Krystyna Mruczek-Nasieniewska and Marek Nasieniewski

BÉZIAU’S LOGICS OBTAINED BY MEANS OF QUASI-REGULAR LOGICS

Abstract
The paper extends the results from [9], [7] and [10] for the case of quasi-regular modal logics. The original idea comes from [3] where the logic Z is formulated with the help of the logic S5. The problem arises when another modal logic is used instead of S5. In the present paper we give a framework for considering logics expressed by means of quasi-regular modal logics, we also give an example of such a logic.

The present paper is a written version of the results presented as [8] and basically relies on [10].

Introduction

In [3] J.-Y. Béziau defined the logic Z with the use of the modal logic S5. The idea was to introduce a negation which was understood as ‘it is possible that not’. In [9], [7] the case of other normal logics was considered instead of S5. In [10] some examples of regular logics were used to obtain propositional logics with Béziau’s negation. As regards semantics, the considered frames had also non-normal worlds. The truth in a model was defined by means of truth in all worlds. A natural question arises: what will change if we restrict ourselves to the normal worlds, i.e., if we consider the case of some quasi-regular modal logics. Here we provide an answer to this question. We give an example of such logic presenting its syntactical and semantical characterization. This logic appears to be paraconsistent.
We refer to the standard notions and results from the field of modal logic\(^1\).

1. Class QT

To keep the paper self-contained we have to recall some notions and results from [9], [10].

**Definition 1.** Let For be a set of all propositional formulae in the language with connectives \(\sim, \land, \lor, \rightarrow, \leftrightarrow\) and the set \(\text{Var}\) of propositional variables.

We consider a logic belonging to the following set.

**Definition 2.** Let QT be the class of all logics being sets strictly contained in For, which including the full positive classical logic (FPCL) in the language \(\{\land, \lor, \rightarrow, \leftrightarrow\}\), the law of excluded middle:

\[A \lor \sim A,\]  

\[(\text{EM})\]

the following version of de Morgan’s law:

\[\sim(A \land B) \rightarrow (\sim A \lor \sim B),\]  

\[(\text{dM1}_\rightarrow)\]

for any axiom \(A\) being an instance of (FPCL), (dM1\(_\rightarrow\)) or (EM) and for any variable \(a\) which does not appear in \(A\) the following formula:\(^2\)

\[\sim A \rightarrow a,\]  

\[(\text{RN}^-)\]

which are moreover closed under modus ponens (MP), uniform substitution (US) and

\[
\frac{\sim(A \rightarrow B) \rightarrow a}{\sim(\sim B \rightarrow \sim A) \rightarrow a},
\]

\[(\text{BR})\]

where \(a\) is a variable not appearing in \(\Gamma \rightarrow A \rightarrow B\)\(^3\).

One can easily see that by (US), (EM) and positive logic we obtain:

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\(^1\)See, for example [4].

\(^2\)Notice that the results of applying of (RN\(^-\)) are not meant here as axioms.
Lemma 1. For any logic $L \in \mathcal{QT}$, any formula $A$ and a variable $b$ not appearing in $A$ if $\vdash A \rightarrow b \in L$, then $\neg A \in L$.

Remark 1. For any logic $L \in \mathcal{QT}$ the following rule is inferable:

$$\frac{\Gamma A \rightarrow B \in \text{FPCL}, \text{(dM1\rightarrow)}, \text{(EM)}}{\neg B \rightarrow \neg A} \quad (\text{CONTR}^-)$$

Proof: By (RN^-) we have $\vdash (A \rightarrow B) \rightarrow a \in L$. Using (BR) we obtain that $\vdash (\neg B \rightarrow \neg A) \rightarrow a \in L$, while by Lemma 1 we have that $\vdash (\neg B \rightarrow (\neg A) \in L$.

Since the classical logic belongs to $\mathcal{QT}$, we have that $\mathcal{QT} \neq \emptyset$. We will need a reference to the following set of logics $\mathcal{RT}$ being a subset of the set $\mathcal{R}$ defined in [10], Definition 2.

Definition 3. Let $\mathcal{RT}$ be the class of all logics being sets strictly contained in $\text{For}$, including (FPCL), (dM1\rightarrow), (EM) and are closed under (MP), (US), and

$$\frac{\vdash A \rightarrow B}{\vdash \neg B \rightarrow \neg A} \quad (\text{CONTR})$$

Remark 2. For any $L \in \mathcal{QT}$, $L$ contains the following versions of de Morgan’s laws:

$$\frac{(\neg p \vee \neg q) \rightarrow (\neg (p \land q))}{(\text{dM1\rightarrow})}$$

$$\frac{(\neg (p \vee q) \rightarrow (\neg (p \land \neg q))}{(\text{dM2\rightarrow})}$$

Proof: One can repeat the proof from [9], p. 231 with the observation that (CONTR^-)—a restricted version of the (CONTR)—is sufficient.

We have obvious:

Lemma 2. For any $L \in \mathcal{QT}$, $L$ contains the following version of Ex falso quodlibet:

$$\vdash (p \rightarrow \neg p) \rightarrow q \quad (\text{EFQq})$$

Proof: It is enough to mention that $\vdash p \rightarrow p \in \text{positive logic}$, so one can use (RN^-).
2. Semantics for QT. The class $\mathcal{P}$ and logic $Q_{S2}$

We will need the notion of a frame, a valuation, a model, truth and validity from [10], Definitions 3–8:

**Definition 4.** A relational frame (a frame, for short) is a triple $\langle W, R, N \rangle$ consisting of a nonempty set $W$, a subset $N$ of $W$, and a binary relation $R$ on $W$. Elements of the sets $W$, $N$, and $W \setminus N$ are called worlds, normal worlds, and non-normal worlds, respectively, while $R$ is the accessibility relation.

**Definition 5.** A valuation is any function $v \colon \text{Var} \rightarrow 2^W$.

**Definition 6.** A model is a quadruple $\langle W, R, N, v \rangle$, where $\langle W, R, N \rangle$ is a frame and $v$ is a valuation. We say that $\langle W, R, N, v \rangle$ is based on the frame $\langle W, R, N \rangle$.

**Definition 7.** A formula $A$ is true in a world $w \in W$ under a valuation $v$ (notation: $w \models_v A$) iff

1. if $A$ is a propositional variable, $w \models_v A \iff w \in v(A)$.
2. if $A$ has a form $\sim B$, for some formula $B$, then
   for $w \in N$:
   $w \models_v \sim B \iff$ there is a world $w'$ such that $wRw'$ and it is not the case that $w' \models_v B$ (for short),
   for $w \in W \setminus N$:
   $w \models_v \sim B$
3. if $A$ is of the form $\neg B \land C$, for some formulae $B$ and $C$, then
   $w \models_v \neg B \land C \iff w \models_v B$ and $w \models_v C$.
4. if $A$ is of the form $\neg B \lor C$, for some formula $B$ and $C$, then
   $w \models_v \neg B \lor C \iff w \models_v B$ or $w \models_v C$.
5. if $A$ is of the form $\neg B \rightarrow C$, for some formulae $B$ and $C$, then
   $w \models_v \neg B \rightarrow C \iff w \models_v B$ or $w \models_v C$.
6. if $A$ is of the form $\neg B \leftrightarrow C$, for some formulae $B$ and $C$, then
   $w \models_v \neg B \leftrightarrow C \iff (w \models_v B$ and $w \models_v C)$ or $(w \models_v B$ and $w \models_v C)$.

**Definition 8.** A formula $A$ is $R$-true in a model $M = \langle W, R, N, v \rangle$ (notation $M \models_R A$) iff $w \models_v A$ for each $w \in W$.
Definition 9. A formula \( A \) is \( \mathcal{R} \)-valid in a frame \( \langle W, R, N \rangle \) iff it is \( \mathcal{R} \)-true in all models based on \( \langle W, R, N \rangle \).

In the present paper we will use a modified understanding of truth and validity:

Definition 10. A formula \( A \) is \( \mathcal{Q} \)-true in a model \( M = \langle W, R, N, v \rangle \) (notation \( M \models_{\mathcal{Q}} A \)) iff \( w \models_{v} A \) for each \( w \in N \).

Definition 11. A formula \( A \) is \( \mathcal{Q} \)-valid in a frame \( \langle W, R, N \rangle \) iff it is \( \mathcal{Q} \)-true in all models based on \( \langle W, R, N \rangle \).

We have

Lemma 3 ([10], Lemma 1). For any model \( \langle W, R, N, v \rangle \) and \( w \in N: w \not\models_{v} \sim(p \to p) \).

We have the following extension of Lemma 2 from [10] as well as the fact (here point c) indicated in the proof of Corollary 7 in [10]. We will give only the proofs of new observations.

Lemma 4.

a) The axiom \( (dM_{1}+) \) is \( \mathcal{R} \)-valid \( (\mathcal{Q} \)-valid) in every frame, while \( (EFQq) \) is \( \mathcal{Q} \)-valid in every frame.

b) All positive classical theses are true in every world (including non-normal ones) of any model based on any frame.

c) The axiom \( (EM) \) is \( \mathcal{R} \)-valid \( (\mathcal{Q} \)-valid) in every frame with accessibility relation reflexive on the set of all normal worlds.

d) The set of all formulae true in a given world of any model based on any frame is closed under \( (MP) \).

e) The set of all formulae \( \mathcal{R} \)-true \( (\mathcal{Q} \)-true) in a given frame is closed under substitution.

f) The set of all formulae \( \mathcal{R} \)-true in a given model based on a frame is closed under \( (CONTR) \).

g) All formulae obtained by \( (RN^{-}) \) are \( \mathcal{Q} \)-valid in every frame with accessibility relation reflexive on the set of all normal worlds.

h) The set of all formulae \( \mathcal{Q} \)-valid in a given frame is closed under \( (BR) \).

Proof: a) Let us consider any model \( \langle W, R, N, v \rangle \) and a world \( w_{0} \in N \). Since in every world \( w \) of a model \( w \models_{v} \sim p \to p \) thus \( w_{0} \not\models_{v} \sim(p \to p) \) and \( w_{0} \models_{v} \sim(p \to p) \to q \).
e) Standard.

g) Consider a model \((W, R, N, v)\), where \(R|_{N \times N}\) is reflexive. By points a), b), and c) we have \(w \not \models v \sim A\) and \(w \models v \vdash A \rightarrow \sim v\) for any normal world \(w\), any axiom \(A\) being an instance of (FPCL), (dM1\(\ldots\)) or (EM), and any variable \(a\).

h) Let us assume that a formula \(\vdash (A \rightarrow B) \rightarrow \sim v\) is \(Q\)-valid in a given frame \((W, R, N)\). Consider a model \((W, R, N, v)\). Let \(w \in N\) and \(w \models v \sim (\sim B \rightarrow \sim A)\). Then there is \(w' \in W\) such that \(w'Rw'\) and \(w' \models v \sim B\) and \(w' \not \models v \sim A\). This means that \(w' \in N\), thus there is \(w''\) such that \(w''Rw''\), \(w'' \not \models v \sim B\) and \(w'' \models v \sim A\), i.e. \(w'' \models v \vdash A \rightarrow \sim v\), so \(w'' \models v \vdash (A \rightarrow B)\).

By the initial assumption we have that \(w'' \models v \vdash A\). Consider a valuation \(v' : \text{Var} \rightarrow 2^W\) such that \(v'|_{\text{Var}\setminus\{a\}} = v|_{\text{Var}\setminus\{a\}}\) while \(v'(a) = v(a)\setminus\{w'\}\).

Since \(a\) does not appear in \(\vdash A \rightarrow \sim v\), we have that \(w'' \models v \vdash (A \rightarrow B)\).

Of course, \(w'' \not \models v \vdash A\). Contradiction. Thus for any \(w \in N\) it holds that \(w \models v \vdash (\sim B \rightarrow \sim A)\), so \(w \models v \vdash (\sim B \rightarrow \sim A) \rightarrow \vdash \sim v\).

Notice that if in a given frame the set \(N\) were empty, every formula would be \(Q\)-valid in that frame. By Lemmas 3 and 4 we have:

**Corollary 1.** Let \((W, R, N)\) be any frame, where \(N \neq \emptyset\).

(a) The set of all formulae \(Q\)-valid in \((W, R, N)\) does not equal \(\text{For}\).

(b) If \(R\) is reflexive, then the set of all formulae \(Q\)-valid in \((W, R, N)\) belongs to \(\text{QT}\).

We will consider the smallest logic in \(\text{QT}\). Let us denote it by \(Q_{S2}\). Let us recall that \(R_{E2}\) is the smallest logic in \(\text{RT}\).\(^3\) Directly from the definition of \(Q_{S2}\), by Lemma 4 we have:

**Corollary 2** (Soundness). For any \(A \in Q_{S2}\), \(A\) is \(Q\)-valid in every frame with reflexive accessibility relation\(^4\).

We recall a notion of a paraconsistent logic in a well known version:

**Definition 12.** A logic \(L\) is paraconsistent iff the formula

\[
p \rightarrow (\sim p \rightarrow q)
\]

does not belong to \(L\).

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\(^3\)See [10], p. 194.

\(^4\)Of course, we can assume that reflexivity concerns the set of all normal worlds.
Let $\mathcal{P}$ be a class of all paraconsistent logics.

**Corollary 3.** $Q_{S2} \in \mathcal{P}$.

**Proof:** Let $W = \{w_1, w_2\}$, $R = \{(w_1, w_1), (w_1, w_2)\}$, $N = \{w_1\}$ and $v(p) = \{w_1\}$, $v(q) = \emptyset$. It is easy to see that the formula $(\dagger)$ is not $Q$-true in the model $(W, R, N, v)$, thus, $(\dagger)$ is not $Q$-valid in $(W, R, N)$, i.e. $(\dagger) \notin Q_{S2}$.

Since $R$-validity is just validity from [10] we can recall the unaltered completeness result for $R_{E2}$.

**Fact 1 ([10], Corollary 7).** A formula $A$ is $R$-valid in every frame with a reflexive accessibility relation iff $A \in R_{E2}$.

### 3. Canonical models for $QT$

**Definition 13.** Let $L \in QT \cup RT$. A set $X$ of formulae is inconsistent with respect to the logic $L$ (or shortly $L$-inconsistent) iff for any formula $B$ there are $n \geq 1$ and formulae $A_1, \ldots, A_n \in X$, such that $\vdash_L \neg (A_1 \land \cdots \land A_n) \to B$. The set $X$ is consistent with respect to the logic $L$ (or, shortly, $L$-consistent) iff it is not $L$-inconsistent.

By Lemma 2 and substitution we have:

**Lemma 5.** If $\neg (p \to p) \in X$, then $X$ is $L$-inconsistent, for any logic $L \in QT$.

**Definition 14.** Let $L \in QT \cup RT$. A set $X$ is maximally consistent with respect to the logic $L$ (or, shortly, maximally $L$-consistent) iff $X$ is $L$-consistent and for any formula $A \notin X$ the set $X \cup \{A\}$ is $L$-inconsistent.

Below we have a counterpart of Theorem 5 in [6]:

**Lemma 6.** For any formula $A$ the following holds: $A \in R_{E2}$ iff for some variable $a$, which does not appear in $A$ it holds that $\neg \neg A \to \neg (a \to a) \in R_{E2}$.

**Proof:** Left-to-right implication is obvious by positive logic and the rule (CONTR). For the reverse implication assume that $\neg \neg A \to \neg (a \to a) \in R_{E2}$. By Fact 1 the formula $\neg \neg A \to \neg (a \to a) \in R_{E2}$. By Fact 1 the formula $\neg \neg A \to \neg (a \to a)$ is valid in every frame with
the reflexive accessibility relation. Assume that there are $M = (W, R, N, v)$, where $R$ is reflexive on $N$, and $w \in W$ such that $w \not\psi \ A$. Now if $w \not\in N$ then since $\langle w, w \rangle \in R$ we have that $w \not\models \sim (a \rightarrow a)$ – a contradiction with Lemma 3. If $w \not\in W \setminus N$ then we consider a model $\mathcal{M} = (W \cup \{w\}, R, N \cup \{w\}, \overline{v})$, where $\overline{v} \not\in W$. $\overline{R} = R \cup \{(w, \overline{w}), (\overline{w}, w)\}$ and $\overline{v}(a) = v(a)$ for any $a \in \text{Var}$. It is easy to see that for any formula $B$: $w \not\models_a B$ iff $w \not\models_B B$. Thus $w \not\models \overline{\tau} A$, therefore $\overline{w} \not\models_\tau \sim A$. Of course, every thesis of $R_{E2}$ is valid in $\mathcal{M}$, so $\overline{w} \not\models_\tau \sim (a \rightarrow a)$ – again a contradiction with Lemma 3.

Since by (CONTR) we can obtain $\tau \sim (a \rightarrow a) \rightarrow \sim (b \rightarrow b) \preceq$ for any variables $a$ and $b$, thus we directly obtain:

**Corollary 4.** For any formula $A$ and a variable $b$, such that $\tau \sim A \rightarrow \sim (b \rightarrow b) \preceq \in R_{E2}$, it holds that $A \in R_{E2}$.

The following Lemma together with Corollary 7 state a reformulation of Theorem 20 in [6].

**Lemma 7.** If $B \in Q_{S2}$ then $\tau B \lor \sim (a \rightarrow a) \preceq \in R_{E2}$, where $a$ is a variable which does not appear in $B$.

**Proof:** Let us consider a proof $C_1, \ldots , C_n$ of a formula $B$ in $Q_{S2}$. We will prove by the induction on $i$ that the respective formulae belong to $R_{E2}$.

If $C_i$ is an axiom then the required thesis follows by positive logic, (MP) and the fact that $R_{E2}$ and $Q_{S2}$ are defined by the same set of axioms.

If $C_i$ arises by (MP) then there are $1 \leq j, k < i$ such that $C_k$ has a form $\tau C_j \rightarrow C_i \preceq$ and by the inductive hypothesis we have $\tau (C_j \rightarrow C_i) \lor A \rightarrow a \preceq \in R_{E2}$ and $\tau C_j \lor \sim (b \rightarrow b) \preceq \in R_{E2}$ for some variables $a$ and $b$ which do not appear, respectively, in $\tau C_j \rightarrow C_i \preceq$ and $C_j$. Thus, by (US) and positive logic, we have that $\tau C_i \lor \sim (a \rightarrow a) \preceq \in R_{E2}$.

Consider the case of $\text{RN}^-$ i.e., $C_i$ has a form $\tau A \rightarrow \sim e \preceq$ where $A$ is an axiom of $Q_{S2}$ and $e$ is a variable not appearing in $A$. Since $A \in R_{E2}$, so by positive logic and (CONTR), we have $\tau A \rightarrow \sim (b \rightarrow b) \lor \sim c \preceq \in R_{E2}$ for any variables $b$ and $c$. Thus, by (FPCL) and (US), we obtain that $\tau (\sim A \rightarrow e) \lor \sim (a \rightarrow a) \preceq \in R_{E2}$, for given $e$ and any variable $a$ not appearing in $\tau A \rightarrow e \preceq$.

Consider the case of (BR). Thus, $C_i$ is of the form $\tau (\sim B \rightarrow \sim A) \rightarrow e \preceq$ where $e$ does not appear in $\sim (B \rightarrow \sim A)$, while by inductive hypothesis we have $\tau (\sim (A \rightarrow B) \rightarrow e) \lor \sim (a \rightarrow a) \preceq \in R_{E2}$ where $a$ does not appear
in \( \vdash (A \rightarrow B) \rightarrow e^\gamma \). By (US) and positive logic we have that \( \vdash (A \rightarrow B) \rightarrow \neg (a \rightarrow a)^\gamma \in R_E \). Therefore, thanks to Lemma 6 we obtain that \( A \rightarrow B \in R_E \). Thus, we have \( \vdash \neg B \rightarrow \neg A \in R_E \) by (CONTR), since by positive logic we have \( \vdash (c \rightarrow c) \rightarrow (\neg B \rightarrow \neg A)^\gamma \in R_E \), therefore by (CONTR) we obtain \( \vdash (\neg B \rightarrow \neg A) \rightarrow (\neg e \rightarrow c)^\gamma \in R_E \) for any variable \( c \), so by positive logic we get \( \vdash \neg (\neg B \rightarrow \neg A) \rightarrow \neg (e \rightarrow c) \lor e^\gamma \in R_E \). Thus, by (FPCL), \( \vdash (\neg (\neg B \rightarrow \neg A) \rightarrow e) \lor \neg (e \rightarrow c)^\gamma \in R_E \).

Finally, notice that in the case of (US) it is enough to use inductive hypothesis and substitution.

\[\square\]

**Lemma 8.** If \( \vdash A \rightarrow a^\gamma \in Q_{S2} \) for some variable \( a \) not appearing in \( A \), then \( A \in Q_{S2} \).

**Proof:** Follows by positive logic, substitution and (EM).

\[\square\]

**Lemma 9.** If \( A \in R_E \) then \( \vdash A \rightarrow a^\gamma \in Q_{S2} \) for some variable \( a \) not appearing in \( A \).

**Proof:** We reason by the induction on the length of a proof. Let \( C_1, \ldots, C_i \) be a proof in \( R_E \). If \( C_i \) is an axiom then the thesis follows by (RN'). If \( C_i \) arises by substitution from some \( C_j \) where \( j < i \), then by the inductive hypothesis we obtain that \( \vdash C_j \rightarrow a^\gamma \in Q_{S2} \) and the thesis follows by substitution. If \( C_i \) arises by (MP), we obtain, by inductive hypothesis and substitution, that \( \vdash C_j \rightarrow a^\gamma \in Q_{S2} \) and \( \vdash (C_j \rightarrow C_i) \rightarrow a^\gamma \in Q_{S2} \) for any variable \( a \). By (BR) we have that \( \vdash (\neg C_i \rightarrow \neg C_j) \rightarrow a^\gamma \in Q_{S2} \). Therefore by Lemma 8 we obtain that \( \vdash C_i \rightarrow C_j \gamma \in Q_{S2} \). Hence by positive logic we have that \( \vdash C_i \rightarrow a^\gamma \in Q_{S2} \). If \( C_i \) arises by (CONTR), then the thesis follows by inductive hypothesis and (BR).

\[\square\]

By Lemmas 8 and 9 we directly have that

**Corollary 5.**

\[ R_E \subseteq Q_{S2} \]

**Corollary 6.** For any formula \( A \) and a variable \( c \) which does not appear in \( A \) the following condition holds:

\[ \vdash A \rightarrow c^\gamma \in Q_{S2} \text{ iff } A \in R_E. \]
Proof: Left-to-right implication follows by Lemmas 7 and 6, and positive logic, if we take \( \neg \gamma A \rightarrow c \gamma \) as \( B \) in Lemma 7 and substitute \( \neg (a \rightarrow a) \) for \( c \).

Right-to-left implication is just Lemma 9. \( \square \)

We have a reverse statement for Lemma 7:

**Corollary 7.** If \( \gamma A \lor \neg (a \rightarrow a) \gamma \in \mathbb{R}_{E_2} \), where \( a \) is some variable which does not appear in \( A \), then \( A \in \mathbb{Q}_{S_2} \).

**Proof:** Assume that \( \gamma A \lor \neg (a \rightarrow a) \gamma \in \mathbb{R}_{E_2} \) and \( a \) is some variable which does not appear in \( A \). By Corollary 5 we obtain that \( \gamma \neg (a \rightarrow a) \gamma \in \mathbb{Q}_{S_2} \). By (RN\(^-\)) and (US) \( \gamma \neg (a \rightarrow a) \rightarrow A \gamma \in \mathbb{Q}_{S_2} \), so by positive logic we obtain that \( A \in \mathbb{Q}_{S_2} \). \( \square \)

We have two slight extensions of Lemmas from [9] and [10]:

**Lemma 10 ([10], Lemma 3).** For \( L \in \mathbb{Q}T \cup \mathbb{R}T \), every maximally \( L \)-consistent set contains \( L \) and is closed under \((MP)\).

**Lemma 11 ([10], Lemma 4).** For \( L \in \mathbb{R}T \cup \mathbb{Q}T \), a maximally \( L \)-consistent set \( X \), and any formulae \( A, B \) we have:

1. \( \gamma A \land B \gamma \in X \iff A \in X \) and \( B \in X \),
2. \( \gamma A \rightarrow B \gamma \in X \iff A \not\in X \) or \( B \in X \),
3. \( \gamma A \lor B \gamma \in X \iff A \in X \) or \( B \in X \),
4. \( \gamma A \leftrightarrow B \gamma \in X \iff A, B \in X \) or \( A, B \not\in X \).

We also recall a version of Lindenbaum (or Loś-Asser) Lemma.

**Lemma 12 (Lindenbaum Lemma).** Let \( L \in \mathbb{Q}T \cup \mathbb{R}T \). If \( A \in \text{For} \), \( X \subseteq \text{For} \), and there are no \( n \geq 0 \) and a set \( \{ A_1, \ldots, A_n \} \subseteq X \) such that \( \gamma L \gamma (A_1 \land \cdots \land A_n) \rightarrow A \gamma^5 \) then there is a maximally \( L \)-consistent set \( Y \) containing \( X \) such that \( A \not\in Y \).

By (FPCL), Definition 14 and Lemmas 10 and 11 we have:

**Lemma 13.** Let \( X \) be a maximally \( \mathbb{R}_{E_2} \)-consistent set. We have:

\( \gamma \neg (p \rightarrow p) \rightarrow A \gamma \in X \), for any formula \( A \) iff \( \gamma \neg (p \rightarrow p) \gamma \not\in X \).

Below we have a version of Lemma 7 in [10]:

\(^5\)The case \( n = 0 \) is treated as equivalent to the fact that \( \not\gamma L A \).
LEMMA 14. 1. The set \{\gamma \vdash \neg(p \to p) \to A \vdash A \in For \} is \text{R}_{E_2}\text{-consistent and there is a maximally \text{R}_{E_2}\text{-consistent set} X containing } \{\gamma \vdash \neg(p \to p) \to A \vdash A \in For \} \text{ such that } \gamma \vdash \neg((p \to p) \to s) \to s^\top \notin X.

2. Let X be a maximally \text{R}_{E_2}\text{-consistent set containing } \{\gamma \vdash \neg(p \to p) \to A \vdash A \in For \} \text{ and such that } \gamma \vdash \neg((p \to p) \to s) \to s^\top \notin X. \text{ The set } \mathcal{Z} = \{A : \gamma \vdash A \notin X\} \cup \{\neg(p \to p)\} \text{ is } \text{R}_{E_2}\text{-consistent.}

PROOF: Ad. 1. Consider a set \(W = \{\gamma \vdash \neg(p \to p) \to A^\top \in X : A \in For\}.

We will show that the assumptions of Lemma 12 are fulfilled. Assume for a contradiction that there are \(n \geq 0\) and \(A_1, \ldots, A_n\) such that \(\gamma \vdash \neg(p \to p) \to A_1 \wedge \cdots \wedge \neg(p \to p) \to A_n \to \neg((p \to p) \to s) \to s) \subseteq \text{R}_{E_2}\). By positive logic we would have that \(\gamma \vdash \neg(p \to p) \to A_1 \wedge \cdots \wedge A_n \to \neg((p \to p) \to s) \to s)

\(\subseteq \text{R}_{E_2}\). But it is not the case. Indeed, by completeness result one can show that \(\gamma \vdash \neg(p \to p) \to A_1 \wedge \cdots \wedge A_n \to \neg((p \to p) \to s) \to s) \subseteq \text{R}_{E_2}\). By transitivity we conclude that

\(\gamma \vdash \neg((A_1 \wedge \cdots \wedge A_n) \to s) \to \neg((p \to p) \to s) \to s) \subseteq \text{R}_{E_2}\).

By (dM1\text{L}_\text{H}) we have that

\(\gamma \vdash (\neg A_1 \vee \cdots \vee \neg A_n) \to s) \to \neg((p \to p) \to s) \to s) \subseteq \text{R}_{E_2}\)

However, by (FPCL) and Lemma 11 we see that \(\gamma \vdash (\neg A_1 \vee \cdots \vee \neg A_n) \to s) \subseteq X\) thus also \(\gamma \vdash (\neg(p \to p) \to s) \to s) \notin X\). Contradiction.

\(\square\)

DEFINITION 15. For any \(\{D_1, \ldots, D_n\} \subseteq \text{For} \) let \text{Q}_{S_2}[D_1 \ldots D_n]\) denote the smallest element in \text{Q} containing \text{Q}_{S_2} \cup \{D_1, \ldots, D_n\}, similarly let \text{R}_{E_2}[D_1 \ldots D_n]\) denote the smallest element in \text{R} containing \text{R}_{E_2} \cup \{D_1, \ldots, D_n\}.
Definition 16. Let \( L = Q_{S2}[D_1 \ldots D_n] \) and \( L^* = R_{E2}[D_1 \ldots D_n] \).

1. Let \( \mathcal{N} \) be the set of all maximally consistent sets \( X \) with respect to \( L^* \) such that for any \( A \in For, \neg \neg (p \rightarrow p) \rightarrow A \neg \in X \), and \( \text{non} \mathcal{N} \) be the set of all other maximally consistent sets with respect to \( L^* \). Let \( \mathcal{W} = \mathcal{N} \cup \text{non} \mathcal{N} \), \( \mathcal{N} \) be the set of normal worlds, and \( R \) be an accessibility relation on \( \mathcal{W} \) defined as follows:

\[
\forall wRw' \iff \forall A(A \notin w' \rightarrow \neg \neg A \in w).
\]

The triple \( (\mathcal{W}, R, \mathcal{N}) \) is called the canonical frame of the logic \( L \).

2. The canonical model of the logic \( L \) is a model \( (\mathcal{W}, R, \mathcal{N}, v) \), where \( (\mathcal{W}, R, \mathcal{N}) \) is the canonical frame of the logic \( L \) and the following condition is satisfied for any variable \( A \):

\[
v(A) = \{w \in \mathcal{W} : A \in w\}.
\]

i.e. \( w \models_v A \iff A \in w \).

Notice that by Lemmas 12 and 14 the set of normal worlds in the canonical frame of \( Q_{S2} \) is nonempty, moreover:

Lemma 15. For some normal world \( w \) in the canonical frame of the logic \( Q_{S2} \) there is a non-normal world \( w' \) such that \( wRw' \).

We have a version of Lemma 6 in [10]:

Lemma 16. Let \( (\mathcal{W}, R, \mathcal{N}, v) \) be the canonical model of \( Q_{S2} \).

a) For each formula \( A \) and \( w \in \mathcal{W} \) it holds: \( w \models_v A \iff A \in w \).

b) A formula is \( Q \)-true in \( (\mathcal{W}, R, \mathcal{N}, v) \) iff it is a theorem of \( Q_{S2} \).


b) (\( \Leftarrow \)). Let \( A \) be a theorem of the logic \( Q_{S2} \) and \( w \) a normal world of \( (\mathcal{W}, R, \mathcal{N}, v) \). We consider a proof \( A_1, \ldots, A_n \) of \( A \). By induction on \( 1 \leq i \leq n \) we show that \( A_i \in w \). If \( A_i \) is an instance of \( (FPCL), (EM) \) or \( (dM1) \), then by Lemma 10 it belongs to \( w \) as a thesis of \( R_{E2} \). If it arises by \( (MP) \), then due to inductive assumption it belongs to \( w \); also by Lemma 10.

If \( A_i \) arises by \( (RN) \), then it has a form \( \neg \neg A_j \rightarrow a \) (where \( A_j \) is an axiom and \( a \) is a variable which does not appear in \( A_i \)) for some \( j < i \), thus by Corollary 6 we see that \( A_i \in R_{E2} \); thus by positive logic and \( (CONTR) \) we have that \( \neg \neg (p \rightarrow p) \rightarrow (\neg \neg A_j \rightarrow a) \) holds in \( R_{E2} \) for any variable \( a \).
Therefore, by Lemma 10, definition of normal worlds and modus ponens we have that $\Gamma \models A_j \rightarrow a^\top \in w$.

If $A_i$ is introduced by the rule (BR), then it is of the form $\Gamma \models (\sim B \rightarrow \sim A) \rightarrow a^\top$ (where $a$ does not appear in $\Gamma \models B \rightarrow A^\top$) and there is $I < i$ such that $A_i$ is of the form $\Gamma \models (A \rightarrow B) \rightarrow a^\top$. By Corollary 6 and (CONTR) we obtain that $\Gamma \models B \rightarrow A^\top \in R_{E_2}$, so by positive logic and (CONTR) we have that $\Gamma \models (\sim B \rightarrow \sim A) \rightarrow (\sim (p \rightarrow p)) \rightarrow (\sim (\sim B \rightarrow \sim A) \rightarrow a) \rightarrow a \in R_{E_2}$ for any variable $a$. Then, by Lemma 10 and normality of $w$, we have that $A_i \in w$.

Finally, if $A_i$ is obtained by substitution into $A_j$ (where $j < i$), then by inductive hypothesis $A_j \in w$. However due to formulation of $\mathcal{Q}T$ the only question concerns the cases of substitution to results of application of (RN$^-$) and (BR), and one can easily see that analyzing the last two cases the required instances can be obtained by positive logic thanks to the fact that the formula $\Gamma \models (p \rightarrow p) \rightarrow A^\top$ belongs to $w$ for any formula $A$.

Thus, we have that $A_n \in w$ where $w$ is a normal world of $\langle W, R, N, v \rangle$. But by the case a) it means that $A$ is $\mathcal{Q}$-true in the canonical model.

($\Rightarrow$). Let us assume that $A \not\in Q_{S_2}$. Then, by Corollary 7 we have that $\Gamma \models A \lor (a \rightarrow a) \not\in R_{E_2}$ for some variable $a$ which does not appear in $A$. By Lemma 12 there is a maximally $R_{E_2}$-consistent set $^w$ such that $\Gamma \models A \lor (a \rightarrow a) \not\in w$. Therefore, due to Lemma 11 we obtain that $A \not\in w$ and $\Gamma \models (a \rightarrow a) \not\in w$. Hence $w$ is normal, by Lemma 13. This means by the case a) of the present lemma that $A$ is not $\mathcal{Q}$-true in the canonical model.

By the above lemma we obtain:

**Corollary 8 (Completeness).** If a formula is $\mathcal{Q}$-valid in every frame with a reflexive accessibility relation then it is a theorem of the logic $Q_{S_2}$.

Due to Lemma 15, Corollaries 2 and 8, we can reformulate completeness theorem:

**Corollary 9.** A formula $A$ is $\mathcal{Q}$-valid in every frame with a reflexive accessibility relation fulfilling the condition that there are $w \in N$ and $w' \in W \setminus N$ such that $wRw'$ iff $A \not\in Q_{S_2}$.

\footnote{It is enough to consider in Lemma the set of all theses of $R_{E_2}$ as $X$.}
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References


Department of Logic
Nicolaus Copernicus University of Toruń
e-mail: mruczek@uni.torun.pl and nnasien@uni.torun.pl