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ABSOLUTELY UBIQUITOUS STRUCTURES AND $\aleph_0$-STABILITY

Abstract

Continuing investigations initiated in [3], we will prove that a certain subclass of absolutely ubiquitous structures are $\aleph_0$-stable. This confirms a special case of a conjecture of Macpherson [5].

Keywords: absolutely ubiquitous structures, stability.

1. Introduction

Throughout $J(\mathcal{A})$ denotes the set of isomorphism types of finite substructures of a first order structure $\mathcal{A}$. Recall from [5] that a countably infinite first order structure $\mathcal{A}$ with a finite language is defined to be *absolutely ubiquitous* iff $\mathcal{A}$ is uniformly locally finite and for any countable locally finite $\mathcal{B}$, $J(\mathcal{A}) = J(\mathcal{B})$ implies that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

A theory T is model-complete iff whenever $\mathcal{A}$ is a substructure of $\mathcal{B}$ such that $\mathcal{A}, \mathcal{B} \models T$ then $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. By Lemma 2.1 of [5] the first order theory of an absolutely ubiquitous structure is model-complete and $\aleph_0$-categorical. In addition, in [5] it was proved that absolutely ubiquitous groups are $\aleph_0$-stable and it was conjectured that every absolutely ubiquitous structure is stable (see the conjecture after Theorem 1.1 on page 484 of [5]). This conjecture has been proved for

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structures with finite relational languages in [2] and some results related to this problem can be found in [3]. We recall further related results at the end of this section.

Since absolutely ubiquitous structures are $\aleph_0$-categorical, these structures are uniformly locally finite. That is, if $\mathcal{A}$ is an absolutely ubiquitous structure then there is a function $f : \omega \to \omega$ such that if a substructure $\mathcal{A}_0$ of $\mathcal{A}$ can be generated by $n \in \omega$ elements then $|A_0| \le f(n)$. We will say that $\mathcal{A}$ is *linearly locally finite* if there is a number $\beta \in \omega$ such that if $X \subseteq A$ is finite, $a \in A$ and $\mathcal{A}_0$, $\mathcal{A}_1$ are the substructures of $\mathcal{A}$ generated by X and $X \cup \{a\}$ respectively, then $|A_1 - A_0| \le \beta$. Such a β will be called a bound for $\mathcal{A}$. If the language of $\mathcal{A}$ contains relation symbols only, or it is $\aleph_0$-categorical (or even uniformly locally finite) and has at most unary function symbols then $\mathcal{A}$ is linearly locally finite, but it is easy to construct $\aleph_0$-categorical linearly locally finite structures having basic operations with arbitrarily large finite arities. An easy induction shows that the cardinality of an n-generated substructure of a linearly locally finite structure is asymptotically linear in n.

Here we prove that absolutely ubiquitous and linearly locally finite structures (for an arbitrary finite first order language) are $\aleph_0$-stable.

We mention the following earlier related results. Let $\mathcal{A}$ be $\aleph_0$-categorical. The algebraic closure operation $acl^{\mathcal{A}}$ is called n-degenerate iff for every $X \subseteq A$ we have $acl(X) = \cup\{acl(Y) : Y \subseteq X, |Y| \le n\}$. In [8] Vassiliev studies $\aleph_0$-categorical structures whose algebraic closure is n-degenerate for some $n \in \omega$. In [8] it was shown, that absolutely ubiquitous structures with n-degenerate algebraic closure are stable. Note, that if $acl^{\mathcal{A}}$ is n-degenerate and $X \subseteq A$ is finite, then $|acl^{\mathcal{A}}|$ is asymptotically $|X|^n$ (that is, polynomial in the size of X). There are two main difference between this result and the present one: our condition bounds the sizes of the substructures generated by X (rather than $acl^{\mathcal{A}}(X)$). The generated substructure may remain relatively small with a relatively large algebraic closure. The other main difference is that we prove $\aleph_0$-stability rather than stability.

In [4] $\aleph_0$-categorical structures are studied whose lattice of algebraically closed subsets are distributive. In [4] it was shown, that absolutely ubiquitous structures with a distributive lattice of algebraically closed sets are $\aleph_0$-stable. Again, this condition is different from ours (as our condition applies to substructures that are not necessarily algebraically closed).

Our notation is mostly standard. Every ordinal is the set of smaller ordinals and natural numbers are identified with finite ordinals. Throughout ω denotes the smallest infinite ordinal. If A and B are sets then ${}^A B$ denotes the set of functions whose domain is A and whose range is a subset of B. If $\mathcal{A}$ is a structure and $X \subseteq A$ then $S^{\mathcal{A}}(X)$ denotes the set of types of $\mathcal{A}$ over X. Let $A \subseteq B$ be sets. As usual, by a slight abuse of notation, $\bar{c} \in B$ means that $\bar{c}$ is a tuple of elements of B and $\bar{c} \notin A$ means that (the range of) $\bar{c}$ does not meet A.

2. Linearly locally finite structures

Here we prove some basic properties of linearly locally finite structures. In order to keep formulation simpler we introduce the following notation. Suppose $\mathcal{A}$ is a structure and $\mathcal{A}_0$ is a substructure of it. If $X \subseteq A$ then $\mathcal{A}_0(X)^{\mathcal{A}}$ or simply $\mathcal{A}_0(X)$ denotes the substructure of $\mathcal{A}$ generated by $A_0 \cup X$.

LEMMA 2.1. *Suppose $\mathcal{A}$ is an absolutely ubiquitous, linearly locally finite structure with bound β. Let $\mathcal{A}_0$ be a countable substructure of $\mathcal{A}$ and let $a \in A$. Then $|A_0(a) - A_0| \leq \beta$.*

PROOF. Let $A_0 = \{a_n : n < \omega\}$ and for any $k \in \omega$ let $\mathcal{B}_k, \mathcal{B}'_k$ be the substructures generated by $\{a_n : n < k\}$ and $\{a\} \cup \{a_n : n < k\}$, respectively. Clearly, $\mathcal{A}_0 = \cup_{k\in\omega}\mathcal{B}_k$ and $\mathcal{A}_0(a) = \cup_{k\in\omega}\mathcal{B}'_k$. Therefore

$$(*)\ A_0(a) - A_0 = \cup_{k\in\omega} B'_k - A_0.$$

In addition, $B'_k - A_0$ is increasing in k and by assumption, for every $k \in \omega$ we have $|B'_k - A_0| \leq |B'_k - B_k| \leq \beta$. Therefore, there is a k such that for all $m \geq k$ we have $B'_m - A_0 = B'_k - A_0$ and hence by $(*)$ it follows that $A_0(a) - A_0 = B'_k - A_0$. As we already observed, $|B'_k - A_0| \leq \beta$. □

By Lemma 1 of [3], for any absolutely ubiquitous structure $\mathcal{A}$ there is a function $s : \omega \to \omega$ such that for every $n \in \omega$ if $\mathcal{B}$ is a substructure of $\mathcal{A}$ containing all isomorphism types of $s(n)$-generated substructures of $\mathcal{A}$ and $\bar{d} \in {}^n B$ then for every 1-type p over $\bar{d}$ there is an element of B which realizes p in $\mathcal{A}$.

LEMMA 2.2. *Suppose $\mathcal{A}$ is an absolutely ubiquitous, linearly locally finite structure with bound β. Let $\bar{b}, \bar{b}' \in {}^l A$ be l-tuples. Assume that $\mathcal{A}_0$ is a finitely generated substructure of $\mathcal{A}$ containing all isomorphism types of $s(2l\beta + 2l)$-generated substructures of $\mathcal{A}$ and suppose that $tp^{\mathcal{A}}(\bar{b}/A_0) =$*

$tp^{\mathcal{A}}(\bar{b}'/A_0)$. If $a \in A - A_0(\bar{b}, \bar{b}')$ then for any formula φ we have $\mathcal{A} \models \varphi(a, \bar{b}) \Leftrightarrow \varphi(a, \bar{b}')$.

Proof. First observe that $|A_0(\bar{b}, \bar{b}') - A_0| \leq 2l\beta$. Let $p = tp(a/(A_0(\bar{b}, \bar{b}') - A_0) \cup \{\bar{b}, \bar{b}'\})$, then by construction there is an $a' \in A_0(\bar{b}, \bar{b}')$ realizing p. Since $a \notin A_0(\bar{b}, \bar{b}')$, it follows that $a' \notin A_0(\bar{b}, \bar{b}') - A_0$, consequently $a' \in A_0$. Therefore

$$\mathcal{A} \models \varphi(a, \bar{b}) \;\Leftrightarrow\; \mathcal{A} \models \varphi(a', \bar{b}) \;\Leftrightarrow\; \mathcal{A} \models \varphi(a', \bar{b}') \;\Leftrightarrow\; \mathcal{A} \models \varphi(a, \bar{b}').$$

□

Lemma 2.3. *Let $\mathcal{A}, \mathcal{B}$ be elementarily equivalent structures, $\mathcal{A}'$ and $\mathcal{B}'$ are substructures of $\mathcal{A}$, $\mathcal{B}$ respectively, let $\mathcal{A}_0 \subseteq \mathcal{A}'$ be a countable elementary substructure of $\mathcal{A}$, let f be an isomorphism between $\mathcal{A}'$ and $\mathcal{B}'$ mapping $\mathcal{A}_0$ onto $\mathcal{B}_0$. Suppose moreover that $\mathcal{A}_0$ is absolutely ubiquitous and linearly locally finite with bound β. Let $a \in A - A', b \in B - B'$ be such that $tp^{\mathcal{B}}(b/B_0) = f[tp^{\mathcal{A}}(a/A_0)]$. Then there is an isomorphism between $\mathcal{A}'(a)$ and $\mathcal{B}'(b)$ extending f and mapping a onto b.*

Proof. For every $u \in A'(a)$ there is a term τ for which $u = \tau^{\mathcal{A}}(a, \bar{d})$ for some $\bar{d} \in A'$. Choose such a term τ and parameter $\bar{d}$ for each $u \in A'(a)$ and let $u' = \tau^{\mathcal{B}}(b, f(\bar{d}))$. The only way of extending f is mapping each u onto u'. We will show that for any $\bar{d} \in A'$ and any atomic formula R one has

$$(*) \qquad \mathcal{A} \models R(a, \bar{d}) \Leftrightarrow \mathcal{B} \models R(b, f(\bar{d})).$$

From $(*)$ it follows that the mapping $u \mapsto u'$ is a well-defined injective function preserving all basic relations and functions of $\mathcal{A}'(a)$. In addition, the range of this mapping contains a set of generators of $\mathcal{B}'(b)$ and hence our mapping is surjective, consequently it is the required isomorphism. Thus, it is enough to prove $(*)$.

This will be done by transfinite induction. Suppose $A' - A_0 = \{a_i : i < \kappa\}$ and let $\mathcal{A}_i = \mathcal{A}_0(\{a_j : j < i\})$. Let $i \leq \kappa$ and assume that for every $j < i$ $(*)$ has already been proved if $\bar{d} \in A_j$. (In case $i = 1$ we have $tp^{\mathcal{B}}(b/B_0) = f[tp^{\mathcal{A}}(a/A_0)]$ by assumption.) If i is a limit ordinal then clearly, $(*)$ remains true for tuples $\bar{d} \in A_i$. Now suppose $i = j + 1$, R is an atomic formula and $\bar{d} \in A_i$. We may assume that the length of $\bar{d}$ is l and $\bar{d} = \bar{e}^\frown \bar{h}$ where $\bar{e} \in A_j(a)$ and $\bar{h} \in A_i - A_j(a)$. Thus, there are $\bar{e}' \in A_j$ and terms τ_v $(v = 0, 1, ..., |\bar{e}| - 1)$ for which $e_v = \tau_v(a, \bar{e}')$. Replacing $\bar{e}$ by $\bar{e}'$ and R by $R(\tau_0, ..., \tau_{|\bar{e}|-1})$ in $(*)$, we may assume $\bar{e} \in A_j$.

Since the isomorphism types of finite substructures of a structure can be completely described by existential and universal formulas, the theory of $\mathcal{A}$ can be axiomatized by such formulas. Therefore, since $\mathcal{A}_0$ is an elementary substructure of $\mathcal{A}$, it follows that $\mathcal{A}'$, $\mathcal{A}_0(\bar{e})$ and $\mathcal{A}_0(\bar{e}, a)$ satisfy the same existential and universal formulas as $\mathcal{A}$ and hence they are elementarily equivalent with $\mathcal{A}$. By Lemma 2.1 of [5] $\mathcal{A}_0$ is model-complete therefore $\mathcal{A}'$, $\mathcal{A}_0(\bar{e})$ and $\mathcal{A}_0(\bar{e}, a)$ are elementary substructures of $\mathcal{A}$. In addition, by Lemma 2.1 $|A_0(\bar{e}, a) - A_0(\bar{e})| \leq \beta$.

Since $\mathcal{A}_0$ is absolutely ubiquitous, there is a function $s = s^{\mathcal{A}_0} : \omega \to \omega$ as described after Lemma 2.1. Similarly to the previous paragraph, $\mathcal{B}_0$ is an elementary substructure of $\mathcal{B}$ and since $\mathcal{A}_0 \cong \mathcal{B}_0$ we have $s = s^{\mathcal{A}_0} = s^{\mathcal{B}_0}$ (where $s^{\mathcal{B}_0}$ is the s-function of $\mathcal{B}_0$).

Since $\mathcal{A}_0(\bar{e}, a)$ is elementarily equivalent with the absolutely ubiquitous $\mathcal{A}_0$, it follows that $\mathcal{A}_0(\bar{e}, a)$ is $\aleph_0$-categorical and hence, for example by Exercise 7.2.11 of [1] it is saturated.

Let $\mathcal{D}$ be a finite substructure of $\mathcal{A}_0$ containing all isomorphism types of $s(2l\beta + 2l)$-generated substructures of $\mathcal{A}_0$. Let $p = tp^{\mathcal{A}}(\bar{h}/D \cup (A_0(\bar{e}, a) - A_0(\bar{e})) \cup \{\bar{e}, a\})$. Then p can be realized in $\mathcal{A}_0(\bar{e}, a)$, say $\bar{g}$ realizes it. Since $\bar{h} \notin A_j(a)$, it follows that $\bar{h} \notin A_0(\bar{e}, a)$ and hence $\bar{g} \notin A_0(\bar{e}, a) - A_0(\bar{e})$, in other words $\bar{g} \in A_0(\bar{e})$. Therefore $\mathcal{D}(\bar{d}, \bar{g})$ is contained in A_i, hence it does not contain a. Therefore, by Lemma 2.2 we have $\mathcal{A} \models R(a, \bar{d}) \Leftrightarrow R(a, \bar{e}^\frown\bar{g})$. By the induction hypothesis $(*)$, it follows that $\mathcal{A} \models R(a, \bar{e}^\frown\bar{g}) \Leftrightarrow \mathcal{B} \models R(b, f(\bar{e}^\frown\bar{g}))$. In addition, since $f : \mathcal{A}' \to \mathcal{B}'$ is an isomorphism and since $tp^{\mathcal{A}}(\bar{h}/D \cup \{\bar{e}\}) = tp^{\mathcal{A}}(\bar{g}/D \cup \{\bar{e}\})$ we have $tp^{\mathcal{B}}(f(\bar{d})/f[D]) = tp^{\mathcal{B}}(f(\bar{e}^\frown\bar{g})/f[D])$. Observe moreover that $b \notin \mathcal{B}_0(f(\bar{e}^\frown\bar{g}), f(\bar{d}))$. Then again by Lemma 2.2 it follows that $\mathcal{B} \models R(b, f(\bar{e}^\frown\bar{g})) \Leftrightarrow R(b, f(\bar{d}))$. Thus $(*)$ is true for i which completes the induction. □

3. Stability

Recall that if T is a theory and κ is a cardinal then $I(T, \kappa)$ denotes the number of pairwise non-isomorphic models of T whose universes are of cardinality κ (see Definition VIII 1.1 of [7]).

We will estimate $I(T, \kappa)$ where T is the theory of a fixed absolutely ubiquitous and linearly locally finite structure. To do this we associate certain "invariant structures" for these structures.

Definition 3.1. *By a frame we mean a triple* $\langle \mathcal{A}_0, \mathcal{A}_1, f\rangle$ *where* $|A_1| \leq 2^{\aleph_0}$, $\mathcal{A}_0$ *is a countable elementary substructure of* $\mathcal{A}_1$ *and* f *is a cardinal-valued function on* A_1. *Let* $\mathcal{A}$ *be any structure. By a frame of* $\mathcal{A}$ *we mean a frame* $\langle \mathcal{A}_0, \mathcal{A}_1, f\rangle$ *where*

- $\mathcal{A}_1$ *is an elementary substructure of* $\mathcal{A}$ *and for any* $p \in S^{\mathcal{A}}(A_0)$ *and* $\kappa \leq 2^{\aleph_0}$ *if* p *is realized in* $\mathcal{A}$ *by at least* κ *different elements then* p *is realized in* $\mathcal{A}_1$ *by at least* κ *different elements,*
- *for any* $p \in S^{\mathcal{A}}(A_0)$, *if the number of realizations of* p *in* $\mathcal{A}$ *is at most* $2^{\aleph_0}$, *then all of these realizations are in* A_1,
- $f : A_1 \to |A|$, *for every* $a \in A_1$ $f(a) = |\{b \in A - A_1 : tp^{\mathcal{A}}(a/A_0) = tp^{\mathcal{A}}(b/A_0)\}|$.

Two frames $\langle \mathcal{A}_0, \mathcal{A}_1, f\rangle$ and $\langle \mathcal{B}_0, \mathcal{B}_1, g\rangle$ are defined to be isomorphic iff there is an isomorphism $\varrho : A_1 \to B_1$ mapping A_0 onto B_0 and for any $a \in dom(f)$ we have $f(a) = g(\varrho(a))$. Clearly, every structure with a countable language has a frame.

Lemma 3.2. *Let* T *be any complete theory and let* K *be the set of models (up to isomorphism, of course) whose universes are of cardinality* $2^{2^{\aleph_0}}$. *Let us fix a frame for every member of* K. *Then, up to isomorphism, the cardinality of these frames is at most* $2^{2^{\aleph_0}}$.

Proof. Observe that a frame can be considered as a pair

$$(*) \qquad \langle\langle A_1, A_0\rangle, f\rangle$$

in which $\langle A_1, A_0\rangle$ is a first order structure with universe A_1 and a certain extra relation A_0. Since we count frames up to isomorphism, we may assume that A_1 is (an initial segment of) $2^{\aleph_0}$. For a countable language, the number of structures on (an initial segment of) $2^{\aleph_0}$ is $2^{2^{\aleph_0}}$. Therefore in $(*)$ the first component $\langle A_1, A_0\rangle$ can be selected in $2^{2^{\aleph_0}}$ ways only. In addition, since $|A_1| \leq 2^{\aleph_0}$, $|dom(f)| \leq 2^{\aleph_0}$ and the cardinality of functions from $\kappa \leq 2^{\aleph_0}$ into $2^{2^{\aleph_0}}$ is $2^{2^{\aleph_0}}$. Therefore there are $2^{2^{\aleph_0}}$ possibilities for choosing f and hence the statement follows. □

We need the following lemma from infinitary combinatorics.

Lemma 3.2. *Let* A *be a set with* $|A| = 2^{2^{\aleph_0}}$, *let* $\lambda \leq 2^{\aleph_0}$ *be a cardinal and let* $\langle A_i : i < \lambda\rangle$ *be a sequence of disjoint subsets of* A *such that* $A = \cup_{i<\lambda} A_i$

and $|A_i| > 2^{\aleph_0}$ *for all* $i < \lambda$. *Then there exists an enumeration* $\{a_j : j < 2^{2^{\aleph_0}}\}$ *of* A *such that for every* $j < 2^{2^{\aleph_0}}$ *if* $a_j \in A_i$ *then* $(2^{\aleph_0} + |j|)^+ \leq |A_i|$.

Proof. For each $i < \lambda$ we define a sequence z^i of disjoint subsets of A_i that cover A_i as follows.

First suppose $|A_i|$ is regular. Then the length of z^i is 1. Let $z^i_0 = A_i$ and let $z^i = \langle z^i_0 \rangle$.

Next suppose $|A_i|$ is singular. Then the length of z^i is the cofinality $cf(|A_i|)$ of $|A_i|$. Fix an increasing sequence of regular cardinals $\langle \mu_k : k < cf(|A_i|)\rangle$ cofinal in $|A_i|$. We may assume $2^{\aleph_0} < \mu_0$ and may also assume that

$$(*) \qquad \Big(\sum_{l<k} \mu_l\Big)^+ = \Big(sup\{\mu_l : l < k\}\Big)^+ = \Big(\sum_{\xi < sup\{\mu_l : l<k\}} \xi\Big)^+ \leq \mu_k$$

holds for all $k < cf(|A_i|)$. Finally, let $\langle z^i_k : k \leq cf(|A_i|)\rangle$ be a partition of A_i such that $|z^i_k| = \mu_k$ holds for all $k < cf(|A_i|)$.

Now let $\langle \nu_m : m < \theta \rangle$ be an increasing enumeration of the set of cardinals $\{|z^i_k| : i \leq \lambda, k < length(z^i)\}$. For each $m < \theta$ let $B_m = \{z^i_k : i < \lambda, k < length(z^i), |z^i_k| = \nu_m\}$ and let $C_m = \cup B_m$. Then $|C_m| = \nu_m$ because $|B_m| \leq \lambda$ and ν_m (the cardinality of elements of B_m) is larger than $2^{\aleph_0}$. In addition, since $A = \cup_{m<\theta} C_m$, it follows, that $2^{2^{\aleph_0}} = \Sigma_{m<\theta} \nu_m$. Now let $\langle I_m : m < \theta \rangle$ be a partition of $2^{2^{\aleph_0}}$ of disjoint intervals such that $|I_m| = \nu_m$ hold for every $m < \theta$. Let $f_m : I_m \to C_m$ be a bijection for all $m < \theta$. Finally, for $j < 2^{2^{\aleph_0}}$ let $a_j = f_m(j)$ where $m < \theta$ is the unique ordinal for which $j \in I_m$.

We claim, that $\{a_j : j < 2^{2^{\aleph_0}}\}$ satisfies the conclusion. To see this, let $j < 2^{2^{\aleph_0}}$ and let $m < \theta$ be the unique ordinal for which $j \in I_m$. If $a_j = f_m(j) \in A_i$ then $A_i \cap C_m \neq \emptyset$ hence $|A_i| \geq |A_i \cap C_m| \geq \nu_m$. On the other hand, since $j \in I_m$, it follows, that by $(*)$ we have

$$|j| = \quad |j - \cup_{l<m} I_l| + |\cup_{l<m} I_l| \quad < \quad \nu_m + \sum_{\xi < sup\{\nu_l : l<m\}} \xi \quad = \quad \nu_m.$$

Hence, $|j|^+ \leq |A_i|$. Finally, $(2^{\aleph_0})^+ \leq |A_i|$ by assumption. □

Theorem 3.4. *Let* T *be the first order theory of an absolutely ubiquitous, linearly locally finite structure. Then* $I(T, 2^{2^{\aleph_0}}) \leq 2^{2^{\aleph_0}}$.

Proof. Let K be the set of models of T (up to isomorphism), whose universes are of cardinality $2^{2^{\aleph_0}}$ and fix a frame for every element of K. By Lemma 3.2 it is enough to prove that two members of K are isomorphic whenever they have isomorphic frames. So suppose $\mathcal{A}, \mathcal{B} \in K$ with frames $\langle \mathcal{A}_0, \mathcal{A}_1, f\rangle, \langle \mathcal{B}_0, \mathcal{B}_1, g\rangle$, respectively and suppose $\varrho_1 : \mathcal{A}_1 \to \mathcal{B}_1$ is an isomorphism mapping $\mathcal{A}_0$ onto $\mathcal{B}_0$ and preserving f and g.

We will extend ϱ_1 to an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ by transfinite recursion.

Let $\langle p_i^\kappa : i < \lambda\rangle$ be the sequence of types of $\mathcal{A}$ over A_0 which have at least $(2^{\aleph_0})^+$ different realizations in $\mathcal{A}$. Clearly, $\lambda \leq 2^{\aleph_0}$. For each $i < \lambda$ let $T_i = \{a \in A - A_1 : a \text{ realizes } p_i\}$. Applying Lemma 3.3 to $\{T_i, i < \lambda\}$ we get an enumeration $A - A_1 = \{a_j : j < 2^{2^{\aleph_0}}\}$ such that for each $j < 2^{2^{\aleph_0}}$, $tp^{\mathcal{A}}(a_j/A_0)$ has at least $(2^{\aleph_0} + |j|)^+$ different realizations in $\mathcal{A}$.

Since $\mathcal{A}$ and $\mathcal{B}$ have isomorphic frames, there is an enumeration $B - B_1 = \{b_j : j < 2^{2^{\aleph_0}}\}$ such that for every $j < 2^{2^{\aleph_0}}$ we have $\varrho_1[tp^{\mathcal{A}}(a_j/A_0)] = tp^{\mathcal{B}}(b_j/B_0)$.

For each $i < 2^{2^{\aleph_0}}$ let $\mathcal{A}_i = \mathcal{A}_1(\{a_j : j < i\})$. Suppose $i \leq 2^{2^{\aleph_0}}$ and there is an increasing sequence $\langle \varrho_j, j < i\rangle$ of isomorphisms with $A_j \subseteq dom(\varrho_j)$ and $\{b_k : k < j\} \subseteq range(\varrho_j)$. If i is a limit ordinal then let $\varrho_i = \cup_{j<i}\varrho_j$. Suppose $i = j + 1$ is a successor ordinal. If $A_i \subseteq dom(\varrho_j)$ then let $\eta = \varrho_j$. Otherwise $a_j \notin A_j$ and $\mathcal{A}_i$ can be generated by $A_j \cup \{a_j\}$. Then $|A_j| \leq 2^{\aleph_0} + |j|$ and therefore, because of our special enumeration, $tp^{\mathcal{A}}(a_j/A_0)$ has at least $(2^{\aleph_0} + |j|)^+$ different realizations in $\mathcal{A}$. Since the frames of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, there is a $b \in B - range(\varrho_j)$ for which $\varrho_1[tp^{\mathcal{A}}(a_j/A_0)] = tp^{\mathcal{B}}(b/B_0)$. Therefore by Lemma 2.3 there is an isomorphism η extending ϱ_j with $dom(\eta) = A_i$. Interchanging the role of $\mathcal{A}$ and $\mathcal{B}$, (the inverse of) η can be extended to an isomorphism ϱ_i for which $b_j \in range(\varrho_i)$ in the similar way. Finally, $\varrho_{2^{2^{\aleph_0}}}$ is the required isomorphism between $\mathcal{A}$ and $\mathcal{B}$. □

Now we present the main result of the paper which confirms a special case of a conjecture of Macpherson [5].

Corollary 3.5. *If $\mathcal{A}$ is absolutely ubiquitous and linearly locally finite then it is $\aleph_0$-categorical and $\aleph_0$-stable.*

Proof. By Lemma 2.1 of [5] $\mathcal{A}$ is $\aleph_0$-categorical. Let $T = Th(\mathcal{A})$. Then T is superstable otherwise by Corollary VIII 3.4 of [7] (or by Theorem

11.3.10 of [1]) $I(T, 2^{2^{\aleph_0}}) = 2^{2^{2^{\aleph_0}}}$ would follow, contradicting to Theorem 3.4. Finally, for example, by Proposition 5.16 of [6] an $\aleph_0$-categorical, superstable structure is $\aleph_0$-stable. □

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