A NOTE ON GENERALIZED FUNCTIONAL
COMPLETENESS IN THE REALM OF ELEMENTARY
LOGIC

Abstract

In “Logicality and Invariance” (2008), Denis Bonnay introduced a generalized notion of functional completeness. In this note we call attention to an alternative characterization that is both natural and elementary.

1. Maximizing the expressive power of a logic

Let $L$ be a logic whose logical vocabulary contains truth-functional connectives (possibly infinitary ones), the first-order quantifiers and possibly some generalized quantifiers $Q_1, \ldots, Q_n$. Semantically, quantifiers are identified with classes of structures in a standard manner. Satisfaction in a structure is defined in conformity with the meaning of the logical constants chosen, e.g., $\mathcal{M} \models Qx\phi(x)$ iff $\langle \mathcal{M}, \{a : \mathcal{M} \models \phi(a)\} \rangle \in Q$. Given those constraints, a logic $L$ can be identified with a set of logical constants. Naturally associated with $L$, we have

1. an elementary equivalence relation between structures ($\equiv_L$), and
2. the class of elementary classes of $L$ ($El_L$).

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1See e.g. [9], p.235. For instance $\forall$ is the class of structures $\langle M, A\rangle$ such that $M = A$.

2Note that this definition of a logic is not purely semantic, and thus is less general that the one usually found in abstract model theory. See for instance [4], Definition 1.1.1. for the general definition.

3A class of structures is $L$-elementary in our terminology iff it is definable by a single sentence of $L$. 
The two concepts are related to the intuitive notion of the expressive power of a logic: on the one hand, the more expressive a logic, the more fine-grained the partition of the universe of structures induced by the associated elementary equivalence relation. (Think of two non-isomorphic first-order equivalent models of \( PA \). They are obviously not second-order equivalent.) On the other hand, the larger the collection of elementary classes associated with a logic, the more expressive the logic. (Adding second-order logical constants to \( \text{FOL} \) makes new classes of structures definable by a sentence; for instance the isomorphism class of \( \mathbb{N} \) is second-order elementary but not first-order elementary).\(^4\)

Given a logic \( L \), all elementary classes are of course closed under elementary equivalence; but what about the converse? In first-order logic, not all classes of structures closed under elementary equivalence are elementary.\(^5\) However, we have no reason to rule out the possibility that for some logic \( L \) all classes of structures closed under elementary equivalence for \( L \) are elementary in \( L \), and such a case seems to be of conceptual interest. Thinking intuitively of the notion of \( L \)-elementary equivalence between structures as a criteria of (relative) identity between structures from the point of view of that logic, it seems natural to think of logics such that all classes closed under elementary equivalence are elementary as meeting a requirement of "expressive completeness" or "internal completeness". We shall call such logics \( \Delta\Sigma \)-closed.

Given the fact that \( \Delta\Sigma \)-closed logics seem to form a natural object of attention, it may be worth remarking upon the fact that they also arise from other directions. The rest of this note will be devoted to the task of making precise the following remark that this simple condition of \( \Delta\Sigma \)-closure of a logic is equivalent to one of expressive completeness introduced recently by Denis Bonnay (in \([3]\), p.35).\(^6\) This remark may enforce the idea of generalized functional completeness" as both natural and elementary.\(^7\)

\(^4\)See for instance \([4]\), Def. 3.1.4 and 3.1.5 for a similar distinction between the two notions of expressive strength of a logic.
\(^5\)Think of the fact that, in first-order logic, when a theory has models of arbitrary finite cardinality it necessarily has a model of infinite cardinality. But a finite and an infinite structure cannot be elementary equivalent in \( \text{FOL} \).
\(^6\)From a historical point of view, the notion is rooted in the work of Tarski on logical constants \([8]\).
\(^7\)The phrase "elementary logic" in the title of this paper refers both to the informal sense of "simple" and to first-order logic.
We first recast our foregoing discussion with precise definitions. The following is standard (see [2], p. 141, for the first-order version):

**Definition 1.** Given a logic \( L \) and a fixed first-order signature:

- An elementary class of structures \( (\text{El}_L) \) is one axiomatizable by a sentence of \( L \).
- A \( \Delta \)-elementary class \( (\text{El}^\Delta_L) \) is the intersection of a set of elementary classes, i.e. is axiomatisable by a theory of \( L \).
- A \( \Sigma \)-elementary class \( (\text{El}^\Sigma_L) \) is the union of a set of elementary classes.
- A \( \Delta\Sigma \)-elementary class \( (\text{El}^{\Delta\Sigma}_L) \) is class which is the union of a collection of \( \Delta \)-elementary class.

Given the previous definition, the following fact is trivial:

**Proposition 1.** Let \( \mathcal{K} \) be a class of structures. \( \mathcal{K} \in \text{El}^{\Delta\Sigma}_L \) iff \( \mathcal{K} \) is closed under elementary equivalence.

**Proof.** The proof is similar to the first-order case (see [2], Lemma 1.11 p.143). □

Proposition 1 justifies our talking of \( \Delta\Sigma \)-closed logic in the opening paragraph. Remark that, like \( \Delta \)-interpolation \(^8\), \( \Delta\Sigma \)-closure also seems to record a kind of balance between the syntax and the semantics of the logic, albeit in a quite different sense than \( \Delta \)-interpolation is usually understood to do: here the balance is between the power to define (classes of structures) and the power to discriminate (between structures).

### 2. Generalized functional completeness

I now briefly recall the motivation and definition of generalized functional completeness given by Bonnay in [3]. The problem is best understood by analogy with Propositional logic. Propositional logic is the logic of truth-functions (of finitely many arguments). In view of this general characterization of propositional logic, two things could naturally be required of any

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\(^8\)A logic has the \( \Delta \)-interpolation property iff every class \( \mathcal{K} \) of structures such that both \( \mathcal{K} \) and its complement (in the given signature) are projectible class in \( L \) is also elementary in \( L \). See for instance [4], 7.2 for details.
particular propositional logic: first, every logical form in the propositional logic should determine a truth-function (expressive adequacy), and second every truth-function should be expressible in the propositional logic (expressive completeness). Usual propositional logic based on the set of logical constants \( \{\neg, \lor\} \) is expressively adequate and complete in this sense, while \( \{\land, \lor\} \) and \( \{\forall, \lor\} \) are not (not complete and not adequate respectively).

The study of functional completeness is of conceptual interest. First, it is \textit{prima facie} a fundamental property that a logic may lack or enjoy, and we would like to understand how a property which is so congenial in the propositional case is supposed to be applied in non-propositional cases. Second, and more importantly, it is fundamental to understand how the many superficially different logics we know of connect to each other and to identify, beyond their variety, some illuminating classification. As we contend that \( \{\lor\} \) and \( \{\neg, \land\} \) are not really different logics, one may well ask, for example, whether \( L_{\infty, \omega} \) is best understood as an intrinsically justified extension of \textit{FOL} or rather as an entirely different logic. In connection to this, understanding functional completeness may bear on the philosophical issue of determining which properties are logical properties, as opposed to, say, distinctively mathematical properties. For instance, starting from the assumption that properties definable in \textit{FOL} are purely logical properties, it would not seem reasonable to deem every non-first-order definable property as non-logical in the eventuality that \textit{FOL} were not functionally complete. Rather, it would seem that accepting \textit{FOL} as logic commits one to accepting every property definable in the functional completion of \textit{FOL} as being also a logical property.

Now, there is no widely accepted extension of the notion of functional completeness beyond the propositional case. This state of affair is explained by Bonnay thus, focusing on case of \textit{FOL}:

\begin{quote}
There is no such standard result for \textit{FOL}, essentially because there is no standard answer to what \textit{FOL} is about, which would be similar to the claim that \textit{PC} is about truth-functions. ([3], p. 35)
\end{quote}

It is one of Tarski’s merits to have set up a framework for studying precisely this question in rigorous terms. Tarski did it in the spirit of Klein’s \textit{Erlanger program}: because logic is the most general science, what a logic
is about are the properties invariant under the largest class of transformations.\footnote{Tarski was interested in the demarcation of the set of logical constants or, to put it otherwise, he wanted to determine which logic is the true logic, if any. G. Sher \cite{Sher}, S. Feferman \cite{Feferman} and D. Bonnay \cite{Bonnay} have followed his path, improving on the framework and conceptual motivation.}

In order to give a precise definition of the notion of expressive completeness as worked out by Bonnay, we first recall briefly his generalized framework for studying invariance properties \footnote{For reasons lying outside the scope of this paper, it has become clear that it is not satisfactory to restrict the notion of invariance to one of invariance under a class of transformation over a structure, as done by Tarski. See \cite{Bonnay} for details.}. A similarity relation $S$ between structures is a binary relation over structures of same signature, and a class $K$ of structures is said to be invariant under $S$ if it is closed under $S$.\footnote{That is $K$ is $S$-invariant iff, for every $M, M'$, if $M S M'$ then $M \in K$ iff $M' \in K$.} Moreover, given a relation of similarity $S$, we can define the operation $\text{Inv}(S)$ taking $S$ to be the collection of classes of structures invariant under $S$, that is the collection of quantifiers invariant under $S$. Associated with $\text{Inv}(S)$ is $\text{Inv}(\text{Inv}(S))$, the class of logical expressions whose elements denote the members of $\text{Inv}(S)$.

We are now in a position to state the generalization of the foregoing notions of adequacy and expressive completeness of a logic in complete analogy with the propositional case. A logic is to be understood as the logic of $S$-invariant notions, for a suitable $S$. Moreover two things are required of a logic $L$, understood as a set of logical constants, to ensure its adequacy and completeness relative to the target notion of logicality: every logical form in $L$ should express an $S$-invariant notion (adequacy) and every $S$-invariant notions should be definable in $L$ (completeness). Remark that on this view, for some choice of a similarity relation $S$, there may not exist a complete and adequate logic of $S$-invariant notions.\footnote{The case of $S = \equiv_{\text{FOL}}$ is a case in point. See below.} For an adequate $S$-logic to exist, $S$ must be closed under definability, which means the following: the classes of structures definable by a sentence of $L_{\text{Inv}(S)}$ should themselves be $S$-invariant (i.e. logical forms in $L_{\text{Inv}(\text{Inv}(S))}$ should express $S$-invariants).\footnote{See \cite{Bonnay}, p. 50 for the definition of closure under definability of a similarity relation. In contradistinction to $\equiv_{\text{FOL}}$, the relations of isomorphism and potential isomorphism between structures are closed under definability. See \cite{Bonnay}, p.51.} Finally, define a logic $L$ to be functionally complete...
complete if, and only if, it is adequate and complete relative to $S$-invariants, for some $S$. It is easily seen that this definition can be rephrased thus:

**Definition 2.** $L$ is functionally complete iff for some $S$, $E_L = Inv(S)$

3. A logic is $\Delta \Sigma$-closed iff functionally complete

The following is almost immediate given the definitions:

**Proposition 2.** The following are equivalent:

1. $L$ is $\Delta \Sigma$-closed
2. $L$ is functionally complete

**Proof.** It is obvious that $\Delta \Sigma$-closure implies functional completeness: If $L$ is $\Delta \Sigma$-closed then $E_L = Inv(\equiv_L)$, because it is the same to speak of the classes of structures invariant under elementary equivalence and to speak of the class of structures closed by the same elementary equivalence relation, so that $Inv(\equiv_L)$ and $E_L^{\Delta \Sigma}$ are just notational variants. Hence there is an $S$ such that $E_L = Inv(S)$.

To prove the converse implication, remark first that

**Proposition 3.** If $S$ is a similarity relation and $L^{Inv(S)}$ the associated logic, then $S$ is closed under definability iff $S = Inv^{\equiv L^{Inv(S)}}$.

**Proof.** Only if: Assume that $S$ is closed under definability, and that it is not the case that $M \equiv L^{Inv(S)} M'$. Then there is $\phi$ s.t. $M \in Mod(\phi)$ and $M' \notin Mod(\phi)$. Since $S$ is closed under definability, $Mod(\phi)$ is $S$-invariant. Hence it is not the case that $MSM'$. The converse inclusion is obvious since if it is not the case that $MSM'$, there is a quantifier $Q$ in $L^{Inv(S)}$ such that $M \in Q$ and $M' \notin Q$, and a formula $\phi$ in $L^{Inv(S)}$ s.t. $M \models \phi$ and $M \not\models \phi$.

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14$L$ is functionally complete iff adequate and complete relative to $S$-invariants of some $S$. Moreover $L$ is adequate to $S$-invariance iff $E_L \subseteq Inv(S)$. By definition $L$ is complete relative to $S$-invariance iff all $S$-invariants are definable in $L$, that is $E_L \supseteq Inv(S)$. 

For the if part: assume that $S$ is not closed under definability. There is a $\phi$ in $L_{Inv(S)}$ which distinguishes two similar structures. Hence they are not elementary equivalent in the sense of $L_{Inv(S)}$. Hence $S \not\equiv L_{Inv(S)}$.

Now remark that if $L$ is functionally complete, there is an $S$ such that $El_L = Inv(S)$ (by definition). But $S =\equiv L$, so that $El_L = Inv(\equiv L)$ (i.e. $L$ is $\Delta\Sigma$-closed).

So the notion of functional completeness, which grew out from fine-grained considerations pertaining to the denotation of logical constants and the nature of logic, has a straightforward counterpart in model-theoretical terms, namely in terms of the relation between elementary classes and the relation of elementary equivalence associated with a logic.

We conclude with some remarks and questions. We have already remarked that FOL is not $\Delta\Sigma$-closed. On the other hand it can be shown that it is sometimes possible to enrich the class of elementary classes of a logic while keeping its associated elementary equivalence relation fixed. For instance, it can be shown $^{15}$ that $El_{L,\infty,G} \supset El_{L,\omega}$ while on the other hand it follows from a result of Barwise that $L_{\infty,G}$ has the Karp property$^{16}$ and then, by a well-known theorem of Karp, that $\equiv_{L,\omega} = \equiv_{L,G}$.

In general, however, it is not possible to improve freely on the elementary classes of $L$ while keeping $\equiv_L$ fixed, for the process of adding more logical constants to $L$ modifies $\equiv_L$ as it modifies $El_L$. This is true even if the added quantifiers are invariant under the relation of elementary equivalence in the base logic: the quantifier There exists infinitely many ($Q_{\geq \aleph_0}$) is $FOL$-invariant, but adding it to $FOL$ modifies the elementary equivalence relation.$^{17}$ This fact suggests the following question: is there a

$^{15}$See [6], Remark 1.1.4 p. 370

$^{16}$See [6], p. 398-399, Th. 3.15 and 3.2.1. Incidentally, this fact implies that the game quantifier $G$ is logical in the sense of Bonnay. In this connection it is interesting to note that in $L_{\infty,G}$ one can write a statement asserting that $< \text{ is a well-ordering of type } \gamma + \gamma \text{ for some ordinal } \gamma$, a statement not expressible in $L_{\infty,\omega}$ (by a result of Malitz, 1966). Thus "game quantifiers give rise to infinitary logics which are different from the usual infinitary logics $L_{\lambda,\omega}$" ([6], p. 370).

$^{17}$Let $M = \langle M, R \rangle$ and $M' = \langle M, R' \rangle$ be such that $R$ and $R'$ are equivalence relations with $R$ having an infinite number of equivalence class of arbitrarily big finite cardinality, and $M'$ being just like $M$ except for the fact that it contains also an infinitary equivalence class. Let $\phi$ be the sentence "$\exists R$ is an equivalence relation and $\exists Q_{\geq \aleph_0}$, $x R y$". Then $M \not\models \phi$ and $M' \models \phi$. See [3], p. 51.
natural property of generalized quantifiers (Q) which would guarantee that elementary equivalence for the extended logic (L+Q) is the same as the elementary equivalence for the base logic (L)?

Moreover, it follows from the fact that $\equiv_{FOL}$ is not closed under definability that there is no $\Delta\Sigma$-extension $L$ of FOL such that $\equiv_L = \equiv_{FOL}$. Given the centrality of FOL to our ordinary logical theorizing, the question arises whether there are natural $\Delta\Sigma$-extension of FOL at all. It is shown in [3] that the least fine-grained similarity relation whose invariants give rise to a functionally complete logic is the relation of potential isomorphism between structures. Hence the logic of $Iso_p$ invariance would be a possible candidate. Is there any known logic which is functionally complete relative to $Iso_p$-invariants? It is proved in [1] that an operator $Q$ is $Iso_p$-invariant if and only if for any set $M$, $Q_M$ is definable in $L_{\infty,\omega}$. A result involving uniform definability of $Iso_p$ invariants and $L_{\infty,\omega}$ would have proved the $\Delta\Sigma$-closure of $L_{\infty,\omega}$. But the already mentioned fact that $El_{L_{\infty,\omega}} \supset El_{L_{\infty,\omega}}$, while $\equiv_{L_{\infty,\omega}} = \equiv_{L_{\infty,\omega}}$, shows that this is not possible. Remark also that since it is possible to define game theoretic logics stronger than $L_{\infty,\omega}$ and enjoying the Karp property, we know that $L_{\infty,G}$ is not itself $\Delta\Sigma$-closed. However, we do not know whether there are $\Delta\Sigma$-closed logics with the Karp property, nor do we know, more generally, whether there are any (natural?) $\Delta\Sigma$-closed extension of $FOL$.  

References


\[^{18}\textrm{I thank a referee for pointing out this question.}\]

\[^{19}\textrm{p. 51, Theorem 3.10}\]

\[^{20}\textrm{See again [6], p. 396-399 for an example.}\]

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