to C) \to [A \to (B \wedge C)]]$$
$$\text{A11.}\ (A \to C) \to [(B \to C) \to [(A \vee B) \to C)]]$$
$$\text{A12.}\ [(C \vee A) \wedge B] \to [(A \wedge B) \vee C]$$
$$\text{A13.}\ [A \to (B \to C)] \to [(A \to B) \to (A \to C)]$$

together with MP as the sole rule of inference.

As was pointed out above, the primary aim of this note is to prove that $S4_+$ is deductively included in $S4B_+$, but at the request of a referee of the BSL, we show that the converse holds too. So, we shall prove that A3, A5, A6 and A7 are theorems of $S4_+$, that Suf and Pref are derived rules of $S4_+$ and, finally, that Adj is an admissible rule of $S4_+$.

References to MP will usually be omitted.

Pref is a derived rule of $S4_+$:

Proof: Suppose

$$1.\ \vdash B \to C$$

By A8 and MP,

$$2.\ \vdash A \to (B \to C)$$

By A13,

$$3.\ \vdash (A \to B) \to (A \to C)$$

□

Then, note that the rule Trans is immediate by Pref and MP.

Next, we have:

t1. $(A \to B) \to [A \to (B \to C)] \to (A \to C)]$

Proof: By A13,

$$1.\ \{[A \to (B \to C)] \to (A \to B)\} \to \{[A \to (B \to C)] \to (A \to C)\}$$

Then, t1 follows by A8 and Trans. □

Now,

$$\text{A7. } [A \to (A \to B)] \to (A \to B)$$

is immediate by A1 and t1.

Suf is a derived rule of $S4_+$:

Proof: Suppose

$$1.\ \vdash A \to B$$

by t1

$$2.\ \vdash [A \to (B \to C)] \to (A \to C)$$

Then, by A8, 2 and Trans,

$$3.\ \vdash (B \to C) \to (A \to C)$$

□

Next, as is remarked below (cf. the proof of T7),

$$\text{t3. } [A \to (B \to C)] \to [(A \wedge B) \to C]$$

is provable by A2, A7, Pref and Suf. Then, by using t3,

$$\text{A3. } [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$$

and

$$\text{A5. } [(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$$

are immediate by A11 and A13, respectively.

Now, in order to prove A6, we need

$$\text{t3. } (A \wedge B) \to (B \wedge A)$$
$$\text{t4. } (A \vee B) \to (B \vee A)$$

and the rules

$$\text{r}\wedge.\ \text{If } \vdash A \to B, \text{ then } \vdash (A \wedge C) \to (B \wedge C)$$

and

$$\text{r}\vee.\ \text{If } \vdash A \to B, \text{ then } \vdash (A \vee C) \to (B \vee C)$$

whose proofs are as follows: t3 and r$\wedge$ are easy by A2 and A10; t4 and r$\vee$ are easy by A4 and A11.

Next, we write A12 in a more convenient form.

$$\text{t5. } [A \wedge (B \vee C)] \to [(A \wedge B) \vee C]$$

Proof: By t3, t4 and r$\wedge$,

$$\text{1. } [A \wedge (B \vee C)] \to [(C \vee B) \wedge A]$$

By A12,

$$\text{2. } [(C \vee B) \wedge A] \to [(B \wedge A) \vee C]$$

By t3 and r$\vee$

$$\text{2. } [(B \wedge A) \vee C] \to [(A \wedge B) \wedge C]$$

Now, t5 follows by applying Trans (twice) to 1, 2 and 3. □

Then, we prove

$$\text{A6. } [A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$$

Proof: By t5,

$$\text{1. } \{A \wedge [B \vee (A \wedge C)]\} \to [(A \wedge B) \vee (A \wedge C)]$$

By t3 and r$\vee$

$$2.\ \ [A \wedge (B \vee C)] \rightarrow [A \wedge (C \vee B)]$$

By t5,

$$3.\ \ [A \wedge (C \vee B)] \rightarrow [(A \wedge C) \vee B]$$

By t3,

$$4.\ \ [(A \wedge C) \vee B] \rightarrow [B \vee (A \wedge C)]$$

Then by using Trans twice in 2, 3, 4,

$$5.\ \ [A \wedge (B \vee C)] \rightarrow [B \vee (A \wedge C)]$$

On the other hand, by A2,

$$6.\ \ [A \wedge (B \vee C)] \rightarrow A$$

So, by A10,

$$7.\ \ \{[A \wedge (B \vee C)] \rightarrow [B \vee (A \wedge C)]\} \rightarrow \{[A \wedge (B \vee C)] \rightarrow \{A \wedge [B \vee (A \wedge C)]\}\}$$

By MP 5, 7

$$8.\ \ [A \wedge (B \vee C)] \rightarrow \{A \wedge [B \vee (A \wedge C)]\}$$

Then, A6 follows by 1, 8 and Trans. □

Now, in order to prove the admissibility of Adj, it is convenient to prove the admissibility of rule K first.

K. If $\vdash A$, then $\vdash B \rightarrow A$

In the proof of this rule, we shall rely on [6]. In this paper, a number of logics among which $S4_+$ is to be found are proven metacomplete, and consequently, are shown to have the intuitionistic disjunction property, to wit,

idp. If $A \vee B$ is a theorem, so is at least one of A or B.

PROOF OF THE ADMISSIBILITY OF RULE K: Induction on the length of the proof of A.

1. *A is of the form $C \rightarrow D$*: then, $\vdash B \rightarrow A$ follows by A8.

2. *A is of the form* $C \wedge D$: by A2, C and D are theorems. By hypothesis of induction,

$$1.\ \vdash B \to C$$
$$2.\ \vdash B \to D$$

Then, by A10 and 1,

$$3.\ \vdash (B \to D) \to [B \to (C \wedge D)]$$

and so, by MP (2, 3),

$$4.\ \vdash B \to (C \wedge D)$$

3. *A is of the form* $C \vee D$: by idp, one of C or D is a theorem. Suppose C is a theorem. (If it is D which is a theorem, the proof of such case is similar.) Then, by hypothesis of induction,

$$1.\ \vdash B \to C$$

whence by A4,

$$2.\ \vdash B \to (C \vee D)$$

□

Finally, we prove:

Adj is admissible in $S4_+$:

Proof: Suppose that A and B are theorems. By K,

$$1.\ \vdash (A \to A) \to A$$
$$2.\ \vdash (A \to A) \to B$$

whence by A10,

$$3.\ \vdash (A \to A) \to (A \wedge B)$$

Then,

$$4.\ \vdash A \wedge B$$

by MP (A1, 3). □

3. $S4B_+$ is deductively equivalent to $S4_+$

In this section we prove that $S4_+$ is deductively included in $S4B_+$. First, notice that A12 is a theorem of B_+ (cf. [9]). So, we prove that A10, A11 and A13 are theorems of $S4B_+$. In fact, we shall prove A10 (T4 below), A11 (T6) and the suffixing axiom (T13). Then, by using a result in [5], A13 is derivable. In order to do this, some preliminary theorems of $S4_+$ are needed.

$$\text{T1. } B \to (A \to A)$$

Proof: By A1 and A8. □

$$\text{T2. } (A \to B) \to [A \to (A \wedge B)]$$

Proof:

1. $[(A \to A) \wedge (A \to B)] \to [A \to (A \wedge B)]$ A3

By A1, T1 and A3:

2. $(A \to B) \to [(A \to A) \wedge (A \to B)]$

whence T2 follows by 1 and Trans. □

$$\text{T3. } (A \to B) \to [(A \to C) \to [(A \to B) \wedge (A \to C)]]$$

Proof:

1. $[(A \to C) \to (A \to B)] \to [(A \to C) \to [(A \to C) \wedge (A \to B)]]$ T2

By commutativity of conjunction (c$\wedge$), Pref and Trans,

2. $[(A \to C) \to (A \to B)] \to [(A \to C) \to [(A \to B) \wedge (A \to C)]]$
3. $(A \to B) \to [(A \to C) \to (A \to B)]$ A8

whence T3 follows by 1, 3 and Trans. □

$$\text{T4. } (A \to B) \to [(A \to C) \to [(A \to (B \wedge C)]]$$

Proof: By A3 and Pref,

1. $\{(A \to C) \to [(A \to B) \wedge (A \to C)]\} \to \{(A \to C) \to [(A \to (B \wedge C)]\}$

whence T4 is immediate by T3. □

$$\text{T5. } (A \to C) \to [(B \to C) \to [(A \to C) \wedge (B \to C)]]$$

Proof: Similar to that of T3. □

$$\text{T6. } (A \to C) \to [(B \to C) \to [(A \vee B) \to C)]]$$

Proof: Similar to that of T4 using now T5. □

Now, in order to prove A13, more preliminary theorems are needed.

$$\text{T7. } [A \to (B \to C)] \to [(A \wedge B) \to C]$$

Proof: By A2 and Suf,

$$1.\ [A \to (B \to C)] \to [(A \wedge B) \to (B \to C)]$$

By A2 and Suf,

$$2.\ (B \to C) \to [(A \wedge B) \to C]$$

By 2 and Pref,

$$3.\ [(A \wedge B) \to (B \to C)] \to [(A \wedge B) \to [(A \wedge B) \to C]]$$

whence by A7,

$$4.\ [(A \wedge B) \to (B \to C)] \to [(A \wedge B) \to C]$$

and finally, T7 by 1 and 4. □

$$\text{T8. } [(A \to B) \wedge (C \to D)] \to \{E \to [[(A \to B) \wedge (C \to D)] \wedge E]\}$$

PROOF:

1. $(A \to B) \to [E \to (A \to B)]$ A8
2. $(C \to D) \to [E \to (C \to D)]$ A8

By T2 and commutativity of conjunction (c$\wedge$),

3. $[E \to (A \to B)] \to [E \to [(A \to B) \wedge E]]$

and

4. $[E \to (C \to D)] \to [E \to [(C \to D) \wedge E]]$

Then,

5. $(A \to B) \to [E \to [(A \to B) \wedge E]]$ Trans 1, 3
6. $(C \to D) \to [E \to [(C \to D) \wedge E]]$ Trans, 2, 4

By A2, 5 and Trans,

7. $[(A \to B) \wedge (C \to D)] \to [E \to [(A \to B) \wedge E]]$

By A2, 6 and Trans,

8. $[(A \to B) \wedge (C \to D)] \to [E \to [(C \to D) \wedge E]]$

By A3, 7 and 8,

9. $[(A \to B) \wedge (C \to D)] \to \{[E \to [(A \to B) \wedge E]] \wedge [E \to [(C \to D) \wedge E]]\}$

Now,

10. $\{[E \to [(A \to B) \wedge E]] \wedge [E \to [(C \to D) \wedge E]]\} \to \{E \to [[(A \to B) \wedge E] \wedge [(C \to D) \wedge E]]\}$ A3

By B_+ and 10,

11. $\{[E \to [(A \to B) \wedge E]] \wedge [E \to [(C \to D) \wedge E]]\} \to \{E \to [[(A \to B) \wedge (C \to D)] \wedge E]\}$

whence T8 follows by 9 and Trans. □

T9. $\vdash \{[(A \to B) \wedge (C \to D)] \wedge E\} \to F \Rightarrow \ \vdash [(A \to B) \wedge (C \to D)] \to (E \to F)$

PROOF:

1. $\vdash \{[(A \to B) \wedge (C \to D)] \wedge E\} \to F$ Hip.
2. $\vdash \{E \to [[(A \to B) \wedge (C \to D)] \wedge E]\} \to (E \to F)$ Pref, 1
3. $\vdash \{[(A \to B) \wedge (C \to D)] \to [E \to [[(A \to B) \wedge$ $(C \to D)] \wedge E]\} \to \{[(A \to B) \wedge (C \to D)] \to (E \to F)\}$ Pref, 2
4. $\vdash [(A \to B) \wedge (C \to D)] \to (E \to F)$ MP, T8, 3

□

T10. $[(A \to B) \wedge A] \to B$

PROOF: Immediate by A1 and T7. □

T11. $\{[(A \to B) \wedge (B \to C)] \wedge A\} \to C$

PROOF:

1. $[(A \to B) \wedge A] \to B$ T10
2. $[(B \to C) \wedge B] \to C$ T10

By B$_+$ and 1,

3. $\{[(A \to B) \wedge (B \to C)] \wedge A\} \to B$

By B$_+$

4. $\{[(A \to B) \wedge (B \to C)] \wedge A\} \to (B \to C)$

By A3, 3 and 4,

5. $\{[(A \to B) \wedge (B \to C)] \wedge A\} \to [(B \to C) \wedge B]$

whence T11 follows by 2, 5 and Trans. □

T12. $[(A \to B) \wedge (B \to C)] \to (A \to C)$

PROOF: By T9 and T11. □

T13. $(A \to B) \to [(B \to C) \to (A \to C)]$

PROOF: By T9 and T12. □

Now, the implicative fragment of S4, S4$_\to$, can be axiomatized with A1, A8, A13 and MP (cf. [3]). And in [5], it is shown that the system axiomatized with A1, A7, A8, T13 and MP is deductively equivalent to $S4_{\rightarrow}$. Consequently, $S4_{+}$ is deductively included in $S4B_{+}$, as was to be proved.

On the other hand, it is known that positive intuitionistic logic J_{+} can be axiomatized when changing A8 in $S4_{+}$ by A9. Therefore, it is clear that J_{+} can be axiomatized by adding A7 and A9 to B_{+}, although in this case the proof can be considerably simplified as sketched below.

$$1.\ A \rightarrow [B \rightarrow (A \wedge B)]$$

PROOF: By T2, A9. □

$$2.\ \vdash (A \wedge B) \rightarrow C \Rightarrow\ \vdash A \rightarrow (B \rightarrow C)$$

PROOF: By 1, Pref. □

$$3.\ [[(A \rightarrow B) \wedge (B \rightarrow C)] \wedge A] \rightarrow C$$

PROOF: Similar to that of T11. □

Then, the suffixing axiom T13 follows by 2 and 3.

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Dpto. de Historia y Filosofía de la CC, la Ed. y el Leng.
Universidad de La Laguna, Facultad de Filosofía
Campus de Guajara, 38071, La Laguna, Tenerife, Spain
http://webpages.ull.es/users/grobles
e-mail: gemmarobles@gmail.com

Universidad de Salamanca
Campus Unamuno, Edificio FES, 37007, Salamanca, Spain
http://web.usal.es/~sefus
e-mail: sefus@usal.es