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**M-HYPERQUASI-IDENTITIES OF FINITE ALGEBRAS**

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**Abstract**

In [4] (see [2]) the notion of a *hypersubstitution of a given type* and the notion of a *derived variety* $D(V)$ of a given variety $V$ of a fixed type $\tau$ were invented. Suitable *hyperequational calculus* appeared as a modification of G. Birkhoff calculus with additional rule called *hypersubstitution rule* (6). In Graczyńska E. and Schweigert D. [6] we considered further generalization to *M-hyperquasivarieties* of a given type. Basing on the result of A. Wroński [11] we modify A. Selman calculus [10] and the Gentzen-style calculus of [11] presenting a finite axiomatization of *M-hyperquasi-identities* of a finite algebra (of a finite type $\tau$). The results were presented on the conference AAA79 and CYA25 Olomouc (Czech Republic), February 12–14, 2010.

1. **Notation**

Our nomenclature and notation is basically those of G. Birkhoff [1], E. Graczyńska and D. Schweigert [4], A. Selman [10] and A. Wroński [11]. We recall some fundamental concepts.

**Definition 1.1.** A type $\tau$ of an algebra $A = (A, \{f_i : i \in I\}) = (A, F)$ is a function $\tau : I \to \mathbb{N}$ from the indexing set $I$ into the set $\mathbb{N}$ of natural numbers, where $\tau(i) = n_i$ if $f_i$ is an $n_i$-ary operation. A type $\tau$ is called *finite* if the set $I$ is finite.

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*AMS Mathematical Subject Classification 2010:*

Primary: 08B05, 08C15, 08B99, 08C99
We deal only with universal algebras of a given finite type $\tau : I \to \mathbb{N}$, where $I$ is a nonempty set and $\mathbb{N}$ denotes the set of all integers.

Hypersubstitutions of a given type $\tau$ were invented by D. Schweigert and the author in [4] (cf. [2]). Shortly speaking, they are mappings sending terms to terms by substituting variables by (the same) variables and fundamental terms of the form $f_i(x_0, ..., x_{\tau(i)-1})$ by terms of the same arieties, i.e. $\sigma(x) = x$ for any variable $x$, and for a given operation symbol $f_i$, for $i \in I$ assume that $\sigma(f_i(x_0, ..., x_{\tau(i)-1}))$ is a given term of the same arity as $f_i$, then $\sigma$ acts on all terms of a given type $\tau$ in an inductive way:

$$\sigma(f_i(p_0, ..., p_{\tau(i)-1})) = \sigma(f_i(x_0, ..., x_{\tau(i)-1}))(\sigma(p_0), ..., \sigma(p_{\tau(i)-1})),$$

for $i \in I$, (where $f_i(x_0, ..., x_{\tau(i)-1})$ denotes a fundamental term).

Consider an algebra $A$ and the equivalence relation $\equiv_A$ on the set $H(\tau)$ defined by: $\sigma_1 \equiv_A \sigma_2$ in $H(\tau)$ iff the identity $\sigma_1(f_i(x_0, ..., x_{\tau(i)-1})) \approx \sigma_2(f_i(x_0, ..., x_{\tau(i)-1}))$ is satisfied in $A$ for every $i \in I$.

The following can be easily proved by induction on the complexity of term $p$:

**Theorem 1.2.** If $\sigma_1 \equiv_A \sigma_2$ in $H(\tau)$ then for every term $p$ of type $\tau$ the identity $\sigma_1(p) \approx \sigma_2(p)$ is satisfied in $A$.

It is clear that in a finite algebra $A$ of a finite type $\tau$ there are only finitely many non-equivalent hypersubstitutions of type $\tau$.

$H(\tau) = (H(\tau), \circ, \sigma_{id})$ denotes the monoid of all hypersubstitutions $\sigma$ of a given type $\tau$ with the operation $\circ$ of composition and the identity hypersubstitution $\sigma_{id}$. $M$ is a submonoid of $H(\tau)$.

## 2. Derived algebras

Let $V$ be a class of algebras of type $\tau$. Derived algebras of a given type $\tau$ were defined in [4].

**Definition 2.1.** Let $A = (A, F)$ be an algebra in $V$ and $\sigma$ a hypersubstitution in $H(\tau)$. Then the algebra $B = (A, (F)^\sigma)$ is a derived algebra of $A$, with the same universe $A$ and the set $(F)^\sigma$ of all derived operations of $F$ by $\sigma$. $B$ is then denoted as $A^\sigma$.

$D(V)$ denotes the class of all derived algebras of type $\tau$ of all algebras of $V$. 

**Definition 2.2.** A variety $V$ is called *solid* if and only if $D(V) \subseteq V$.

Let us note, that for a finite algebra $A$ of a finite type $\tau$, the class $D(A)$ is finite.

### 2.1. Quasi-identities and quasivarieties of algebras

We recall the notion invented by A. I. Mal’cev from [8]:

**Definition 2.3.** A *quasi-identity* $e$ is an implication of the form:

$$ (1.2.1) \ (t_0 \approx s_0) \land ... \land (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n). $$

where $t_i \approx s_i$ are $k$-ary identities of a given type, for $i = 0, ..., n$.

A quasi-identity above is *satisfied in an algebra* $A$ of a given type if and only if the following implication is satisfied in $A$:

$$(t_0 \approx s_0) \land ... \land (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

A quasi-identity $e$ is *satisfied in a class* $V$ of algebras of a given type, if and only if it is satisfied in all algebras $A$ belonging to $V$.

A hyperquasi-identity $e$ (of a type $\tau$) is the same as quasi-identity (of type $\tau$). The difference between quasi-identities and hyperquasi-identities is in satisfaction. Similarly as in [5] we accept the following modification of the satisfaction of a quasi-identity to the notion of a *hypsatisfaction* in the following way:

**Definition 2.4.** A hyperquasi-identity $e$ is *satisfied* (is hyper-satisfied, holds) in an algebra $A$ if and only if the following implication holds: if $\sigma$ is a hypersubstitution of type $\tau$ and the elements $a_1, ..., a_k \in A$ satisfy the equalities $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in $A$, for $i \in \{0, 1, ..., k - 1\}$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ holds in $A$. In symbols $A \models_{H} e$.

If $e$ is satisfied (as a hyperquasi-identity) in a class $V$ of algebras of type $\tau$, then we write $V \models_{H} e$. 

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3. Solid quasivarieties

A reformulation of the notion of quasivariety invented by A. I. Mal’cev in [8], p. 210 to the notion of hyperquasivariety of a given type $\tau$ was considered by D. Schweigert and the author in [5] in a natural way:

**Definition 3.1.** A class $K$ of algebras of type $\tau$ is called a hyperquasivariety if there is a set $\Sigma$ of quasi-identities of type $\tau$ such that $K$ consists exactly of those algebras of type $\tau$ that hypersatisfy all the quasi-identities of $\Sigma$.

**Definition 3.2.** A quasivariety $V$ is called solid if and only if $D(V) \subseteq V$.

It was proved in [5] that the notion of hyperquasivariety and solid quasivariety coincides. We presented there some examples and theorems of Mal’cev type for solid quasivarieties.

**Definition 3.3.** A hyperquasi-identity $e$ is $M$-hyper-satisfied (holds) in an algebra $A$ if and only if the following implication is satisfied: If $\sigma$ is a hypersubstitution of $M$ and the elements $a_1,\ldots,a_n \in A$ satisfy the equalities $\sigma(t_i)(a_1,\ldots,a_k) = \sigma(s_i)(a_1,\ldots,a_k)$ in $A$, for $i = 0,1,\ldots,n-1$, then the equality $\sigma(t_n)(a_1,\ldots,a_k) = \sigma(s_n)(a_1,\ldots,a_k)$ holds in $A$. We say then, that $e$ is an $M$-hyperquasi-identity of $A$ and write:

$$A \models^M_{H} (t_0 \approx s_0) \land \ldots \land (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

$MHQId(A)$ denotes the set of all hyperquasi-identities satisfied in $A$ (as $M$-hyperquasi-identities).

Following [11] in the sequel we use the notation:

$$\Delta \rightarrow \alpha,$$

for a set $\Delta = \{p_i \approx q_i : 0 \leq i \leq n - 1\}$ and $\alpha = p_n \approx q_n$ instead of the quasi-identity: $p_0 \approx q_0 \land \ldots \land p_{n-1} \approx q_{n-1} \rightarrow p_n \approx q_n$.

We adopt the convention, that an identity $p \approx q$ may be regarded as a quasi-identity $e$ of the form $\emptyset \rightarrow p \approx q$, where $\emptyset$ denotes the empty set. A hyperquasi-identity $e$ is $M$-hyper-satisfied (holds) in a class $V$ if and only if it is $M$-hypersatisfied in any algebra of $V$. We write then: $V \models^M_{H} e$. 
We modify Selman calculus [10] by adding a new rule:

(4) an $M$-hypersubstitution rule

$$\frac{(t_0 \approx s_0) \land \ldots \land (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n)}{\sigma(t_0) \approx \sigma(s_0) \land \ldots \land \sigma(t_{n-1}) \approx \sigma(s_{n-1}) \rightarrow \sigma(t_n) \approx \sigma(s_n)},$$

or in an equivalent notation:

(4) an $M$-hypersubstitution rule

$$\frac{\{\gamma_0, \ldots, \gamma_{n-1}\} \rightarrow \beta}{\sigma(\gamma_0), \ldots, \sigma(\gamma_{n-1}) \rightarrow \sigma(\beta)},$$

where $\sigma \in \{\sigma_j : j = 1, \ldots, m\}$ for any hypersubstitutions $\sigma_1, \ldots, \sigma_m \in M$ of $H(\tau)$ such that $M/\equiv_A = \{[\sigma_1] \equiv_A, \ldots, [\sigma_m] \equiv_A\} \subseteq H(\tau)/\equiv_A$. We assume that $\sigma_j$ are pairwise nonequivalent.

In the sequel we will use the symbol $\Gamma$ for a set $\{\gamma_0, \ldots, \gamma_{n-1}\}$ and the set $\{\sigma(\gamma_0), \ldots, \sigma(\gamma_{n-1})\}$ will be denoted by $\sigma(\Gamma)$, for $\sigma \in M$.

We modify the rule (4) to the following rule:

$$(4)_A \quad \frac{\Gamma \rightarrow \delta}{\sigma_j[\Gamma] \rightarrow \sigma_j[\delta]}, \text{ for every } j = 1, 2, \ldots, m.$$
consists exactly of those algebras of type $\tau$ that $M$-hypersatisfy all the hyperquasi-identities of $\Sigma$.

We accept the following definition:

**Definition 4.2.** Let $QV$ be a quasivariety, then $QV$ is $M$-solid if and only if every $M$-derived algebra $A^\sigma$ belongs to $QV$, for every algebra $A$ in $QV$ and $\sigma$ in $M$, i.e.

$$D_M(QV) \subseteq QV$$

The following was proved in [6]:

**Theorem 4.3.** A (quasi)variety $K$ of algebras of a given type is an $M$-hyper(quasi)variety if and only if it is $M$-solid.

In [6] we presented some Mal’cev type theorems for $M$-hyperquasivarieties.

Let us recall from [6] that:

**Remark.** $MHQId(A) = QId(D_M(A))$

5. **Gentzen-style calculus**

We use the notation of [11], p. 74. For every $n = 1, 2, \ldots$ we accept the rule $G_n$ invented by A. Wroński:

$$(G_n) \quad \frac{x_0 \approx x_1, \Gamma \rightarrow \delta, \ldots, x_0 \approx x_n, \Gamma \rightarrow \delta, x_{n-1} \approx x_n, \Gamma \rightarrow \delta}{\Gamma \rightarrow \delta}.$$ 

For every $n = 1, 2, \ldots$ let $G_n$ be Gentzen-style calculus invented in [11] by adding the rule $G_n$ to the calculus of A. Selman [10].

We modify Gentzen-style axiomatization $G_n(A)$ of quasi-identities A. Wroński to the system $MHG_n(A)$ resulting from Selman calculus by adding the rules $G_n$ and (4)$_A$ and all the axioms (1), (2), (3) to obtain a finite axiomatization of all $M$-hyperquasi-identities of the algebra $A$. In fact we could agree to call this axiomatization as hyper Gentzen-style, as we use an additional inference rule of hypersubstitution.

Let $n$ be a natural number and $F(x_1, \ldots, x_n)$ be absolutely free algebra of terms of type $\tau$ in variables $x_1, \ldots, x_n$. Let $A$ be an algebra of type $\tau$ whose number of elements does not exceed $n$. The quotient algebra
\( F(x_1, \ldots, x_n)/MHId(A) \cong F(x_1, \ldots, x_n)/Id(D_M(A)) \) is an \( n \)-generated free algebra in the variety \( HSPD_M(A) \). For every term \( t \in F(x_1, \ldots, x_n) \) we pick a term \( r(t) \in [t]MHId(A) \) and call it the representative of \( t \). Thus whenever \( A \models M_{\tau} t_1 \approx t_2 \) and \( Var(t_1, t_2) \subseteq \{ x_1, \ldots, x_n \} \) then \( t_1 \) and \( t_2 \) must have the same representative i.e. \( r(t_1) \approx r(t_2) \).

In order to axiomatize the \( M \)-hyperquasi-identities of \( A \) we adopt similar axioms (1), (2) and (3) as in [11], p. 74, 75:

1. For every term \( t \) of type \( \tau \) in variables \( x_1, \ldots, x_n \), the identity \( t \approx r(t) \) can be derived from axioms (1), (2) by Birkhoff calculus.
2. For all terms \( t_1, t_2 \) of type \( \tau \) in variables \( x_1, \ldots, x_n \), if \( A \models M_{\tau} t_1 \approx t_2 \) then \( A \models M_{\tau} \sigma(t_1) \approx \sigma(t_2) \) and all the identities \( \sigma(t_1) \approx \sigma(t_2) \) can be derived from axioms (1), (2) by Birkhoff calculus, for every \( \sigma \in M \subseteq H(\tau) \).

Similarly as in A. Wroński [11], in addition to (1) and (2) we adopt as axioms all quasi-identities of the form:

3. \( \Gamma \rightarrow \delta \) such that \( A \models M_{\tau} \Gamma \rightarrow \delta \) and each member of \( \Gamma \cup \{ \delta \} \) is an identity of the form \( r(t_1) \approx r(t_2) \) where \( t_1, t_2 \) are terms of type \( \tau \) in variables \( x_1, \ldots, x_n \).

It is clear from [11], p. 75 that the cardinality of axioms (1)–(3) is finite. Moreover, for a finite algebra \( A \) of a finite type there are only finitely many nonequivalent hypersubstitutions of type \( \tau \). For a given monoid \( M \subseteq H(\tau) \) we choose for one representative from \( M \) in every equivalence class by the relation \( \equiv_A \) in \( M \). Assume that \( \sigma_1, \ldots, \sigma_m \) are all representatives, where \( \sigma_1 = \sigma_{id} \). Therefore the cardinality of \( (4)_A \) is also finite.

The following observation is similar to (iii) of [11], p. 75:

(iii) For every quasi-identity \( \Gamma \rightarrow \delta \) of type \( \tau \) in variables \( x_1, \ldots, x_n \), if \( A \models M_{\tau} \Gamma \rightarrow \delta \) then the quasi-identity \( \sigma(\Gamma) \rightarrow \sigma(\delta) \) can be derived from axioms (1), (2) and (3) by Selman calculus, for every \( \sigma \in M \).

**Proof.** For every \( \sigma \in M \) the quasi-identities of the form \( \sigma(\Gamma) \rightarrow \sigma(\delta) \) we get that \( A \models M_{\tau} \sigma(\Gamma) \rightarrow \sigma(\delta) \) by the definition of \( M \)-hyperquasi-satisfaction.
From the other hand they are consequences of quasi-identities of the form \( \Gamma \rightarrow \delta \) by the inference rules (1), (2) and (3) by Selman calculus, in the following way: assume \( \sigma \equiv A \sigma_j \) in \( M \), for some \( j \): \( 1 \leq j \leq m \). Then \( A \models_M \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \), for \( 0 \leq j \leq m \). By Selman calculus to each of such quasi-identity one may adopt the proof of (iii) of [11], p. 75 in order to obtain that it can be derived from axioms (1), (2) and (3) as follows:

First, using (1), (2) and Selman calculus we derive \( \sigma_j(\Gamma) \rightarrow r(\sigma_j(\Gamma)) \) and \( r(\sigma_j(\delta)) \rightarrow \sigma_j(\delta) \). We also have \( r(\sigma_j(\Gamma)) \rightarrow r(\sigma_j(\delta)) \) by (3). Thus the result \( \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \) can be achieved by cut and the desired result by Selman calculus. □

We have the following slight modification of Theorem 2 of A. Wroński [11], p.77:

**Theorem 5.1.** \( A \models_M \Gamma \rightarrow \delta \) iff \( \mathcal{MHG}_n(A) \) proves \( \Gamma \rightarrow \delta \)

iff \( \mathcal{MHG}_n(A) \) proves \( \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \), for \( j = 1, \ldots, m \)

iff \( \mathcal{MHG}_n(A) \) proves \( \sigma(\Gamma) \rightarrow \sigma(\delta) \), for every \( \sigma \in M \).

**Proof.** The part “if” follows from the fact that if \( \mathcal{MHG}_n(A) \) proves \( \Gamma \rightarrow \sigma \) then it proves \( \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \) for all \( 1 \leq j \leq m \) by the inference rule (4)_A. Moreover, each quasi-identity \( \sigma(\Gamma) \rightarrow \sigma(\delta) \), for \( \sigma \in M \) is a consequence of quasi-identities of the form \( \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \) for some \( 1 \leq j \leq m \) by Selman calculus. We conclude that every implication \( \sigma(\Gamma) \rightarrow \sigma(\delta) \) is provable from \( \mathcal{MHG}_n(A) \), for \( \sigma \in M \). Thus \( A \models_M \Gamma \rightarrow \sigma \), as all the rules are admissible in \( A \).

To prove the “only if” part suppose that \( \Gamma \rightarrow \delta \) is a quasi-identity in variables \( x_1, \ldots, x_k \), for \( k \in \mathbb{N} \), such that \( A \models_M \Gamma \rightarrow \delta \), thus \( A \models \Gamma \rightarrow \delta \) by the definition of satisfaction \( \models_M \) therefore by a similar argument as in the proof of Theorem 2 of [11], p. 77 we conclude that \( \Gamma \rightarrow \delta \) is provable by the calculus \( \mathcal{MHG}_n(A) \) (without an application of the rule (4)_A). Therefore every implication of the form \( \sigma_j(\Gamma) \rightarrow \sigma_j(\delta) \) is provable by the calculus \( \mathcal{MHG}_n(A) \), for all \( 0 \leq j \leq m \). In consequence every implication of the form \( \sigma(\Gamma) \rightarrow \sigma(\delta) \) is provable by the calculus \( \mathcal{MHG}_n(A) \), for all \( \sigma \in M \), by the similar argument as in the part “if”. □

**Remark.** Let us note that the calculus \( \mathcal{MHG}_n(A) \) reduces to the calculus \( G_n(A) \) of A. Wroński [11] for the trivial monoid \( M \).
The next Theorem follows from Theorem above by a finite application of hypersubstitution rule \((4)_A\) to each axiom obtained in Theorem 2. Then the rule \((4)_A\) may be omitted as the resulting set is already closed under that rule. This fact follows from an observation that the rule \((4)_A\) commutes with all the rules of Selman calculus and the rule \((G_n)\). Therefore the closure of a set of axioms closed under \((4)_A\) under Selman and \((G_n)\) inference rules is already closed under that rule.

The following is a slight modification of Theorem 1 of A. Wroński [11], p. 74:

**Theorem 5.2.** For every \(n = 1, 2, \ldots\) the calculus \(G_n\) allows for a finite axiomatization of all \(M\)-hyperquasi-identities of every algebra of a finite type whose number of elements does not exceed \(n\).

**References**


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