FORMALIZATION OF LEIBNIZ’S NOTION OF EXISTENCE AND OF GOD WITH SUSZKO’S NON-FREGEAN LOGIC

Abstract
The present study contains an attempt to formalize the Leibniz’s notion of existence and of the notion of God. To this end we use the non-Fregean sentential calculus with identity (SCI) with quantifiers to build Leibniz’s Language (LL), that allows us to formulate the formal definition of existence (in the sense of Leibniz) for the universe of situations and next – of the Leibniz’s notion of God. On the ground of these definitions we prove several ontological theorems, including - the theistic thesis.

1. Introduction
The present study contains an attempt to formalize the essential part of Leibniz’s metaphysics, and namely: of the notion of existence and of the notion of God. To this end we use the non-Fregean logic created by Roman Suszko under the direct inspiration of Ludwig Wittgenstein’s Tractatus Logico-Philosophicus. In particular, we use the non-Fregean sentential calculus with identity (SCI), which constitutes the core of W-languages, created by Suszko in order to formalize Wittgenstein’s ontology of situations ([1], p. 197–247). In the present work we use the SCI-calculus with quantifiers to build Leibniz’s Language (LL), that allows us to formulate the formal definition of existence (in the sense of Leibniz) for the universe of situations and next – of the Leibniz’s notion of God. On the ground of these definitions we prove several ontological theorems, including - the theistic thesis.
2. **LL-Language - Logical Part**

First of all, the LL contains the following (commonly used) basic symbols:

1. propositional variables: “p”, “q”, “r”, . . . ,
2. one-place predicate: “¬”,
4. quantifiers: “∀”, “∃”,
5. auxiliary signs (brackets): “(,)”.

In order to formalize philosophical intuitions of Leibniz we add to the set of basic symbols the following auxiliary symbols:

1. Propositional constants:
   “1” to be read as “logical truth”
   “0” to be read as “logical false”
2. One-place predicates:
   “□” to be read as “it is necessary, that”
   “♦” to be read as “it is possible, that”
   “EX” to be read as “really exists” or “is real”
   “PW” to be read as “is a possible world”
   “RW” to be read as “is a real world”
   “GL” to be read as “is God (in the sense of Leibniz)”
3. Two-place predicates:
   “<” to be read “contains” (ontological inclusion)
   “∩” to be read “is possible with” or “is co-possible with”
   “BL” to be read “is bigger in number than”.
4. Predicate of higher order:
   “M” to be read as “is the situation, which contains all such situations that satisfy the given propositional function”.

We also use small letters of Greek alphabet as variables of LL-formulas.

If we want to specify how many propositional variables occur in a given formula, we add brackets with relevant propositional variables, e.g.: α(p, q).

Among the formulas of LL, the following ones are distinguished as logical axioms:
(AL1) \( \alpha \Rightarrow (\beta \Rightarrow \alpha) \)
(AL2) \( (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\Rightarrow \gamma)) \)
(Al3) \( \neg \alpha \Rightarrow (\alpha \Rightarrow \beta) \)
(Al4) \( (\alpha \Rightarrow \beta) \Rightarrow ((\neg \alpha \Rightarrow \beta) \Rightarrow \beta) \)
(Al5) \( (\alpha \equiv \beta) \Rightarrow (\alpha \Rightarrow \beta) \)
(Al6) \( (\alpha \equiv \beta) \Rightarrow (\beta \Rightarrow \alpha) \)
(Al7) \( (\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \alpha) \Rightarrow (\alpha \equiv \beta)) \)
(Al8) \( (\alpha \wedge \beta) \equiv \neg (\alpha \Rightarrow \neg \beta) \)
(Al9) \( (\alpha \vee \beta) \equiv \neg (\neg \alpha \Rightarrow \beta) \)
(Al10) \( \alpha = \alpha \)
(Al11) \( (\alpha = \beta) \Rightarrow (\neg \alpha \Rightarrow \neg \beta) \)
(Al12) \( ((\alpha = \beta) \wedge (\gamma = \delta) \wedge \ldots \wedge (\phi = \gamma)) \Rightarrow F(\alpha, \gamma, \ldots, \phi) = F(\beta, \delta, \ldots, \gamma) \)

where \( F \) is any functor of the LL-language.
(Al13) \( (\alpha = \beta) \Rightarrow (\alpha \Rightarrow \beta) \)
(Al14) \( \forall p \alpha(p) \Rightarrow \alpha(p) \)
(Al15) \( \forall p (\alpha \Rightarrow \beta) \Rightarrow (\forall p \alpha \Rightarrow \forall p \beta) \)
(Al16) \( \alpha \Rightarrow \forall p \alpha \) (if \( p \) is not a free-variable in \( \alpha \))
(Al17) \( \exists p \alpha \Rightarrow \neg \forall \neg \alpha \)
(Al18) \( \forall p (\alpha = \beta) \Rightarrow (\forall p \alpha = \forall p \beta) \)
(Al19) \( \forall p (\alpha = \beta) \Rightarrow (\exists p \alpha = \exists p \beta) \)

We can see, that our system of axioms (given schematically) contains:
(1) axioms of the classical sentential calculus (from (AL1) to (AL9); (2) axioms for the identity predicate (from (AL10) to (AL13), and (3) axioms for the generalisation of formulas (from (AL14) to (AL19). This set of axioms is the same as that of Suszko’s SCI system as presented by Omlya ([2], p. 86).

On the basis of logical axioms given above, we define non-Fregean operation of consequence:
For any formula \( \alpha \in F \) (where \( F \) is the set of all LL-formulas) and for any set \( X \subset L \), \( \alpha \in Cn(X) \) if and only if \( \alpha \) is derivable from the set \( AL \cup X \) (where \( AL \) is the set of logical axioms) in a finite number of steps with the aid of the following rule (modus ponendo ponens):

(R1) \( \alpha, \alpha \Rightarrow \beta \Rightarrow \beta \)

All our definitions are identity formulas, i.e. they are specific propositional formulas, which have the form \( \alpha = \beta \). Definitions of propositional constants, of individual constants and of first-order predicates are singu-
lar identity formulas, whereas definitions of higher-order predicates are schemas of identity.

\[(D1)\] \[1 = \forall p(p \lor \neg p)\]

\[(D2)\] \[0 = \exists p(p \land \neg p)\]

\[(D3)\] \[\Box p = (p = 1)\]

\[(D4)\] \[\diamond p = \neg(p = 0)\]

\[(D5)\] \[(p < q) = \Box(p \Rightarrow q)\]

\[(D6)\] \[p \circ q = \Diamond(p \land q)\]

\[(D7)\] \[Mr(\varphi(r), p) = \forall r(\varphi(r) \Rightarrow (r < p)) \land \forall q(q < \forall r(\varphi(r) \Rightarrow (p < q)))\]

We read “\(Mr(\varphi(r), p)\)” as “\(p\) is the situation containing all such situations \(r\), that satisfy propositional function \(\varphi(r)\)”.

Further auxiliary terms of LL-language will be defined in the context of Leibniz’s notions.

3. Definition Of (Real) Existence in the Sens of Leibniz

Let us concentrate on the definition of real existence which Leibniz formulated in his work “Generales induisitiones de analysi notionum et veritum” in 1868. German translation of it sounds: “‘Existierendes’ definiert werden könnte als das, was mit mehr Dingen vereinbar [compatibile] ist als irgendeines anderer Ding, das mit ihm unvereinbar ist’ ([3], p. 110]. In English: “What really exists, can be defined as something, that is compatible with more objects, than any other object, which is not compatible with it [the given object].”

The functor of existence will be defined in several steps. First of all we define the relation of ideal correspondence, i.e. the relation of one-to one correspondence:

\[(D8)\] \[OD(p, q) = (((\alpha(p, q) \land \alpha(p, r)) \Rightarrow (q = r)) \land (((\alpha(p, r) \land \alpha(q, r)) \Rightarrow (p = q))\]

Then we define the relation \(RL(p, q)\) which we read as “\(p\) is equal in number with \(q\):”

\[(D9)\] \[RL(p, q) = ((\alpha(p, q) = OD(p, q)) \land \forall r(p < r \Rightarrow \exists s(\alpha(r, s) \land q < s)) \land \forall s(q < s \Rightarrow \exists r(\alpha(r, s) \land p < r)))\]
and relation $BL(p, q)$ that we read as “$p$ is more in number than $q$”:

\[(D10) \quad BL(p, q) = \exists r (p < r \land (r \neq p) \land RL(r, q))\]

Finally, if we take “compatibility” occurring in his verbal definition as having the same sense as “co-possibility”, the Leibniz’s notion of existence may be explicated in $LL$ as follows:

\[(D11) \quad EXp = \Diamond p \land \forall q (\neg (p \circ q) \Rightarrow BL((Mr((p \circ r), u), Ms((q \circ s), v)))\]

where “$Mq((p \circ q), u)$” stands for “$u$ is the situation that is an upper limit (in other words: a maximal situation) containing all situations which are co-possible with situation $p$”.

It must be noted that the quoted Leibniz’s definition of real existence refers to all objects of ideas; term “object” used by Leibniz should be interpreted as the (extensionally) universal name. Our definition, however, applies just to the certain subset of universal class, namely, to the class of situations.

Finally, we formulate the definition of real world:

\[(D12) \quad RWp = EXp \land \forall q (EXq \Rightarrow (p < q))\]

Real world is a situation that really exists and contains all real situations.

4. **$LL$-Language – Ontological Axioms**

In order to complete $LL$ we formulate ontological axioms. All of them are identical with ontological axioms of Suszko’s $W$-languages ([1], pp. 197–247).

First ontological axiom can be called anti-Fregean axiom:

\[(AO0) \quad \exists p (\neg (p = 1) \land \neg (p = 0))\]

According to it, there are situations, which are possible and not-necessary at the same time. Leibniz calls such a kind of object “contingent object” and this term comprises everything that is given to us empirically.

Next axiom:

\[(AO1) \quad \alpha = 1, \text{ where } \alpha \text{ is any formula belonging to } Cn(\emptyset).\]
constitutes the rule of identity between tautological situations, because formulas of the form $\alpha \iff 1$ are tautologies for every formula belonging to $Cn(\emptyset)$. ($AO1$) may be called the axiom of tautology.

Next axiom has the following form:

$$(AO2) \quad ((p \iff q) = 1) \Rightarrow (p = q)$$

It is the **axiom of Boolean algebra**. ($AO2$) puts the structure of Boolean algebra on the universe of situations. Situations 1, 0 constitute zero and unit of this algebra; operations of negation, conjunction and alternative satisfy all laws of Boolean algebra; ontological inclusion constitutes an ordering relation of the Boolean algebra of situations. Hence we have:

$$(AO2') \quad (p = q) \iff ((p < q) \Rightarrow (q < p))$$

Next let us adopt:

$$(AO3) \quad ((p = q) = 1) \lor (p = q) = 0)$$

i.e. the **axiom of bivalence of identity operation**.

According to the next axiom – the **axiom of the real world** – there is a real world.

$$(AO4) \quad \exists p (RW p)$$

Last but not least, we formulate two axiom schemas: that of the lowest limit and that of the upper limit:

$$(AO5) \quad \exists p (\forall q (\alpha(q) \Rightarrow (q < p)) \land \forall r (\forall q (\alpha(q) \Rightarrow (q < r)) \Rightarrow (p < r)))$$

and

$$(AO6) \quad \exists p (\forall q (\alpha(q) \Rightarrow (p < q)) \land \forall r (\forall q (\alpha(q) \Rightarrow (r < q)) \Rightarrow (r < p)))$$

According to them, for any condition (constraint) put on situations with the aid of an expression $\alpha(p)$, there are situations which are a lower and an upper limit satisfying this condition.
5. Conventions on Proofs

Before starting we have to agree on the way in which the proofs will be presented.

The main purpose of the present work is to explicate, in a formal way, certain philosophical intuitions. In order to facilitate reading – to make the proofs intuitively as intelligible as possible – we will not apply the rule (R1) only. We will also refer to those tautologies that constitute a stable elements of philosophical organon, such as: *reductio ad absurdum*, syllogistic laws, de Morgan’s laws, law of transposition, *dictum de omni* etc. As it is well known, all tautologies of classical sentential calculus are derivable with the aid of the above defined operation of consequence (i.e. (R1)) and the axioms above given (from (AL1) to (AL19)). Then, let us consider those commonly used tautologies as formulas belonging to LL without proof, and let us use them in our explications as secondary inference rules.

In order to facilitate reading, we also adopt certain notation for proofs. Typical step of a proof looks like that:

3) $\exists p \exists q \neg ((EXp \land \Box(p \Rightarrow q)) \Rightarrow EXq)$
   3* Implication Negation *4
4) $\exists p \exists q((EXp \land \Box(p \Rightarrow q)) \land \neg EXq)$

It should be read as follows: formula in the line number 4) is derived from the formula in the line 3) with the aid of Implication Negation.

Final remark: we may omit the most obvious proof steps – especially at the end of a given proof.

6. Singularity of the Real World

For the beginning we will prove two theses connected with the notion of real existence.

First of them sounds: every situation contained in an real situation, really exists.

\[(T1) \quad \forall p (EXp \Rightarrow \forall q((p < q) \Rightarrow EXq))\]
Proof:

1) $\forall p \forall q ((EXp \land (p < q)) \Rightarrow EXq)$

1* $(D5)$ *2

2) $\forall p \forall q ((EXp \land \Box(p \Rightarrow q)) \Rightarrow EXq)$

2* negation for *ad absurdum* proof *3

3) $\forall p \forall q ((EXp \land \Box(p \Rightarrow q)) \Rightarrow EXq)$

3* De Morgan’s Law and Negation of Implication Law *4

4) $\exists p \exists q ((EXp \land \Box(p \Rightarrow q)) \land \neg EXq)$

4* $(D11)$; $\forall w (\neg (p \circ w) \Rightarrow BL(Mr((p \circ r), u), Ms((w \circ s), v))/B_q)$

$\forall t (\neg (q \circ t) \Rightarrow BL(Mk((q \circ k), l), Mm((t \circ m), n))/B_q)$ *5

Remark: $B_p$ and $B_q$ are used for the sake of brevity; they are admissible because expressions $B_p$ and $B_q$ do not play any direct role in the following proving procedure.

5) $\exists p \exists q ((\Diamond p \land B_p \land (\Box p \Rightarrow \Box q) \land \neg (\Diamond q \land B_q))$)

5* modality thesis (9.8) by Suszko ([1], p. 234) *6

6) $\exists p \exists q ((\Diamond p \land B_p \land (\Box p \Rightarrow \Box q) \land \neg (\Diamond q \land B_q))$)

6* Suszko’s modality law ([1], p. 234) *7

7) $\exists p \exists q (((\Diamond p \land B_p \land (\Diamond p \Rightarrow \Diamond q)) \land \neg (\Diamond q \land B_q))$)

7* Replacement of Implication Law *8

8) $\exists p \exists q ((\Diamond p \land B_p \land (\neg (\Diamond p \lor \Diamond q) \land \neg (\Diamond q \land B_q)))$

Now we check the logical value of 8) for every combination of values for $B_p$ and $B_q$. For $B_p = 1$ and $B_q = 1$ and $B_p = 0$ and $B_q = 0$ and also for $B_p = 0$ and $B_q = 1$ formula 8) is counter-tautology. For $B_p = 1$ and $B_q = 0$ we need a slightly longer deductive chain.

9) Let $B_p = 1$ and $B_q = 0$.

If $B_p = 1$ and $B_q = 0$ then expression 8) is false for every combination of values of $\Diamond p$ and $\Diamond q$, except of situation when $\Diamond p = 1$ and $\Diamond q = 1$. Hence, the presupposition taken by us for the purpose of *redactio ad absurdum* in line 3) leads us to the following conclusion:

10) $(\Diamond p = 1) \land (\Diamond q = 1)$

9* decoding $B_p$ and $B_q$ taken in step 4) *11

11) $\forall w (\neg (p \circ w) \Rightarrow BL(Mr((p \circ r), u), Ms((w \circ s), v))) = 1$

11* tautological transformation of implication *12

12) $\forall w (p \circ w) \lor BL(Mr((p \circ r), u), Ms((w \circ s), v))) = 1$

12* $(D6)$ i De Morgan’s Law *13
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13) ∀w(◊¬p ∨ ◊¬w) ∨ BL(Mr((p ◦ r), u), Ms((w ◦ s), v)) = 1

13* features of alternative *14

14) ◊¬p = 1

14* definition (D4) *15

15) Formula 14) contradicts the formula 10) Hence: for every possible combination of values BP i Bq, the expression 8) takes the value 0. And since formula 8) is the deductive conclusion derived from the negation of (T1), then on the ground of conjunctional reduction ad absurdum we get the conclusion that (T1) is true. QED.

We prove now that what is real world is one:

(T2) ∀p(RWp ⇒ (∀q(RWq ⇒ (p = q))))

Proof:

(T2)* laws of quantification *1

1) ∀p∀q((RWp ∧ RWq) ⇒ (p = q)))

1* Axiom (AO2’) * 2

2) ∀p∀q∀r((RWp ∧ RWq) ⇒ ((p < r) ⇔ (q < r)))

2* negation in order to carry out ad absurdum proof *3

3) ¬∀p∀q∀r((RWp ∧ RWq) ⇒ ((p < r) ⇔ (q < r)))

3* De Morgans Law and Negation of Implication *4

4) ∃p∃q∃r((RWp ∧ RWq) ∧ ¬((p < r) ⇔ (q < r)))

5* definition (D12) *5

5) ∃p∃q∃r((EXp ∧ (EXq ⇒ (r < p)) ∧ EXq ∧ (EXr ⇒ (q < r))) ∧ ¬((p < r) ⇔ (q < r)))

5* Composition Law *6

6) ∃p∃q∃r((EXp ∧ EXq ∧ (EXr ⇒ (p < r) ∧ (q < r))) ∧ ¬((p < r) ⇔ (q < r)))

6* let us suppose, that (a) (p < r) = 1 i (q < r) = 0 * 7

7) ∀p∀q∀r((EXp ∧ (EXq ⇒ 1) ∧ EXq ∧ (EXr ⇒ 0)))

7* Composition Law *8

8) ∀p∀q∀r((EXp ∧ EXq ∧ (EXr ⇒ (1 ∧ 0))))

8* tautological transformation *9

9) ∀p∀q∀r((EXp ∧ EXq ∧ (EXr ⇒ 0)))

9* (T1) and supposition 6) *10
10) \((1 \land EXq \land (1 \Rightarrow 0))\) albo \((0 \land EXq \land (0 \Rightarrow 0))\)

We have got falsity.

By analogy, if we suppose, that:

(b) \((p < r) = 0\) i \((q < r) = 1\)

we receive a false formula.

In case of (c) and (d), i.e. when \((p < r)\) and \((q < r)\) have the same logical value (both are true or both are false), formula from 6) becomes obviously false.

Summing up, the supposition, that \((T1)\) is false leads to false conclusion, hence \((T1)\) must be true. QED.

Having this done, we have got the proof, that the real world is one.

7. Definition of God in the Sense of Leibniz

On the ground of Leibniz’s writings, we can collect a list of features that Leibniz attributed to God. According to Leibniz, God is: (a) real (existing really), (b) one, (c) necessary, (d) ontologically perfect, (e) compatible with everything and (f) making possible everything (including himself), and – last but not least – (g) is a subject of all positive qualities (“perfections”). Translated into the language of the presented ontology of situations, this list sounds as follows: God is the situation, which is (a) real (existing really), (b) one, (c) necessary, (d) ontologically perfect, (e) co-possible with every other situation (including himself). The feature (g) is not expressible in our ontology.

As it will turn out below, on the ground of the presented ontology of situations, one of these attributes – and namely: being ontologically perfect – is fundamental in relation to all others. From the statement that God is an ontologically perfect situation, it is possible to deduce all the remaining attributes. Hence being ontologically perfect may be considered as an essential (and defining) feature of God.

Leibniz considered every object (thus also situation) as ontologically perfect when its (this object’s) possibility was logically equivalent to its (this object’s) real existence. Hence we get the definition of God in the sense
of Leibniz: God is the situation whose possibility is logically equivalent to its real existence. Written in $LL$:

$$(D13) \, GLp = (\lozenge p \iff EXp)$$

What we will aim at now is to prove that God as defined in $(D13)$ really exists. It will be necessary to prove several theses first.

8. The Proof of Theistic Thesis

8.1 There is at Least One Ontologically Perfect Situation

The question occurs whether there is – according to our ontology – at least one ontologically perfect situation. The answer depends on whether we can prove the following theorem:

$$(T2) \, \exists p (\lozenge p \iff EXp)$$

**Proof:**

From definition $(D11)$ we see directly the truth of:

1) $\exists p (EXp \Rightarrow \lozenge p)$

   In turn, let us concentrate on the proof of:

2) $\exists p (\lozenge p \Rightarrow EXp)$

   2* in order to carry out an ad absurdum proof, let us suppose that

   3) $\neg \exists p (\lozenge p \Rightarrow EXp)$

   3* De Morgan’s Law and Implication Negation *4

   4) $\forall p (\lozenge p \wedge \neg EXp)$

   4* Definition $(D11)$ and after replacement in $(D11)$ – for the sake of brevity – “$BL(Mr((p \circ r), u), Ms((q \circ s), v))$” with “$A$” *5

   5) $\forall p (\lozenge p \wedge (\neg \lozenge p \vee \neg \forall q (\neg (p \circ q) \Rightarrow A)))$

   5* Distribution *6

   6) $\forall p (\lozenge p \wedge \neg \lozenge p) \vee (\lozenge p \wedge \neg \forall q (\neg (p \circ q) \Rightarrow A))$

   6* after removing $(\lozenge p \wedge \neg \lozenge p)$ *7

   7) $\forall p (\lozenge p \wedge \neg \forall q (\neg (p \circ q) \Rightarrow A))$

   7* De Morgan’s Law *8
8) \( \forall p(\Diamond p \land \exists q(\neg(p \circ q) \Rightarrow A)) \)
8* Implication Negation *9
9) \( \forall p(\Diamond p \land \exists q(\neg(p \circ q) \land \neg A)) \)
9* Definition (D6) *10
10) \( \forall p(\Diamond p \land \exists q(\neg(\neg p \circ q) \land \neg A)) \)
10* De Morgan’s Law *11
11) \( \forall p(\Diamond p \land \exists q(\neg(\neg p \circ q) \land \neg A)) \)
11* Theorem \( \Diamond (p \lor q) \Rightarrow (\Diamond p \lor \Diamond q) \) (proven by Suszko ([1], p. 237)) *12
12) \( \forall p(\Diamond p \land \exists q(\neg(\neg p \lor \neg q) \land \neg A)) \)
12* moving quantifier *13
13) \( \forall p(\exists q(\Diamond p \land \neg(\Diamond (p \land q) \land \neg A)) \)
13* De Morgan’s Law *14
14) \( \forall p(\exists q(\Diamond p \land \neg(\Diamond \neg p \lor \Diamond \neg q) \land \neg A)) \)
14* Definition (D4) *15
15) \( \forall p(\exists q(\neg p = 0 \land \neg(p = 0) \land \neg(\neg q) \land \neg A)) \)
16) For both \( p = 0 \) and \( p = 1 \) the formula 15) is false.

Then we have concluded that \( \neg\exists p(\Diamond p \Rightarrow EXp) \) is false, hence (on the ground of Contradiction Law) \( \exists p(\Diamond p \Rightarrow EXp) \) must be true.

Having formulas 1) and 2) proven, we conclude, on the ground of Biconditional Introduction, that \( \exists p(\Diamond p \Leftrightarrow EXp) \) is true. QED.

So the statement that there is at least one ontologically perfect situation belongs to theses of the presented ontology of situation.

### 8.2 Relations Between Real World and God

Let us prove that if a given situation is the real world, than it (this situation) is God:

\( (T3) \ \ \forall p(RWp \Rightarrow GLp) \)

**Proof:**

\( T3 \ast (D12) \ast 1 \)

1) \( \forall p((EXp \land \forall q(EXq \Rightarrow (p < q))) \Rightarrow (\Diamond p \Leftrightarrow EXp)) \)
1* from the tautology \( (p \Rightarrow q) \Leftrightarrow (\neg p \lor q) \) *2
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2) \( \forall p (\neg (EXp \land \forall q(Exq \Rightarrow (p < q))) \lor (\Diamond p \Leftrightarrow EXp)) \)

2* Biconditional Introduction *3

3) \( \forall p (\neg (EXp \land \forall q(Exq \Rightarrow (p < q))) \lor ((\Diamond p \Rightarrow Exp) \land (Exp \Rightarrow \Diamond p))) \)

3* On the ground of \((D11) EXp \Rightarrow \Diamond p\) is obviously true *4

4) \( \forall p (\neg (EXp \land \forall q(Exq \Rightarrow (p < q))) \lor (\Diamond p \Rightarrow EXp)) \)

4* De Morgan’s Law and \((p \Rightarrow q) \Leftrightarrow (\neg p \lor q)\) *5

5) \( \forall p (\neg EXp \lor \neg \forall q(Exq \Rightarrow (p < q))) \lor (\neg \Diamond p \lor EXp) \)

Now, since in 5) we have got a tautology, \( \forall p (RWp \Rightarrow GLp) \) is true. QED.

Let us now prove that if a given situation is God, it (this situation) is the real world. Relevant formula is:

\((T4) \ \forall p (GLp \Rightarrow RWp)\)

Proof:

\((T4) \ast (D12) \ast 1\)

1) \( \forall p ((\Diamond p \Rightarrow EXp) \Rightarrow RWp) \)

1* \((D11) \ast 2\)

2) \( \forall p ((\Diamond p \Rightarrow EXp) \Rightarrow (EXp \land \forall q(Exq \Rightarrow (p < q)))) \)

2* Biconditional Introduction *3

3) \( \forall p (((\Diamond p \Rightarrow EXp) \land (EXp \Rightarrow \Diamond p)) \Rightarrow (EXp \land \forall q(Exq \Rightarrow (p < q)))) \)

3* on the ground of \((D11) EXp \Rightarrow \Diamond p\) is obviously true *4

4) \( \forall p (\Diamond p \Rightarrow EXp) \Rightarrow (EXp \land \forall q(Exq \Rightarrow (p < q))) \)

4* from the tautology \((p \Rightarrow q) \Leftrightarrow (\neg p \lor q)\) *5

5) \( \forall p (\neg (\Diamond p \lor EXp) \lor (EXp \land \forall q(Exq \Rightarrow (p < q)))) \)

5* De Morgan’s Law *6

6) \( \forall p (\neg (\Diamond p \lor EXp) \lor \neg EXp \lor \neg \forall q(Exq \Rightarrow (p < q))) \)

Because what we have gained in 6) is tautology, \( \forall p (GLp \Rightarrow RWp) \) is true. QED.

From \((T3)\) and \((T4)\) on the ground of Biconditional Introduction follows, that

\((T5) \ \forall p (RWp \Leftrightarrow GLp)\)

is true. It means that being God in the sense of Leibniz is logically equivalent to being the real world. (Note, that in non-Fregean logic used by us, logical equivalence is not a synonym for being identical).
8.3 Necessity of God

We shall prove, that (Leibniz’s) God is a necessary situation:

\((T6)\) \(\forall p (GLp \Rightarrow \Box p)\)

**Proof:**

\((T6) \ast (T5)\) and Conjunctive Hypothetical Syllogism \(\ast 1\)

1) \(\forall p (RWp \Rightarrow \Box p)\)
   1* negation of \((T6)\) for ad absurdum prof; De Morgan’s Law \(\ast 2\)

2) \(\exists p (RWp \Rightarrow \Box p)\)
   2* Implication Negation \(\ast 3\)

3) \(\exists p (RWp \land \neg \Box p)\)
   3* \((D12)\) \(\ast 4\)

4) \(\forall p (((EXp \land \forall q(EXq \Rightarrow (p < q))) \land \neg \Box p)\)
   4* \((D11)\) and replacement – for brevity sake – “\(\forall q((p \circ q) \Rightarrow \Psi > (M_r(p \circ r), u), Ms((q \circ s), v)))\)” with “\(B\)” \(\ast 5\)

5) \(\forall p (\Diamond p \land B \land \forall q(EXq \Rightarrow (p < q))) \land \neg \Box p)\)

Since \((\Diamond p \land \neg \Box p)\) is a counter-tautology, formula 5) is false. Since presupposed negation of \((T6)\) has led us deductively to a false statement, then \((T6)\) must be true. QED.

8.4. Co-Possibility Between God and Himself

If God is co-possible (compatible) with himself, the following formula:

\((T8)\) \(\forall p (GLp \Rightarrow (p \circ p))\)

must be true.

**Proof:**

\((T7)\) negation for the *ad absurdum* proof \(\ast 1\)

1) \(\exists p (GLp \Rightarrow (p \circ p))\)
   1* De Morgan’s Law and Implication Negation \(\ast 2\)

2) \(\exists p (GLp \land \neg (p \circ p))\)
   2* \((D4)\) i \((D4)\) and Double Negative Elimination \(\ast 3\)
3) $\exists p (GLp \land (p \land p) = 0)$
   3* Quantifier Distribution *4
4) $\exists p (GLp \land \exists p ((p \land p) = 0)$
   4) $(T6)$ * 5
5) $\forall p (GLp \Rightarrow \Box p)$
   5* modal law, subalternation *6
7) $\exists p (GLp \Rightarrow (p = 1))$
   4, 6 * (R1) * 8
8) $\exists p ((p = 1) \land (p \land p) = 0)$

formula 8) is false, which brings the ad absurdum proof to the conclusion:
$(T6)$ is true. QED.

8.5 God is One

Thesis, that God is one has the following formal paraphrase:

$(T9)$ $\forall p \forall q ((GLp \Leftrightarrow GLq) \Rightarrow (p = q))$

Proof. From $(T6)$ (necessity of God) and $(T9)$ we get:
$\forall p \forall q (\Box p \Leftrightarrow \Box q) \Rightarrow (p = q))$
which on the ground of laws of modal sentences is equivalent to:
$\forall p \forall q (\Box (p \Leftrightarrow q) \Rightarrow (p = q))$.
This formula is the Boolean ontological axiom $(AO2)$. QED

8.6 Conditional Theistic Thesis

Now let us, prove the statement: if a situation is God then this situation really exists:

$(T10)$ $\forall p (GLp \Rightarrow EXp)$

Proof:

$(T10)$ * $(T4)$ and Conjunction Hypothetical Syllogism * 1
1) $\forall p (RWp \Rightarrow EXp)$
   1 * $(D12)$ * 2
2) $\forall p ((EXp \land \forall q (EXq \Rightarrow (p < q))) \Rightarrow EXp)$

Since formula 2) is tautology, $(T10)$ is true. QED.
8.7 Absolute Theistic Thesis

Finally, we prove the thesis: God exists:

\[(T11) \exists p(GLp \land EXp)\]

**Proof:**

From \((T4)\)

\[\forall p(RWp \Rightarrow GLp)\]

we get:

\[\exists p(RWp \Rightarrow GLp)\].

Taking into account the ontological axiom \((AO4)\):

\[\exists pRWp\]

we derive:

\[(T4')\exists pGLp.\]

From conjunction of \((T4')\) with

\[(T10)\forall p(GLp \Rightarrow EXp)\]

we have:

\[\exists p(GLp \land EXp)\]

QED.

References

