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NEW AXIOMATIZATIONS OF THE WEAKEST REGULAR MODAL LOGIC DEFINING JASKOWSKI’S LOGIC $D_2$

Abstract
In [3] the weakest regular modal logic $rS5^M$ defining $D_2$ was indicated. The logic $rS5^M$ was defined by means of a specific rule of inference ($RM_2$): $\Box\Box A/\Diamond A$. This rule was used in [5] to define $S5^M$ – the weakest normal modal logic defining $D_2$. In [1], [2] axiomatizations of $S5^M$ without this rule were given. In the present paper we axiomatize also $rS5^M$ without the rule ($RM_2$). The present paper is a continuation of [3].

Key words: the discussive logic $D_2$, regular modal logics for $D_2$.

1. Introduction

Let $For_m$ be the set of all formulae of modal propositional language, while $For_d$ be the set of all formulae of the discussive language.¹

Jaśkowski used notation ‘$D_2$’ referring to a logic, i.e. a set of formulas. This set is defined as follows:

$$D_2 := \{ A \in For_d : (\Diamond A^*) \in S5 \},$$

where $(\cdot)^*$ is a translation of discussive formulae into the modal language, i.e. $(\cdot)^*$ is a function from $For_d$ into $For_m$.

¹In the appendix of [3] we recall some chosen facts concerning modal logic. Also the function $(\cdot)^*$, the discussive language and other notions used in the present paper are defined there.
Definition 1.1. Let \( L \) be any modal logic.

(i) We say that \( L \) defines \( D_2 \) iff \( D_2 = \{ A \in \text{For}^d : \square \diamond A \in L \} \).

(ii) Let \( S5_\diamond \) be the set of all modal logics which have the same theses beginning with ‘\( \diamond \)' as \( S5 \), i.e., \( L \in S5_\diamond \) iff \( \forall A \in \text{For}_m (\square \diamond A \in L \iff \square \diamond A \in S5) \).

(iii) Let \( RS5_\diamond \) (resp. \( NS5_\diamond \)) be the set of all regular (resp. normal) logics from \( S5_\diamond \).

Fact 1.1 ([3]). For any congruent (classical) modal logic \( L \): \( L \) defines \( D_2 \) iff \( L \in S5_\diamond \).

In [5] the logic \( S5^M \) is defined as the smallest normal logic containing

\[
\diamond (p \rightarrow p)^2, \quad (P) \\
\diamond \square (\diamond \square p \rightarrow \square p), \quad (ML5) \\
\diamond \square (\square p \rightarrow p), \quad (MLT)
\]

and closed under the following rule:

if \( \square \diamond A \in S5^M \) then \( \square A \in S5^M \). \( (RM^2_1)\)

Fact 1.2 ([5]). \( S5^M \) is the smallest logic in \( NS5_\diamond \).

In [3] it was observed that one can drop two out of the three axioms of the original formulation of \( S5^M \) (cf. Fact 1.4ii). Besides, in [1], [2] it was proved that one can define the logic \( S5^M \) without the rule \( (RM^2_1) \).

In the present paper we prove that \( S5^M \) has other axiomatizations (for the proof of Fact 1.3iv see p. 49).

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As it is well known, in all regular logics (and so in normal ones) the formula \( (P) \) is equivalent to the following formula:

\[ \square p \rightarrow \diamond p \quad (D) \]

The smallest normal logic containing \( (D) \) (equivalently \( (P) \)) is denoted by ‘\( KD \)' or simply by ‘\( D \)'. Thus, of course \( D \subseteq S5^M \).
FACT 1.3 ([1], [2], [3]). SSM is the smallest normal logic which:

(i) contains (MLT) and “semi-4”

\[ \Box p \to \Diamond \Box \Box p \]  \hspace{1cm} (4e)

i.e. SSM = K4(MLT).\(^3\)

(ii) contains (4e) and the converse of (5)

\[ \Box p \to \Diamond \Box p \]  \hspace{1cm} (5e)

i.e. SSM = K4,5c;

(iii) contains (MLT) and is closed under (RM\(_2\)1);

(iv) contains (5c) and is closed under (RM\(_2\)1).

In [3] a regular version of the logic SSM was considered. It was proved that while defining the logic D\(_2\) one can use weaker modal logic than SSM.

DEFINITION 1.2. Let rSSM be the smallest regular logic which contains (MLT) and is closed under the rule (RM\(_2\)1).

FACT 1.4 ([3]).

(i) The logic rSSM is not normal. Thus, rSSM \(\subseteq\) SSM.

(ii) (P), (D), (ML5) \(\in\) rSSM.

(iii) rSSM is the smallest logic in RS5\(_5\).

(iv) rSSM is the smallest regular logic defining D\(_2\).

COROLLARY 1.1. For any modal logic L: if rSSM \(\subseteq\) L \(\subseteq\) S5, then L \(\in\) S5\(_5\).

Besides, we have the upward analogon of the result from Fact 1.4iv.

FACT 1.5 ([3]). If L is a regular logic defining D\(_2\), then L \(\subseteq\) S5.

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\(^3\)To simplify naming of normal (resp. regular) logics we use the Lemmon code KX\(_1\) \ldots X\(_n\) (resp. CX\(_1\) \ldots X\(_n\)) to denote the smallest normal (resp. regular) logic containing formulae (X\(_1\)), \ldots, (X\(_n\)) (cf. e.g. appendices in [3], [4]).
2. New facts about the logic $\text{rS}5^M$

Firstly, we show that the rule $(R^M_2)$ has not to be a primitive rule of $\text{rS}5^M$.

**Lemma 2.1.** Every regular logic containing $(4_s)$ is closed under $(R^M_2)$.

**Proof:** We have the following proof:

1. $\Diamond\Diamond A$ \hspace{1cm} assumption
2. $\Diamond\Diamond A \to (T \to \Diamond\Diamond A)$ \hspace{1cm} PL
3. $T \to \Diamond\Diamond A$ \hspace{1cm} 1, 2 and modus ponens
4. $\Box T \to \Box\Diamond\Diamond A$ \hspace{1cm} 3 and the rule of monotonicity
5. $\Box\Diamond\Diamond A \to \Diamond A$ \hspace{1cm} $(4_s)$ and laws of regular logics
6. $\Box T \to \Diamond A$ \hspace{1cm} PL, 4 and 5
7. $\Diamond(T \to A)$ \hspace{1cm} 6, regularity and PL
8. $(T \to A) \to \Diamond A$ \hspace{1cm} PL
9. $\Diamond(T \to A) \to \Diamond A$ \hspace{1cm} 8 and monotonicity
10. $\Diamond A$ \hspace{1cm} (MP), 7 and 9

Secondly, we show (cf. [2], p. 61) that

**Lemma 2.2.** $(\text{MLT})$ is a thesis of any regular logic containing the following formula

$$\Box\Diamond p \to \Diamond p$$

So $(\text{MLT})$ belongs to any regular logic containing $(5_5)$.

**Proof:** Consider the following inference:

1. $\neg \Diamond\Box(\Box p \to p) \to \Box\Diamond(\Box p \land \neg p)$ \hspace{1cm} PL and laws of regular logics
2. $\Box(\Box p \land \neg p) \to \Diamond\Box p \land \Diamond\neg p$ \hspace{1cm} PL and laws of regular logics
3. $\Box\Diamond(\Box p \land \neg p) \to \Box(\Box\Box p \land \Diamond\neg p)$ \hspace{1cm} 2 and monotonicity
4. $\Box(\Box\Box p \land \Diamond\neg p) \to \neg \Box(\Box\Box p \to \Box p)$ \hspace{1cm} PL and laws of regular logics
5. $\neg \Box\Box(\Box p \to p) \to \neg \Box(\Box\Box p \to \Box p)$ \hspace{1cm} 1, 3, 4 and PL
6. $\Diamond\Box p \to \Box p$ \hspace{1cm} 5 and PL
7. $\Box\Diamond\Box p \to \Diamond\Box p$ \hspace{1cm} $(5_5)$: $p/\Box p$
8. $\Box\Box(\Box p \to \Diamond\Box p) \to \Diamond(\Box\Box p \to \Box p)$ \hspace{1cm} laws of regular logics
9. $\Diamond\Box(\Box p \to p)$ \hspace{1cm} 7, 8, 6 and $2\times$ modus ponens

\footnote{For the case of normal logics the proof of Lemma 2.1 can be significantly simplified by the usage of Gödel’s rule (see [2], p. 62).}
Thirdly, in [3], p. 201 it was proved that:

**Lemma 2.3 ([3]).**

(i) \((5_c) \in rS5^M\).

(ii) \((4_s)\) belongs to any regular logic which contains \((5_c)\) and is closed under \((RM^2_1)\). So \((4_s) \in rS5^M\).

**Theorem 2.1.** \(rS5^M\) is the smallest regular logic which:

(i) contains \((4_s)\) and \((MLT)\), i.e. \(rS5^M = C4_s(MLT)\),

(ii) contains \((4_s)\) and \((5_c)\), i.e. \(rS5^M = C4_s5_c\),

(iii) contains \((5_c)\) and is closed under \((RM^2_1)\).

**Proof:** By Lemma 2.1, \(C4_s(MLT)\) is closed under \((RM^2_1)\). Hence we have \(rS5^M \subseteq C4_s(MLT)\). By Lemma 2.2, \(C4_s(MLT) \subseteq C4_s5_c\). By Lemma 2.3, \(C4_s5_c \subseteq rS5^M\). Thus, we have

\[ rS5^M = C4_s(MLT) = C4_s5_c. \]

By Lemma 2.3ii, we have that \(C4_s5_c\) is contained in the smallest regular logic which contains \((5_c)\) and is closed under \((RM^2_1)\). By Lemma 2.1, \(C4_s5_c\) is closed under \((RM^2_1)\). Therefore we have also the reverse inclusion. ⊣

**Proof of Fact 1.3iv:** By Fact 1.3ii, \(S5^M = K4_s5_c\). By Lemma 2.3ii, we have that \(K4_s5_c\) is contained in the smallest normal logic which contains \((5_c)\) and is closed under \((RM^2_1)\). By Lemma 2.1, \(K4_s5_c\) is closed under \((RM^2_1)\). Thus we have also the reverse inclusion. ⊣

References


