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NEW AXIOMATIZATIONS OF THE WEAKEST REGULAR MODAL LOGIC DEFINING JAŚKOWSKI'S LOGIC D_2

Abstract

In [3] the weakest regular modal logic $\mathbf{rS5^M}$ defining $\mathbf{D_2}$ was indicated. The logic $\mathbf{rS5^M}$ was defined by mens of a specific rule of inference $(\mathtt{RM}^2_1)$: $\Diamond\Diamond A/\Diamond A$. This rule was used in [5] to define $\mathbf{S5^M}$ – the weakest normal modal logic defining $\mathbf{D_2}$. In [1], [2] axiomatizations of $\mathbf{S5^M}$ without this rule were given. In the present paper we axiomatize also $\mathbf{rS5^M}$ without the rule $(\mathtt{RM}^2_1)$. The present paper is a continuation of [3].

Key words: the discussive logic $\mathbf{D_2}$, regular modal logics for $\mathbf{D_2}$.

1. Introduction

Let $\mathrm{For_m}$ be the set of all formulae of modal propositional language, while $\mathrm{For^d}$ be the set of all formulae of the discussive language.[1]

Jaśkowski used notation '$\mathbf{D_2}$' referring to a logic, i.e. a set of formulas. This set is defined as follows:

$$\mathbf{D_2} := \{\, A \in \mathrm{For^d} : \ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5} \,\},$$

where $(-)^\bullet$ is a translation of discussive formulae into the modal language, i.e. $(-)^\bullet$ is a function from $\mathrm{For^d}$ into $\mathrm{For_m}$.

[1]In the appendix of [3] we recall some chosen facts concerning modal logic. Also the function $(-)^\bullet$, the discussive language and other notions used in the present paper are defined there.

Definition 1.1. Let $\boldsymbol{L}$ be any modal logic.

(i) We say that $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^{\bullet} \urcorner \in \boldsymbol{L} \}$.
(ii) Let $\mathsf{S5}_{\Diamond}$ be the set of all modal logics which have the same theses beginning with '$\Diamond$' as $\mathbf{S5}$, i.e., $\boldsymbol{L} \in \mathsf{S5}_{\Diamond}$ iff $\forall_{A \in \mathrm{For_m}} (\ulcorner \Diamond A \urcorner \in \boldsymbol{L} \iff \ulcorner \Diamond A \urcorner \in \mathbf{S5})$.
(iii) Let $\mathsf{RS5}_{\Diamond}$ (resp. $\mathsf{NS5}_{\Diamond}$) be the set of all regular (resp. normal) logics from $\mathsf{S5}_{\Diamond}$.

Fact 1.1 ([3]). *For any congruent (classical) modal logic* $\boldsymbol{L}$: $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ *iff* $\boldsymbol{L} \in \mathsf{S5}_{\Diamond}$.

In [5] the logic $\mathbf{S5^M}$ is defined as the smallest normal logic containing

$$\Diamond(p \to p)^2 \tag{P}$$

$$\Diamond\Box(\Diamond\Box p \to \Box p) \tag{ML5}$$

$$\Diamond\Box(\Box p \to p) \tag{MLT}$$

and closed under the following rule:

$$\text{if } \ulcorner \Diamond\Diamond A \urcorner \in \mathbf{S5^M} \text{ then } \ulcorner \Diamond A \urcorner \in \mathbf{S5^M}. \tag{RM^2_1}$$

Fact 1.2 ([5]). $\mathbf{S5^M}$ *is the smallest logic in* $\mathsf{NS5}_{\Diamond}$.

In [3] it was observed that one can drop two out of the three axioms of the original formulation of $\mathbf{S5^M}$ (cf. Fact 1.4ii). Besides, in [1], [2] it was proved that one can define the logic $\mathbf{S5^M}$ without the rule (RM^2_1).

In the present paper we prove that $\mathbf{S5^M}$ has other axiomatizations (for the proof of Fact 1.3iv see p. 49).

[2] As it is well known, in all regular logics (and so in normal ones) the formula (P) is equivalent to the following formula:

$$\Box p \to \Diamond p \tag{D}$$

The smallest normal logic containing (D) (equivalently (P)) is denoted by '**KD**' or simply by '**D**'. Thus, of course $\mathbf{D} \subseteq \mathbf{S5^M}$.

FACT 1.3 ([1], [2], [3]). $\mathbf{S5^M}$ *is the smallest normal logic which*:

(i) *contains* (MLT) *and "semi-4"*

$$\Box p \to \Diamond\Box\Box p \qquad (4_s)$$

i.e. $\mathbf{S5^M} = \mathbf{K4_s(MLT)}$;[3]

(ii) *contains* (4_s) *and the converse of* (5)

$$\Box p \to \Diamond\Box p \qquad (5_c)$$

i.e. $\mathbf{S5^M} = \mathbf{K4_s5_c}$;

(iii) *contains* (MLT) *and is closed under* (RM^2_1);

(iv) *contains* (5_c) *and is closed under* (RM^2_1).

In [3] a regular version of the logic $\mathbf{S5^M}$ was considered. It was proved that while defining the logic $\mathbf{D_2}$ one can use weaker modal logic than $\mathbf{S5^M}$.

DEFINITION 1.2. Let $\mathbf{rS5^M}$ be the smallest regular logic which contains (MLT) and is closed under the rule (RM^2_1).

FACT 1.4 ([3]).

(i) *The logic* $\mathbf{rS5^M}$ *is not normal. Thus,* $\mathbf{rS5^M} \subsetneq \mathbf{S5^M}$.

(ii) (P), (D), (ML5) $\in \mathbf{rS5^M}$.

(iii) $\mathbf{rS5^M}$ *is the smallest logic in* $\mathbf{RS5_\Diamond}$.

(iv) $\mathbf{rS5^M}$ *is the smallest regular logic defining* $\mathbf{D_2}$.

COROLLARY 1.1. *For any modal logic* $\boldsymbol{L}$: *if* $\mathbf{rS5^M} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}$, *then* $\boldsymbol{L} \in \mathbf{S5_\Diamond}$.

Besides, we have the upward analogon of the result from Fact 1.4iv.

FACT 1.5 ([3]). *If* $\boldsymbol{L}$ *is a regular logic defining* $\mathbf{D_2}$, *then* $\boldsymbol{L} \subseteq \mathbf{S5}$.

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[3] To simplify naming of normal (resp. regular) logics we use the *Lemmon code* $\mathbf{KX}_1 \ldots \mathbf{X}_n$ (resp. $\mathbf{CX}_1 \ldots \mathbf{X}_n$) to denote the smallest normal (resp. regular) logic containing formulae ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$) (cf. e.g. appendices in [3], [4]).

2. New facts about the logic $\mathbf{rS5^M}$

Firstly, we show that the rule (RM^2_1) has not to be a primitive rule of $\mathbf{rS5^M}$.

LEMMA 2.1. *Every regular logic containing* (4_s) *is closed under* (RM^2_1).

PROOF: We have the following proof:[4]

1.	$\Diamond\Diamond A$	assumption
2.	$\Diamond\Diamond A \to (\top \to \Diamond\Diamond A)$	**PL**
3.	$\top \to \Diamond\Diamond A$	1, 2 and *modus ponens*
4.	$\Box\top \to \Box\Diamond\Diamond A$	3 and the rule of monotonicity
5.	$\Box\Diamond\Diamond A \to \Diamond A$	(4_s) and laws of regular logics
6.	$\Box\top \to \Diamond A$	**PL**, 4 and 5
7.	$\Diamond(\top \to A)$	6, regularity and **PL**
8.	$(\top \to A) \to A$	**PL**
9.	$\Diamond(\top \to A) \to \Diamond A$	8 and monotonicity
10.	$\Diamond A$	(MP), 7 and 9 ⊣

Secondly, we show (cf. [2], p. 61) that

LEMMA 2.2. (MLT) *is a thesis of any regular logic containing the following formula*

$$\Box\Diamond p \to \Diamond p \qquad (5^{\Diamond}_c)$$

So (MLT) *belongs to any regular logic containing* (5_c).

PROOF: Consider the following inference:

1.	$\neg\Diamond\Box(\Box p \to p) \to \Box\Diamond(\Box p \wedge \neg p)$	**PL** and laws of regular logics
2.	$\Diamond(\Box p \wedge \neg p) \to \Diamond\Box p \wedge \Diamond\neg p$	**PL** and laws of regular logics
3.	$\Box\Diamond(\Box p \wedge \neg p) \to \Box(\Diamond\Box p \wedge \Diamond\neg p)$	2 and monotonicity
4.	$\Box(\Diamond\Box p \wedge \Diamond\neg p) \to \neg\Diamond(\Diamond\Box p \to \Box p)$	**PL** and laws of regular logics
5.	$\neg\Diamond\Box(\Box p \to p) \to \neg\Diamond(\Diamond\Box p \to \Box p)$	1, 3, 4 and **PL**
6.	$\Diamond(\Diamond\Box p \to \Box p) \to \Diamond\Box(\Box p \to p)$	5 and **PL**
7.	$\Box\Diamond\Box p \to \Diamond\Box p$	$(5^{\Diamond}_c)$: $p/\Box p$
8.	$(\Box\Diamond\Box p \to \Diamond\Box p) \to \Diamond(\Diamond\Box p \to \Box p)$	laws of regular logics
9.	$\Diamond\Box(\Box p \to p)$	7, 8, 6 and 2× *modus ponens* ⊣

[4] For the case of normal logics the proof of Lemma 2.1 can be significantly simplified by the usage of Gödel's rule (see [2], p. 62).

Thirdly, in [3], p. 201 it was proved that:

LEMMA 2.3 ([3]).

(i) $(\mathtt{5_c}) \in \mathbf{rS5^M}$.
(ii) $(\mathtt{4_s})$ *belongs to any regular logic which contains* $(\mathtt{5_c})$ *and is closed under* $(\mathrm{R}\mathtt{M}^2_1)$. *So* $(\mathtt{4_s}) \in \mathbf{rS5^M}$.

THEOREM 2.1. $\mathbf{rS5^M}$ *is the smallest regular logic which*:

(i) *contains* $(\mathtt{4_s})$ *and* $(\mathtt{MLT})$, *i.e.* $\mathbf{rS5^M} = \mathbf{C4_s(MLT)}$,
(ii) *contains* $(\mathtt{4_s})$ *and* $(\mathtt{5_c})$, *i.e.* $\mathbf{rS5^M} = \mathbf{C4_s5_c}$,
(iii) *contains* $(\mathtt{5_c})$ *and is closed under* $(\mathrm{R}\mathtt{M}^2_1)$.

PROOF: By Lemma 2.1, $\mathbf{C4_s(MLT)}$ is closed under $(\mathrm{R}\mathtt{M}^2_1)$. Hence we have $\mathbf{rS5^M} \subseteq \mathbf{C4_s(MLT)}$. By Lemma 2.2, $\mathbf{C4_s(MLT)} \subseteq \mathbf{C4_s5_c}$. By Lemma 2.3, $\mathbf{C4_s5_c} \subseteq \mathbf{rS5^M}$. Thus, we have

$$\mathbf{rS5^M} = \mathbf{C4_s(MLT)} = \mathbf{C4_s5_c}\,.$$

By Lemma 2.3ii, we have that $\mathbf{C4_s5_c}$ is contained in the smallest regular logic which contains $(\mathtt{5_c})$ and is closed under $(\mathrm{R}\mathtt{M}^2_1)$. By Lemma 2.1, $\mathbf{C4_s5_c}$ is closed under $(\mathrm{R}\mathtt{M}^2_1)$. Therefore we have also the reverse inclusion. ⊣

PROOF OF FACT 1.3IV: By Fact 1.3ii, $\mathbf{S5^M} = \mathbf{K4_s5_c}$. By Lemma 2.3ii, we have that $\mathbf{K4_s5_c}$ is contained in the smallest normal logic which contains $(\mathtt{5_c})$ and is closed under $(\mathrm{R}\mathtt{M}^2_1)$. By Lemma 2.1, $\mathbf{K4_s5_c}$ is closed under $(\mathrm{R}\mathtt{M}^2_1)$. Thus we have also the reverse inclusion. ⊣

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