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Marek Nasieniewski
Andrzej Pietruszczak

SEMANTICS FOR REGULAR LOGICS CONNECTED WITH JAŚKOWSKI'S DISCUSSIVE LOGIC D_2

Abstract

Jaśkowski's discussive logic $\mathbf{D_2}$ was defined with the help of the modal logic **S5** as follows (see [7], [8]): $A \in \mathbf{D_2}$ iff $\ulcorner \Diamond A^{\bullet} \urcorner \in \mathbf{S5}$, where $(-)^{\bullet}$ is a translation of discussive formulae from $\mathrm{For}^{\mathrm{d}}$ into the modal language. The formulation of $\mathbf{D_2}$ does not change if instead of **S5** we use some weaker logics (see [5], [13], [19], [12]).

In the present paper we give semantics for the weakest regular modal logic defining $\mathbf{D_2}$, we also recall semantic results for some other known logics defining $\mathbf{D_2}$. Due to these results one can partially describe the lattice of logics defining $\mathbf{D_2}$.

Key words: Jaśkowski's discussive logic $\mathbf{D_2}$, semantics for logics defining $\mathbf{D_2}$.

1. Introduction

The logic $\mathbf{D_2}$ is defined as a set of discussive formulae of a certain kind. These formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', …; truth-value operators: '$\neg$' and '$\vee$' (negation and disjunction); discussive connectives: '$\wedge^{\mathrm{d}}$', '$\rightarrow^{\mathrm{d}}$', '$\leftrightarrow^{\mathrm{d}}$' (conjunction, implication and equivalence); and brackets. Let $\mathrm{For}^{\mathrm{d}}$ be the set of all these formulae.

In Appendix A one can find some chosen basic facts and notions concerning modal logic.

Let $\mathrm{For_m}$ be the set of all modal formulae. The logic $\mathbf{D_2}$ was formulated with the help of the modal logic **S5** as follows:

$$\mathbf{D_2} := \{\, A \in \mathrm{For^d} : \ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5} \,\},$$

where $(-)^\bullet$ is a translation of discussive formulae into the modal language, i.e., the function $-^\bullet$ from $\mathrm{For^d}$ into $\mathrm{For_m}$ such that:

1. $(a)^\bullet = a$, for any propositional variable a,
2. for any $A, B \in \mathrm{For^d}$:
   (a) $(\neg A)^\bullet = \ulcorner \neg A^\bullet \urcorner$,
   (b) $(A \vee B)^\bullet = \ulcorner A^\bullet \vee B^\bullet \urcorner$,
   (c) $(A \wedge B)^\bullet = \ulcorner A^\bullet \wedge \Diamond B^\bullet \urcorner$,
   (d) $(A \rightarrow^{\mathrm{d}} B)^\bullet = \ulcorner \Diamond A^\bullet \rightarrow B^\bullet \urcorner$,
   (e) $(A \leftrightarrow^{\mathrm{d}} B)^\bullet = \ulcorner (\Diamond A^\bullet \rightarrow B^\bullet) \wedge \Diamond(\Diamond B^\bullet \rightarrow A^\bullet) \urcorner$.

We say that a modal logic $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{A \in \mathrm{For^d} : \ulcorner \Diamond A^\bullet \urcorner \in \boldsymbol{L}\}$. It is known that there are other modal logics which define $\mathbf{D_2}$.

We see that while expressing logic $\mathbf{D_2}$ we refer to modal logics which (†): have the same theses beginning with '$\Diamond$' as **S5**. Let $\mathsf{S5}_\Diamond$ be the set of all these logics, i.e.,

$$\boldsymbol{L} \in \mathsf{S5}_\Diamond \quad \text{iff} \quad \forall_{A \in \mathrm{For_m}} (\ulcorner \Diamond A \urcorner \in \boldsymbol{L} \iff \ulcorner \Diamond A \urcorner \in \mathbf{S5}).$$

It is obvious that

FACT 1.1. *If $\boldsymbol{L} \in \mathsf{S5}_\Diamond$, then $\boldsymbol{L}$ defines $\mathbf{D_2}$.*

Observe that for all rte-logics[1] the following holds: defining of $\mathbf{D_2}$ is equivalent to having the property (†).

FACT 1.2 ([12]). *For any rte-logic $\boldsymbol{L}$: $\boldsymbol{L}$ defines $\mathbf{D_2}$ iff $\boldsymbol{L} \in \mathsf{S5}_\Diamond$.*

In [12] it was proved that **S5** is the upper bound of all logics from $\mathsf{S5}_\Diamond$ which are closed under congruence (cgr). Hence—by Fact 1.2—the logic **S5** is also the upper bound of all logics which define $\mathbf{D_2}$ and are closed under (cgr).

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[1] We say that a modal logic $\boldsymbol{L}$ is *rte-logic* iff $\boldsymbol{L}$ is closed under replacement of tautological equivalents, i.e., for any $A, B, C \in \mathrm{For_m}$

$$\text{if } \ulcorner A \leftrightarrow B \urcorner \in \mathbf{PL} \text{ and } C \in \boldsymbol{L}, \text{ then } C[{}^{A}/_{B}] \in \boldsymbol{L}. \qquad \text{(rte)}$$

Of course, every modal logic closed under (rep) is also closed under (rte) (see [1] and [12]).

1.1. Smallest normal and regular logics defining $\mathbf{D_2}$

It is known that **S4** belongs to $\mathsf{S5}_\Diamond$ (see [5]). So, to define $\mathbf{D_2}$ one can use weaker than **S5** modal logics. In [13] the smallest normal logic in $\mathsf{S5}_\Diamond$ was given. This logic, denoted by $\mathbf{S5^M}$, was defined as the smallest normal logic containing $\ulcorner\Diamond\mathtt{T}\urcorner$, $\ulcorner\Diamond\Box(\mathtt{5})\urcorner$ and $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$, and closed under the following rule[2]:

$$\frac{\Diamond\Diamond A}{\Diamond A} \qquad (\mathrm{cut}^{\Diamond\Diamond}_{\Diamond})$$

Of course, since $\mathbf{S5^M} \in \mathsf{S5}_\Diamond$, so $\mathbf{S5^M}$ defines $\mathbf{D_2}$, by Fact 1.2. Summarizing,

FACT 1.3 ([13], [10]). (i) $\mathbf{S5^M}$ *is the smallest normal logic in* $\mathsf{S5}_\Diamond$.
(ii) $\mathbf{S5^M}$ *is the smallest normal logic defining* $\mathbf{D_2}$.

Below we recall the results concerning known reformulations of $\mathbf{S5^M}$. These results will be useful in what follows.

FACT 1.4 ([3], [9], [10], [11]). $\mathbf{S5^M}$ *is the smallest normal logic which*:

(i) *contains* $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ *and "semi-4"*

$$\Box p \to \Diamond\Box\Box p \qquad (\mathtt{4_s})$$

i.e. $\mathbf{S5^M} = \mathbf{K4_s} \oplus \Diamond\Box(\mathtt{T})$;

(ii) *contains* $(\mathtt{5_c})$ *and* $(\mathtt{4_s})$, *i.e.* $\mathbf{S5^M} = \mathbf{K5_c4_s}$;

(iii) *contains* $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ *and is closed under* $(\mathrm{cut}^{\Diamond\Diamond}_{\Diamond})$;

(iv) *contains* $(\mathtt{5_c})$ *and is closed under* $(\mathrm{cut}^{\Diamond\Diamond}_{\Diamond})$.

In [10] it was shown that $\mathbf{rS5^M}$—the smallest regular logic defining $\mathbf{D_2}$—is the regular version of the logic $\mathbf{S5^M}$ in the axiomatization from Fact 1.4iii. Thus, we put

$\mathbf{rS5^M}$:= the smallest regular logic containing $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ and closed under $(\mathrm{cut}^{\Diamond\Diamond}_{\Diamond})$.

The logic $\mathbf{rS5^M}$ is properly contained in $\mathbf{S5^M}$. Summarizing,

FACT 1.5 ([10]). (i) *The logic* $\mathbf{rS5^M}$ *is not normal. Thus,* $\mathbf{rS5^M} \subsetneq \mathbf{S5^M}$.
(ii) $\mathbf{rS5^M}$ *is the smallest regular logic in* $\mathsf{S5}_\Diamond$; *moreover* $\mathbf{rS5^M}$ *is the smallest regular logic defining* $\mathbf{D_2}$.

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[2] In [13], [10], and [11] this rule was denoted by 'RM^2_1'.

We have also a counterpart of Fact 1.4:

FACT 1.6 ([11]). $\mathbf{rS5^M}$ *is the smallest regular logic which*:

(i) *contains* $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ *and* $(\mathtt{4_s})$, *i.e.* $\mathbf{rS5^M} = \mathbf{C4_s} \oplus \Diamond\Box(\mathtt{T})$;
(ii) *contains* $(\mathtt{5_c})$ *and* $(\mathtt{4_s})$, *i.e.* $\mathbf{rS5^M} = \mathbf{C5_c4_s}$;
(iii) *contains* $(\mathtt{5_c})$ *and is closed under* $(\mathrm{cut}^{\Diamond\Diamond}_{\Diamond})$.

1.2. Semantics for $\mathbf{S5^M}$

Below we recall semantic results for $\mathbf{S5^M}$ obtained by the canonical model method (see [3], [9]). These results can be also obtained by Theorem 6.5 from [14], pp. 55–62, for formulae falling under the so-called Hintikka schema.

FACT 1.7 ([9], Theorem 14). *A formula is a thesis of* $\mathbf{K5_c}$ *iff it is valid in all normal frames satisfying the following condition*

$$\forall_{w\in W}\exists_{u\in W}\big(w\,R\,u \;\&\; \forall_{x\in W}(u\,R\,x \Rightarrow w\,R\,x)\big). \qquad (\divideontimes)$$

FACT 1.8 ([9], Teorem 15). *A formula is a thesis of* $\mathbf{K4_s}$ *iff it is valid in all normal frames satisfying the following condition*

$$\forall_{w\in W}\exists_{u\in W}\big(w\,R\,u \;\&\; \forall_{x\in W}(\exists_{y\in W}(u\,R\,y \;\&\; y\,R\,x) \Rightarrow w\,R\,x)\big). \qquad (\circledast)$$

By facts 1.4ii, 1.7 and 1.8 we have

COROLLARY 1.9 ([9]). *A formula is a thesis of* $\mathbf{S5^M}$ *iff it is valid in all normal frames fulfilling conditions* $(\divideontimes)$ *and* $(\circledast)$.

Finally, in [9], Corollary 17, it was proved that formulae $(\mathtt{5_c})$ and $(\mathtt{4_s})$ are independent on the basis of $\mathbf{K}$.

2. Semantics for $\mathbf{rS5^M}$

We give conditions for frames which establish the completeness result for the logic $\mathbf{rS5^M}$ $(= \mathbf{C5_c4_s})$.

LEMMA 2.1. $(\mathtt{5_c})$ *is valid in all frames satisfying the following condition*

$$\forall_{w\in N}\exists_{u\in N}\big(w\,R\,u \;\&\; \forall_{x\in W}(u\,R\,x \Rightarrow w\,R\,x)\big). \qquad (\times)$$

PROOF: Let $\langle W, R, N, V\rangle$ be any model based on a frame satisfying $(\times)$. Assume that for $w \in W$ we have $V(\Box p, w) = 1$. Then $w \in N$ and by $(\times)$ for some $u_0 \in N$ we have $w\,R\,u_0$. Besides by $(\times)$ and by the assumption, $V(p, x) = 1$, for any $x \in W$ such that $u_0\,R\,x$. Thus, $V(\Box p, u_0) = 1$ and $V(\Diamond\Box p, w) = 1$. ⊣

LEMMA 2.2. *The frame* $\langle W_{\mathbf{C5_c}}, R_{\mathbf{C5_c}}, N_{\mathbf{C5_c}}\rangle$ *of the canonical model for* $\mathbf{C5_c}$ *satisfies* $(\times)$.

PROOF: For any $w \in W_{\mathbf{C5_c}}$ we put $\Gamma_w := \{A : \ulcorner\Box A\urcorner \in w\} \cup \{\Box B : \ulcorner\Box B\urcorner \in w\}$ (of course, $\Gamma_w \neq \emptyset$ iff $w \in N_{\mathbf{C5_c}}$). Firstly, we show that Γ_w is $\mathbf{C5_c}$-consistent. Assume otherwise, i.e., for some $\ulcorner\Box A_1\urcorner, \ldots, \ulcorner\Box A_n\urcorner, \ulcorner\Box B_1\urcorner, \ldots, \ulcorner\Box B_m\urcorner$ in w we have $\ulcorner\neg(A_1 \wedge \cdots \wedge A_n \wedge \Box B_1 \wedge \cdots \wedge \Box B_m)\urcorner \in \mathbf{C5_c}$. Let us denote $\ulcorner A_1 \wedge \cdots \wedge A_n \wedge B_1 \wedge \cdots \wedge B_m\urcorner$ by C. By the assumption, (R) and Lemma A.6, we obtain that $\ulcorner\Box C\urcorner \in w$. Notice that $\ulcorner(C \wedge \Box C) \rightarrow (A_1 \wedge \cdots \wedge A_n \wedge \Box B_1 \wedge \cdots \wedge \Box B_m)$ belongs to $\mathbf{C2}$. Hence, by the assumption, $\ulcorner\neg(C \wedge \Box C)\urcorner \in \mathbf{C5_c}$. So $\ulcorner C \rightarrow \neg\Box C\urcorner \in \mathbf{C5_c}$. Therefore $\ulcorner\Box C \rightarrow \Box\neg\Box C\urcorner$ and $\ulcorner\Box C \rightarrow \neg\Diamond\Box C\urcorner$ belong to $\mathbf{C5_c}$, by (mon) and (df $\Diamond$). Thus, we obtain that $\ulcorner\neg\Diamond\Box C\urcorner \in w$, but by $(\mathtt{5_c})$ we have also that $\ulcorner\Diamond\Box C\urcorner \in w$. A contradiction.

Let $w \in N_{\mathbf{C5_c}}$. Since the set Γ_w is $\mathbf{C5_c}$-consistent, there is a maximally $\mathbf{C5_c}$-consistent set u_0 including Γ_w. We have that $u_0 \in N_{\mathbf{C5_c}}$, because $\ulcorner\Box\top\urcorner \in w$, so also $\ulcorner\Box\top\urcorner \in \Gamma_w \subseteq u_0$. Besides $w\,R_{\mathbf{C5_c}}\,u_0$, because for any $A \in \mathrm{For_m}$ such that $\ulcorner\Box A\urcorner \in w$ we have $A \in \Gamma_w \subseteq u_0$.

Now suppose that for $x \in W_{\mathbf{C5_c}}$ we have $u_0\,R_{\mathbf{C5_c}}\,x$. We show that $w\,R_{\mathbf{C5_c}}\,x$. Indeed, if $\ulcorner\Box A\urcorner \in w$, then $\ulcorner\Box A\urcorner \in \Gamma_w \subseteq u_0$; but since $u_0\,R_{\mathbf{C5_c}}\,x$, we obtain that $A \in x$. ⊣

THEOREM 2.3. *A formula is a thesis of the logic* $\mathbf{C5_c}$ *iff it is valid in all frames satisfying the condition* $(\times)$.

PROOF: "⇒" By Lemma 2.1. "⇐" Let A be any formula which is valid in all frames satisfying $(\times)$. By Lemma 2.2, A is true in the canonical model of $\mathbf{C5_c}$. So A is a thesis of $\mathbf{C5_c}$, by Lemma A.7. ⊣

LEMMA 2.4. (4_s) *is valid in all frames satisfying the following condition*

$$\forall_{w\in N}\exists_{u\in N^c}\big(w\,R\,u\ \&\ \forall_{x\in W}(\exists_{y\in N}(u\,R\,y\ \&\ y\,R\,x)\Rightarrow w\,R\,x)\big),\qquad(\star)$$

where $N^c := \{w \in N : R(w) \subseteq N\}$ *is the set of all closed normal worlds.*

PROOF: Let $\langle W, R, N, V\rangle$ be any model based on a frame satisfying ($\star$). Assume that for $w \in W$ we have $V(\Box p, w) = 1$. Then $w \in N$. Hence, by ($\star$), for some $u_0 \in N^c$ we have $w\,R\,u_0$. Besides, by ($\star$) and by the assumption, $V(p, x) = 1$, for any $x \in W$ such that for some $y \in N$: $u_0\,R\,y$ and $y\,R\,x$. Thus, $V(\Box\Box p, u_0) = 1$ and $V(\Diamond\Box\Box p, w) = 1$. ⊣

LEMMA 2.5. *The frame* $\langle W_{\mathbf{C4_s}}, R_{\mathbf{C4_s}}, N_{\mathbf{C4_s}}\rangle$ *of the canonical model for* $\mathbf{C4_s}$ *satisfies* ($\star$).

PROOF: For any $w \in W_{\mathbf{C5_c}}$ we put $\Gamma_w := \{A : \ulcorner\Box A\urcorner \in w\} \cup \{\Box\Box B : \ulcorner\Box B\urcorner \in w\}$. Let us start with the observation that Γ_w is $\mathbf{C4_s}$-consistent. Assume otherwise, i.e., for some $\ulcorner\Box A_1\urcorner, \dots, \ulcorner\Box A_n\urcorner, \ulcorner\Box B_1\urcorner, \dots, \ulcorner\Box B_m\urcorner$ in w we have that $\ulcorner\neg(A_1 \wedge \dots \wedge A_n \wedge \Box\Box B_1 \wedge \dots \wedge \Box\Box B_m)\urcorner$ belongs to $\mathbf{C4_s}$. Let us denote $\ulcorner A_1 \wedge \dots \wedge A_n \wedge B_1 \wedge \dots \wedge B_m\urcorner$ by C. By the assumption, (R) and Lemma A.6, we obtain that $\ulcorner\Box C\urcorner \in w$. Notice that $\ulcorner(C \wedge \Box\Box C) \to (A_1 \wedge \dots \wedge A_n \wedge \Box\Box B_1 \wedge \dots \wedge \Box\Box B_m)$ belongs to $\mathbf{C2}$. Therefore by the assumption, $\ulcorner\neg(C \wedge \Box\Box C)\urcorner \in \mathbf{C4_s}$. So $\ulcorner C \to \neg\Box\Box C\urcorner \in \mathbf{C4_s}$. Hence $\ulcorner\Box C \to \Box\neg\Box\Box C\urcorner$ and $\ulcorner\Box C \to \neg\Diamond\Box\Box C\urcorner$ belong to $\mathbf{C4_s}$, by (mon) and (df $\Diamond$). Thus, we obtain that $\ulcorner\neg\Diamond\Box C\urcorner \in w$ and $\ulcorner\Diamond\Box C\urcorner \in w$, by ($4_s$). We have a contradiction.

Let $w \in N_{\mathbf{C4_s}}$. Since the set Γ_w is $\mathbf{C4_s}$-consistent, there is a maximally $\mathbf{C4_s}$-consistent set u_0 including Γ_w. Since $\ulcorner\Box\top\urcorner \in w$, we obtain that $\ulcorner\Box\Box\top\urcorner \in \Gamma_w \subseteq u_0$. Moreover we have that $\ulcorner\Box\Box\top \to \Box\top\urcorner \in \mathbf{C2} \subseteq \mathbf{C4_s}$, thus we see that $\ulcorner\Box\top\urcorner \in u_0$, so $u_0 \in N_{\mathbf{C4_s}}$. Besides, $w\,R_{\mathbf{C5_c}}\,u_0$, because for any $A \in \mathrm{For_m}$, if $\ulcorner\Box A\urcorner \in w$ then $A \in \Gamma_w \subseteq u_0$. Taking into account that $\ulcorner\Box\Box\top\urcorner \in u_0$, we obtain that for any $x \in R_{\mathbf{C4_s}}(u_0)$: $x \in N_{\mathbf{C4_s}}$. So $u_0 \in N^{\mathrm{c}}_{\mathbf{C4_s}}$.

Let $x, y \in W_{\mathbf{C4_s}}$, $u_0\,R_{\mathbf{C4_s}}\,y$ and $y\,R_{\mathbf{C4_s}}\,x$. We show that $w\,R_{\mathbf{C4_s}}\,x$. Indeed, let us assume that $\ulcorner\Box A\urcorner \in w$. Then $\ulcorner\Box\Box A\urcorner \in u_0$. So by the assumption about x and y we get: $\ulcorner\Box A\urcorner \in y$ and $A \in x$. ⊣

By lemmas 2.4 and 2.5, in the standard way, we obtain:

THEOREM 2.6. *A formula is a thesis of* $\mathbf{C4_s}$ *iff it is valid in all frames fulfilling* ($\star$).

From theorems 2.3 and 2.6 we have:

COROLLARY 2.7. *A formula is a thesis of the logic* $\mathbf{rS5^M}$ *iff it is valid in all frames fulfilling conditions* ($\times$) *and* ($\star$).

Since ($\mathtt{5_c}$) and ($\mathtt{4_s}$) are independent on the basis of **K**, thus so are they also on the basis of **C2**.

Remark. The above observations show that formulae ($\mathtt{5_c}$) and ($\mathtt{4_s}$) are connected, respectively, with conditions ($\times$) and ($\star$). The above proofs show that the substitutions of these formulae, $\ulcorner\Box\top \to \Diamond\Box\top\urcorner$ and $\ulcorner\Box\top \to \Diamond\Box\Box\top\urcorner$, correspond, respectively, to the following strengthenings of seriality: $\forall_{w\in N}\exists_{u\in N}\, w\,R\,u$ and $\forall_{w\in N}\exists_{u\in N^c}\, w\,R\,u$.

3. The lattice of regular logics connected with $\mathbf{D_2}$

Using semantics result we will try to situate the logics $\mathbf{S5^M}$ and $\mathbf{rS5^M}$ among normal and—more generally—among regular logics (see Fig. 1). Notice that the logic **KD4** (resp. **CD4**) is quite close to $\mathbf{S5^M}$ (resp. $\mathbf{rS5^M}$) as regards well known normal (resp. regular) modal logics including $\mathbf{S5^M}$ (resp. $\mathbf{rS5^M}$). Indeed, by Lemma A.4, we have $\mathbf{KD4} = \mathbf{K5_c4}$ (resp. $\mathbf{CD4} = \mathbf{C5_c4}$). Therefore, it is enough to notice that, by Lemma 1.4iv (resp. Lemma 1.6iii), $\mathbf{S5^M}$ (resp. $\mathbf{rS5^M}$) is the smallest normal (resp. regular) logic containing ($\mathtt{5_c}$) and closed under ($\mathrm{cut}^{\Diamond\Diamond}_{\Diamond}$). Simply, the rule ($\mathrm{cut}^{\Diamond\Diamond}_{\Diamond}$) replaces the formula ($4^\Diamond$).

Another known logic located above $\mathbf{S5^M}$ is the logic $\mathbf{T^*}$ introduced in [13]. It is the smallest normal logic containing ($\mathtt{T}$) and $\ulcorner\Diamond\Box(\mathtt{5})\urcorner$, and closed under the rule ($\mathrm{cut}^{\Diamond\Diamond}_{\Diamond}$). In [2] it was observed that $\ulcorner\Diamond\Box(\mathtt{5})\urcorner$ can be dropped from the axiomatization of $\mathbf{T^*}$ given in [13]. It was proved (see [2], [3]) that $\mathbf{T^*}$ is the smallest normal logic which contains ($\mathtt{T}$) and ($\mathtt{4_s}$), i.e. $\mathbf{T^*} = \mathbf{KT4_s}$. To obtain Fig. 1 we also use the following fact.

FACT 3.1. (i) ($\mathtt{4_s}$) $\notin$ **KD**; *so* $\mathbf{KD} \subsetneq \mathbf{K4_s}$ *and* $\mathbf{CD} \subsetneq \mathbf{C4_s}$.
(ii) $\Diamond\Box(\mathtt{T}) \notin$ **KD**; *so* $\mathbf{KD} \subsetneq \mathbf{K} \oplus \Diamond\Box(\mathtt{T})$ *and* $\mathbf{CD} \subsetneq \mathbf{C2} \oplus \Diamond\Box(\mathtt{T})$.
(iii) ($\mathtt{5_c}$) $\notin \mathbf{K} \oplus \Diamond\Box(\mathtt{T})$; *so* $\mathbf{K} \oplus \Diamond\Box(\mathtt{T}) \subsetneq \mathbf{K5_c}$ *and* $\mathbf{C2} \oplus \Diamond\Box(\mathtt{T}) \subsetneq \mathbf{C5_c}$.
(iv) ($\mathtt{N^1}$) $\notin \mathbf{CT4_s}$; *so* $\mathbf{CT4_s} \subsetneq \mathbf{CN^1T4_s} = \mathbf{CF} \cap \mathbf{KT4_s}$ *and* $\mathbf{rS5^M} = \mathbf{C5_c4_s} \subsetneq \mathbf{CN^15_c4_s} = \mathbf{CF} \cap \mathbf{S5^M}$.

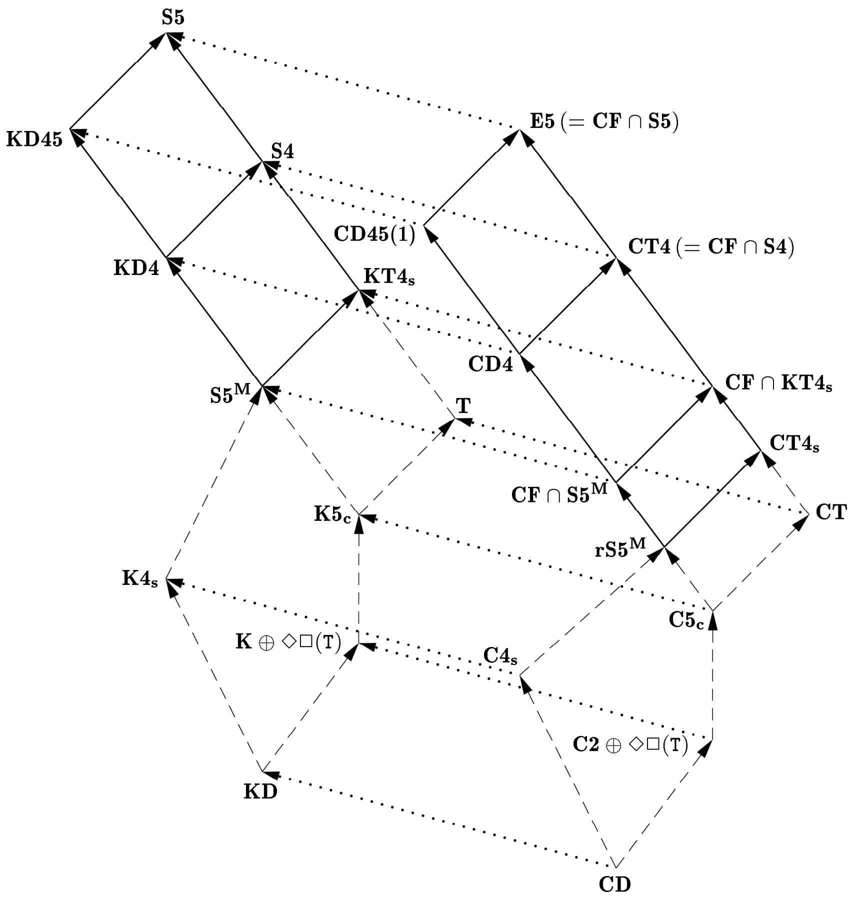

Fig. 1. $\mathbf{S5^M}$ and $\mathbf{rS5^M}$ among some normal and regular logics

PROOF: (i) Take a three-element normal model $\langle W, R, V\rangle$ such that $W = \{a, b, c\}$, $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, c\rangle\}$, and $V(p, w) = 1$ iff $w = b$. The formula ($\mathtt{4_s}$) is false in a, while ($\mathtt{D}$) is valid in the frame.

(ii) Consider a four-element normal model, where $W = \{a, b, c, d\}$, $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, d\rangle, \langle d, d\rangle\}$, and $V(p, w) = 1$ iff $w = d$. The formula $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ is false in a, while again ($\mathtt{D}$) is valid in that frame.

(iii) In a three-element normal frame, in which $W = \{a, b, c\}$ and $R = \{\langle a, b\rangle, \langle b, b\rangle, \langle b, c\rangle, \langle c, c\rangle\}$, the formula $\ulcorner\Diamond\Box(\mathtt{T})\urcorner$ is valid, while $(\mathtt{5_c})$ can be falsified in a.

(iv) In a three-element non-normal frame, in which $W = \{a, b, c\}$, $N = \{a, b\}$ and $R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle, \langle a, c\rangle\}$, the formulae $(\mathtt{4_s})$ and $(\mathtt{T})$ are valid, while $(\mathtt{N^1})$ can be falsified in a.[3] ⊣

A. Appendix: some facts from modal logic

Syntax

As in [4] modal formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\to$', and '$\leftrightarrow$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); modal operators: the necessity sign '$\Box$' and the possibility sign '$\Diamond$'; and brackets. By $\mathrm{For_m}$ we denote the set of modal formulae. Of course, the set $\mathrm{For_m}$ includes the set of all classical formulae (without '$\Box$' and '$\Diamond$'); let **Taut** be the set of all classical tautologies. Let $\top :=$ '$p \to p$'.

Besides, for any $A, B, C \in \mathrm{For_m}$, let $C[^A/_B]$ be any formula that results from C by replacing one or more occurrences of A, in C, by B.

Modal logics are certain sets of formulae. As in [1], we define a *regular modal logic* as a set $\boldsymbol{L}$ of modal formulae satisfying the following conditions:

- **Taut** $\subseteq \boldsymbol{L}$,
- $\boldsymbol{L}$ contains the following formula

$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{df $\Diamond$}$$

- $\boldsymbol{L}$ is closed under the following rules: *modus ponens* for '$\to$':

$$\frac{A \qquad A \to B}{A} \tag{mp}$$

the regularity rule:

$$\frac{A \wedge B \to C}{\Box A \wedge \Box B \to \Box C} \tag{reg}$$

and *uniform substitution*:

$$\frac{A}{\mathrm{s}\, A} \tag{sb}$$

[3]Observe that $a \notin N^{\mathrm{c}}$, so the frame given here is not closed, however it fulfils $(\star)$.

where $\mathrm{s}\,A$ is the result of uniform substitution of formulae for propositional letters in A.

All members of a regular logic are called its *theses*. Of course, by (sb), every regular logic includes the set **PL** of modal formulae which are instances of classical tautologies (i.e. instances of elements of **Taut**).

A set of formulae is a regular modal logic iff it includes **Taut**, contains ($\mathtt{df}\,\Diamond$) and the following formula

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

and is closed under (mp), (sb) and the monotonicity rule:

$$\frac{A \to B}{\Box A \to \Box B} \tag{mon}$$

Equivalently, a set of formulae is a regular modal logic iff it includes **Taut**, contains ($\mathtt{df}\,\Diamond$) and the following formula

$$\Box(p \land q) \leftrightarrow (\Box p \land \Box q) \tag{R}$$

and is closed under (mp), (sb) and the congruence rule

$$\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B} \tag{cgr}$$

Thus, every regular logic is closed under replacement:

$$\frac{A \leftrightarrow B}{C[{}^{A}/{}_{B}] \leftrightarrow C} \tag{rep}$$

so it has the following thesis

$$\Box p \leftrightarrow \neg \Diamond \neg p \tag{$\mathtt{df}\,\Box$}$$

Of course, every regular logic is also closed under the dual form of (mon):

$$\frac{A \to B}{\Diamond A \to \Diamond B} \tag{$\mathrm{mon}_\Diamond$}$$

Every regular logic has the following theses:

$$\Diamond(p \lor q) \leftrightarrow (\Diamond p \lor \Diamond q) \tag{$\mathtt{R}^\Diamond$}$$

$$\Diamond(p \to q) \leftrightarrow (\Box p \to \Diamond q) \tag{$\mathtt{R}^{\Diamond\Box}$}$$

LEMMA A.1. *For any regular logic the following conditions are equivalent*:
(a) *it has at least one thesis of the form* $\ulcorner\Diamond B\urcorner$,
(b) *it contains* $\ulcorner\Diamond\top\urcorner$,
(c) *it contains*

$$\Box p \to \Diamond p \tag{D}$$

LEMMA A.2. *If a regular logic contains the formula* $\ulcorner\Box\top\urcorner$, *then it is closed under the necessity rule*:

$$\frac{A}{\Box A} \tag{nec}$$

Further we will use the following formulae

$$\Box p \to p \tag{T}$$
$$\Box p \to \Box\Box p \tag{4}$$
$$\Diamond\Diamond p \to \Diamond p \tag{$4^\Diamond$}$$
$$\Diamond\Box p \to p \tag{B}$$
$$\Diamond\Box p \to \Box p \tag{5}$$
$$\Box p \to \Diamond\Box p \tag{5_c}$$

LEMMA A.3. *If a regular logic has theses* (5) *and either* $\ulcorner\Diamond\top\urcorner$ *or* (T), *then it closed under the rule* (nec).

A regular logic is *normal* iff it contains $\ulcorner\Box\top\urcorner$. Equivalently, a set of modal formulae is a normal logic iff it includes **Taut**, contains (df $\Diamond$) and (K), and is closed under (mp), (sb) and (nec).

Let **K** (resp. **C2**) be the smallest normal (resp. regular) modal logic. Using names of the above mentioned formulae, to simplify notation of normal (resp. regular) logics we write the *Lemmon code* $\mathbf{KX}_1 \ldots \mathbf{X}_n$ (resp. $\mathbf{CX}_1 \ldots \mathbf{X}_n$) to denote the smallest normal (resp. regular) logic containing the formulae ($\mathbf{X}_1$), ..., ($\mathbf{X}_n$) (see [1], [4], and [14]). Besides, for any formula $A \in \mathrm{For_m}$, let $\mathbf{KX}_1 \ldots \mathbf{X}_n \oplus A$ (resp. $\mathbf{CX}_1 \ldots \mathbf{X}_n \oplus A$) be the smallest normal (resp. regular) logic which includes $\mathbf{KX}_1 \ldots \mathbf{X}_n$ (resp. $\mathbf{CX}_1 \ldots \mathbf{X}_n$) and contains A.

LEMMA A.4. (i) (D) $\in \mathbf{C5_c} \subseteq \mathbf{K5_c}$.
(ii) (5_c) $\in \mathbf{CD4} \subseteq \mathbf{KD4}$.
(iii) $\mathbf{KD4} = \mathbf{K5_c4}$ *and* $\mathbf{CD4} = \mathbf{C5_c4}$.

Proof: (i) '$\Diamond(p \to \Box p)$' belongs to $\mathbf{C5_c}$, by ($\mathtt{R}^{\Diamond\Box}$). So, we use Lemma A.1.

(ii) By (4), (sb), (D) and **PL** we obtain (5_c).

(iii) By (i) and (ii). ⊣

We will use a formula

$$\Diamond(p \wedge \neg p) \tag{F}$$

The regular logic **CF** is called *falsum*. We have $\ulcorner \Diamond A \urcorner \in \mathbf{CF}$, for any $A \in \mathrm{For_m}$.

We standardly put $\mathbf{T} := \mathbf{KT}$, $\mathbf{S4} := \mathbf{KT4}$ and $\mathbf{S5} := \mathbf{KT5} = \mathbf{KT4B} = \mathbf{KD4B} = \mathbf{KD5B}$. As it is known, $\mathbf{T} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$, $\mathbf{CD5} = \mathbf{KD5}$, $\mathbf{CD45} = \mathbf{KD45}$ and $\mathbf{CT5} = \mathbf{KT5} =: \mathbf{S5}$. Thus, to avoid "normalization" of regular logics one has to use some special formulae. We adopt a convention from [14], p. 206 and for any formula (X) we put $(\mathtt{X(1)}) := \ulcorner \Box\top \to (\mathtt{X}) \urcorner$. Notice that in all monotonic logics, any formula of the form $\ulcorner \Box A \to B \urcorner$ is equivalent to $\ulcorner \Box\top \to (\Box A \to B) \urcorner$. Thus, the formulae (T), (D), (4) and (5_c) are equivalent to (T(1)), (D(1)), (4(1)) and (5_c(1)).

Lemma A.5 ([14], vol. II, Corollary 2.4).

$$\mathbf{CN^1X}_1\mathbf{(1)} \ldots \mathbf{X}_n\mathbf{(1)} = \mathbf{CF} \cap \mathbf{KX}_1 \ldots \mathbf{X}_n\,,$$

where

$$\Box\top \to \Box\Box\top \tag{$\mathtt{N}^1$}$$

In [14] Segerberg puts $\mathbf{E5} := \mathbf{CN^1T4B(1)}$. Thus $\mathbf{E5} = \mathbf{CF} \cap \mathbf{KT4B} = \mathbf{CF} \cap \mathbf{S5}$, by Lemma A.5. Notice that $\mathbf{E5} = \mathbf{CT4B(1)}$, since ($\mathtt{N}^1$) is an instance of (4). We also have $\mathbf{E5} = \mathbf{CN^1T5(1)}$ and $\mathbf{E5} = \mathbf{CF} \cap \mathbf{KT4B} = \mathbf{CF} \cap \mathbf{KD4B} = \mathbf{CD4B(1)}$.[4] Moreover, notice that $\mathbf{CD45(1)} = \mathbf{CN^1D45(1)} = \mathbf{CF} \cap \mathbf{KD45}$.

Kripke semantics for regular modal logics

A *relational frame* for regular logics is a triple $\langle W, R, N\rangle$ consisting of a nonempty set W, a binary relation R on W and a subset N of W. Elements of sets W and N we call, respectively, *worlds* and *normal worlds*, while R is the *accessibility relation*. We say that a frame $\langle W, R, N\rangle$ is *normal* iff $W = N$. Of course, any normal frame $\langle W, R, N\rangle$ may be identified with the pair $\langle W, R\rangle$.

[4] In [14] Segerberg also puts $\mathbf{D5} := \mathbf{CN^1D4B(1)} = \mathbf{CD4B(1)}$. So we have $\mathbf{D5} = \mathbf{E5}$.

A *model* for regular logics is any quadruple $\langle W, R, N, V\rangle$, where $\langle W, R, N\rangle$ is a frame and $V\colon \mathrm{For_m} \times W \longrightarrow \{0,1\}$ such that V preserves classical truth conditions for classical constants and for any $A \in \mathrm{For_m}$ and $w \in W$:

$$V(\Box A, w) = 1 \quad \text{iff} \quad w \in N \text{ and } \forall_{x\in R(w)}\, V(A, x) = 1\,,$$
$$V(\Diamond A, w) = 1 \quad \text{iff} \quad w \notin N \text{ or } \exists_{x\in R(w)}\, V(A, x) = 1\,,$$

where $R(w) := \{x \in W : w\, R\, x\}$. Of course, V is uniquely determined by its restriction to the set of all propositional letters. We say that the model $\langle W, R, N, V\rangle$ is based on the frame $\langle W, R, N\rangle$.

We say that a formula A is *true* in a model $M = \langle W, R, N, V\rangle$ iff $V(A, w) = 1$ for each $w \in W$. The set of all formulae true in a given model is closed under (reg).

Moreover, we say that a formula is *valid* in a frame iff it is true in all models based on this frame. (K), (R) and all formulae from **PL** are valid in any frame. The set of all formulae valid in a given frame is closed under (mp), (reg) and (sb). Thus, if a formula A is valid in a given frame and $B \in \mathbf{C2} \oplus A$, then also B is valid in this frame.

We say that a regular logic $\boldsymbol{L}$ is *determined* by a class $\mathscr{C}$ of frames iff $\boldsymbol{L}$ equals the set of all formulae which are valid in all frames from $\mathscr{C}$.

The logic falsum **CF** is determined by the single frame $\langle\{a\}, \emptyset, \emptyset\rangle$, i.e. a frame where $W = \{a\}$ and $R = \emptyset = N$. Moreover, if a normal logic $\mathbf{KX}_1 \ldots \mathbf{X}_n$ is determined by a class $\mathscr{C}$ of normal frames, then a regular logic $\mathbf{CN^1X}_1\mathbf{(1)} \ldots \mathbf{X}_n\mathbf{(1)}$ $(= \mathbf{CF} \cap \mathbf{KX}_1 \ldots \mathbf{X}_n)$ is determined by the class of frames obtained by adding to $\mathscr{C}$ the single frame $\langle\{a\}, \emptyset, \emptyset\rangle$ (see [14], pp. 204–206).

We say that a normal world w is *closed* iff $R(w) \subseteq N$. Let N^{c} be the set of all closed normal worlds, $N^{\mathrm{c}} \subseteq N$. We say that a frame is *closed* iff $N = N^{\mathrm{c}}$. The formula ($\mathtt{N}^1$) is valid in a given frame iff this frame is closed.

Canonical models for regular logics

We use canonical model method (see, for example [6], p. 205 and [14]). Let $\boldsymbol{L}$ be, throughout the rest of Appendix, any regular modal logic.

We say that a set Γ of formulae is $\boldsymbol{L}$*-inconsistent* iff there are A_1, $\ldots$, $A_n \in \Gamma$ such that $\ulcorner\neg(A_1 \wedge \cdots \wedge A_n)\urcorner \in \boldsymbol{L}$. A set of formulae is $\boldsymbol{L}$*-consistent* iff it is not $\boldsymbol{L}$-inconsistent.

LEMMA A.6. *For any maximally $\boldsymbol{L}$-consistent set Γ and any $A, B \in \mathrm{For_m}$:*

(i) $\boldsymbol{L} \subseteq \Gamma$,
(ii) $\ulcorner \neg A \urcorner \in \Gamma$ *iff* $A \notin \Gamma$,
(iii) $\ulcorner A \wedge B \urcorner \in \Gamma$ *iff* $A \in \Gamma$ *and* $B \in \Gamma$,
(iv) $\ulcorner A \vee B \urcorner \in \Gamma$ *iff* $A \in \Gamma$ *or* $B \in \Gamma$,
(v) $\ulcorner A \to B \urcorner \in \Gamma$ *iff* $A \notin \Gamma$ *or* $B \in \Gamma$,
(vi) $\ulcorner A \leftrightarrow B \urcorner \in \Gamma$ *iff* $A, B \in \Gamma$ *or* $A, B \notin \Gamma$.

Let $W_{\boldsymbol{L}}$ be the set of all maximally $\boldsymbol{L}$-consistent sets, $N_{\boldsymbol{L}}$ be the set of all sets from $W_{\boldsymbol{L}}$ containing $\ulcorner \Box \top \urcorner$, and $R_{\boldsymbol{L}}$ be the binary relation on $W_{\boldsymbol{L}}$ defined as follows:

$$w \, R_{\boldsymbol{L}} \, u \quad \text{iff} \quad \forall_{A \in \mathrm{For_m}} (\ulcorner \Box A \urcorner \in w \;\Rightarrow\; A \in u).$$

The *canonical frame* of $\boldsymbol{L}$ is the triple $\langle W_{\boldsymbol{L}}, R_{\boldsymbol{L}}, N_{\boldsymbol{L}} \rangle$. Since $\ulcorner \Box \top \urcorner \in \boldsymbol{L}$ iff $\boldsymbol{L}$ is normal, we obtain that: the canonical frame of $\boldsymbol{L}$ is normal, i.e. $W_{\boldsymbol{L}} = N_{\boldsymbol{L}}$, iff $\boldsymbol{L}$ is normal.

A *canonical model* of $\boldsymbol{L}$ is a model $\mathfrak{M}_{\boldsymbol{L}} = \langle W_{\boldsymbol{L}}, R_{\boldsymbol{L}}, N_{\boldsymbol{L}}, V_{\boldsymbol{L}} \rangle$ such that for any propositional letter a and $w \in W_{\boldsymbol{L}}$: $V_{\boldsymbol{L}}(a, w) = 1$ iff $a \in w$. Of course, there is exactly one canonical model of $\boldsymbol{L}$.

LEMMA A.7. (i) *For all $A \in \mathrm{For_m}$ and $w \in W_{\boldsymbol{L}}$: $V_{\boldsymbol{L}}(A, w) = 1$ iff $A \in w$.*
(ii) *A formula is true in $\mathfrak{M}_{\boldsymbol{L}}$ iff it is a thesis of $\boldsymbol{L}$.*

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Department of Logic
Nicolaus Copernicus University
ul. Asnyka 2, 87-100 Toruń, Poland
e-mail: {mnasien,pietrusz}@umk.pl