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QL-REGULAR QUANTIFIED MODAL LOGICS

Abstract

We show completeness for first-order counterparts of Lemmon’s modal propositional logics C1, D1 and E1 examined by Andrzej Pietruszczak in [4].

1. Preliminaries

The alphabet of quantified modal logic is a union of separate sets: individual variables \( \text{Var} = \{x_1, x_2, x_3, \ldots \} \), logical connectives: \( \{\neg, \land, \lor, \rightarrow, \leftrightarrow, \Diamond, \Box\} \), quantifiers \( \{\forall, \exists\} \), identity predicate \( \{\equiv\} \) and parentheses \( \{, \} \).

A first-order modal language \( \mathcal{L} \) is specified by indicating a set of extralogical constants which may consist of a (possibly empty) set of predicate letters and a (possibly empty) set of individual constants. Sets \( \text{Term}_\mathcal{L} \) of terms and \( \text{Form}_\mathcal{L} \) of formulas are defined in a standard way.

(Classical) quantified logic is a map

\[
\text{QL}: \mathcal{L} \mapsto \text{QL}_\mathcal{L},
\]

where \( \mathcal{L} \) is a language and \( \text{QL}_\mathcal{L} \) is the smallest subset of \( \text{Form}_\mathcal{L} \), such that:

- it contains a set of all substitutions of tautologies of (classical) propositional logic \( \text{PL}_\mathcal{L} \),
- it contains all formulas of the following form:
\[ x_i \equiv x_i \quad (ax1) \]
\[ x_i \equiv x_j \Rightarrow (\varphi \Rightarrow \varphi(x_i/x_j)) \quad (ax2) \]
\[ \forall x_i \varphi \Rightarrow \varphi(x_i/t) \quad (ax3) \]
\[ \varphi(x_i/t) \Rightarrow \exists x_i \varphi \quad (ax4) \]
\[ \forall x_i (\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \forall x_i \psi) \quad (ax5) \]
\[ \forall x_i (\varphi \Rightarrow \psi) \Rightarrow (\exists x_i \varphi \Rightarrow \psi) \quad (ax6) \]

where \( x_i \) and \( x_j \) are any variables, \( \varphi \) and \( \psi \) are any formulas; in \( ax2 \) formula \( \varphi(x_i/x_j) \) is obtained from \( \varphi \) through substitution of some (not necessary every) free occurrences of variable \( x_i \) with variable \( x_j \), whereas formula \( x_j \) is free for every occurrence of variable \( x_i \), in which the substitution was made; \( \varphi(x_i/t) \) stands for a result of simultaneous substitution of term \( t \) for every occurrence of free variables \( x_i \) in \( \varphi \); in \( ax5 \) \( x_i \) is not free in \( \varphi \); in \( ax6 \) \( x_i \) is not free in \( \psi \).

- it is closed under rules of modus ponens and generalization

\[
\begin{align*}
&\text{if } \varphi, \varphi \Rightarrow \psi \in \text{QL}_L, \text{ then } \psi \in \text{QL}_L & (\text{mp}) \\
&\text{if } \varphi \in \text{QL}_L, \text{ then } \forall x_i \varphi \in \text{QL}_L & (\text{rg})
\end{align*}
\]

Obviously, even though the modal operators are present in \( L \), \( \text{QL}_L \) does not contain any formula of one of the following forms: \( \Box \varphi \}, \Box \forall x_i \Box \varphi \), \( \exists x_i \Box \varphi \), \( \Box \exists x_i \varphi \}, \Box \varphi \equiv \neg \Box \neg \varphi \}, \Box \varphi \equiv \neg \varphi \equiv \varphi \} \) or \( \Box \forall x_i \varphi \Rightarrow \forall x_i \Box \varphi \} \).

### 2. Quantified modal logics

All logics to be introduced will contain Barcan Formula, hence for any modal propositional logic \( S \), they should be labelled as \( QSBF \). However, both for neatness and convenience, we will omit the ‘BF’ in the names.

**Definition 2.1.** The smallest \( \text{QL} \)-regular modal logic (with Barcan Formula) is a map

\[ \text{QC1}: \mathcal{L} \mapsto \text{QC1}_L, \]

where \( \mathcal{L} \) is any language and \( \text{QC1}_L \) is the smallest subset of \( \text{Form}_L \), such that
• $\mathcal{QL}_L \subseteq \mathcal{QC}_1_L$.
• it contains all formulas of the following form:

$$\forall x, \Box \varphi \Rightarrow \Box \forall x, \varphi$$  \hspace{1cm} \text{(BF)}

$$\Diamond \varphi \Leftrightarrow \neg \Box \neg \varphi,$$  \hspace{1cm} \text{(def $\Diamond$)}

• it is closed under (mp), (rg) and $\mathcal{QL}$-restricted rule of regularity:

if $\Gamma (\varphi \land \psi) \Rightarrow \chi \in \mathcal{QL}_L$, then $\Gamma (\Box \varphi \land \Box \psi) \Rightarrow \Box \chi \in \mathcal{QC}_1_L$, \text{(rr$_{\mathcal{QL}}$)}

We will consider now two extensions of $\mathcal{QC}_1$.

**Definition 2.2.** Let $\mathcal{L}$ be any language. A first-order counterpart of $D_1$ is a map

$$QD_1 : \mathcal{L} \mapsto QD_1\mathcal{L},$$

where $QD_1\mathcal{L}$ is the smallest subset of $\text{Form}_\mathcal{L}$, such that

• $\mathcal{QC}_1\mathcal{L} \subseteq QD_1\mathcal{L}$,
• it contains all formulas of the following form:

$$\Box \varphi \Rightarrow \Diamond \varphi,$$  \hspace{1cm} \text{(D)}

• it is closed under (mp), (rg) and (rr$_{\mathcal{QL}}$).

**Definition 2.3.** Let $\mathcal{L}$ be any language. A first-order counterpart of $E_1$ is a map

$$QE_1 : \mathcal{L} \mapsto QE_1\mathcal{L},$$

where $QE_1\mathcal{L}$ is the smallest subset of $\text{Form}_\mathcal{L}$, such that

• $\mathcal{QC}_1\mathcal{L} \subseteq QE_1\mathcal{L}$,
• it contains all formulas of the following form:

$$\Box \varphi \Rightarrow \varphi,$$  \hspace{1cm} \text{(T)}

• it is closed under (mp), (rg) and (rr$_{\mathcal{QL}}$).

Any of the above $\mathcal{QL}$-regular logics will be denoted by $\mathcal{L}$. They are proper extensions of $\mathcal{QL}$ in the sense that for any $\mathcal{L}$, $\mathcal{QL} \not\subseteq \mathcal{L}$. Obviously, $\mathcal{QC}_1 \subseteq QD_1 \subseteq QE_1$. 
Lemma 2.4. Let \( L \) be a language and let \( \varphi \in \text{Form}_L \). Then \( \varphi \in L \) iff there is a sequence of formulas \( \chi_1, \ldots, \chi_n \) from \( \text{Form}_L \) such that \( \varphi = \chi_n \) and for every \( 1 \leq i \leq n \) at least one of the following conditions is satisfied:

1. \( \chi_i \in \text{QL}_L \),
2. \( \chi_i \) is of the form (def\( \Diamond \)),
3. \( \chi_i \) is of the form (BF),
4. \( \chi_i \) is of the form (D) (only in case of QD1),
5. \( \chi_i \) is of the form (T) (only in case of QE1),
6. there are \( j, k < i \) such that \( \chi_j = \left[ \chi_k \Rightarrow \chi_i \right] \),
7. there is an \( j < i \) such that \( \chi_j = \left[ \forall x_k \chi_j \right] \), for some variable \( x_k \),
8. there are \( j < i \) and \( \varphi, \psi, \chi \in \text{Form}_L \) such that \( \chi_j = \left[ (\varphi \land \psi) \Rightarrow \chi \right] \in \text{QL}_L \) and \( \chi_i = \left[ (\Box \varphi \land \Box \psi) \Rightarrow \Box \chi \right] \).

Fact 2.5. Let \( L \) be a language. For any QL-regular logic \( L \) and every \( \varphi, \psi, \chi, \varphi_1, \ldots, \varphi_n \in \text{Form}_L \) \((n \geq 1)\), \( L \) contains all formulas of the following form:

\[
\Box \varphi \iff \neg \Diamond \neg \varphi \quad \text{(def\( \Box \))}
\]
\[
(\Box \varphi \land \Box \psi) \Rightarrow \Box (\varphi \land \psi) \quad \text{(C)}
\]
\[
\Box (\varphi \land \psi) \Rightarrow (\Box \varphi \land \Box \psi) \quad \text{(M)}
\]
\[
(\Box \varphi \land \Box \psi) \iff \Box (\varphi \land \psi) \quad \text{(R)}
\]
\[
\Box (\varphi \Rightarrow \psi) \Rightarrow (\Box \varphi \Rightarrow \Box \psi) \quad \text{(K)}
\]

Moreover, \( L \) is closed under classical rules for QL as well as under the following QL-restricted modal rules of extensionality, monotonicity and generalised regularity:

- If \( \varphi \iff \psi \in \text{QL}_L \), then \( \Box \varphi \iff \Box \psi \in L \), \( \text{(req}_{QL} \) \)
- If \( \varphi \Rightarrow \psi \in \text{QL}_L \), then \( \Box \varphi \Rightarrow \Box \psi \in L \), \( \text{(rm}_{QL} \) \)
- If \( \varphi_1 \land \cdots \land \varphi_n \Rightarrow \psi \in \text{QL}_L \), then \( (\Box \varphi_1 \land \cdots \land \Box \varphi_n) \Rightarrow \Box \psi \in L \), \( \text{(grr}_{QL} \) \)

None of the QL-regular logics contains formulas beginning with ‘\( \Box \)’. Indeed, assume that every formula beginning with ‘\( \Box \)’ is false but those beginning with ‘\( \Diamond \)’ are true, keeping standard interpretation of other elements of the language. Then all elements of \( L \) are true.
In particular ‘\(\Box \forall x \equiv x\)’ is not a thesis of any QL-regular logic \(L\). Hence, since ‘\(\forall x \equiv x\)’ belongs to \(\mathcal{QL}_\mathcal{L}\), the set \(L^\mathcal{L}\) is not even closed under the QL-restricted Gödel rule:

\[
\text{if } \varphi \in \mathcal{QL}_\mathcal{L}, \text{ then } \neg \Box \varphi \in L^\mathcal{L},
\]

thus is not QL-normal.

Therefore all of the \(L\) logics are epistemic in the sense of Lemmon (cf. [3], p. 183 n.). Operators ‘\(\Box\)’ and ‘\(\Diamond\)’ can be read as: ‘it is a priori, but not logically, necessary that’ and ‘it is a priori, but not logically, possible that’, respectively. Or they can be applied to Ontological Argument with ‘\(\Diamond\)’ representing Anselmian conceivable (cf. [2]).

3. First-order modal theories

Let \(L\) be QL or any of the QL-regular logics.

**Definition 3.1.** Let \(\mathcal{L}\) be a language and let \(\{\varphi\}, \Psi \subseteq \text{Form}_\mathcal{L}\). We say that \(\varphi\) is a *consequence of* \(\Psi\) *in* \(L\) (in symbols: \(\Psi \vdash_L \varphi\)) iff there is a subset \(\{\psi_1, \ldots, \psi_n\}\) of \(\Psi (n \geq 0)\) such that \(\neg (\psi_1 \land \cdots \land \psi_n) \Rightarrow \varphi \in L^\mathcal{L}\). ■

Let \(A \subseteq \text{Form}_\mathcal{L}\) and let \(\bar{A}\) be a closure of all formulas from \(A\). A *theory based on logic* \(L\) (in short: \(L\)-theory) built in \(\mathcal{L}\) is a set

\[
T_A = \{\varphi \in \text{Form}_\mathcal{L} : \bar{A} \vdash_L \varphi\}.
\]

**Fact 3.2.** For every \(L\)-theory \(T_A\):

- \(L^\mathcal{L} \subseteq T_A\),
- \(A \subseteq T_A\),
- \(T_A\) is closed under (mp) and (rg). ■

We will often write ‘\(T\)’ instead of ‘\(T_A\)’.

We say that an \(L\)-theory \(T\) is *consistent* if there is no \(\varphi \in \text{Form}_\mathcal{L}\) such that \(\Box \varphi \land \neg \varphi \in T\).

\(T\) is *complete* if for any sentence \(\varphi \in \text{Form}_\mathcal{L}\): either \(\varphi \in T\) or \(\neg \varphi \in T\).

\(T\) is a *Henkin theory* if it is consistent, complete and when for every \(\varphi \in \text{Form}_\mathcal{L}\) such that \(\varphi = \{x_i\}\) there is an individual constant \(c \in \text{Term}_\mathcal{L}\) such that \(\exists x_i \varphi \Rightarrow \varphi(x_i/c) \in T\).
Let $T_1$ and $T_2$ be $L$-theories built in languages $L_1$ and $L_2$, respectively. We say that $T_2$ is an extension of $T_1$ if $L_1 \subseteq L_2$ and $T_1 \subseteq T_2$.

The following lemma can be obtained in an old-school way.

**Lemma 3.3.** [Henkin] Any consistent $L$-theory $T$ can be extended to a Henkin theory $T^h$. ■

**Lemma 3.4.** Let $T$ be a complete $L$-theory built in $L$ and let \( \{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \neq \emptyset \). Then for every $\psi \in \text{Form}_L$:

(a) $\Box \Box \psi \in T$ iff $\psi \in T'$ for every complete QL-theory $T'$ such that $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \subseteq T'$,

(b) $\Diamond \psi \in T$ iff $\psi \in T'$ for some complete QL-theory $T'$ such that $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \subseteq T'$.

**Proof.** (a) Assume $\Box \Box \psi \in T$. Let $T'$ be any complete QL-theory and let $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \subseteq T'$. Then $\psi \in T'$.

Now assume that for any complete QL-theory $T'$ we have that if $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \subseteq T'$, then $\psi \in T'$. Thus formula $\psi$ belongs to every complete QL-theory which is built in $L$ and which is an extension of $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \}$, so $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \} \vdash_{QL} \psi$. Hence there is a finite sequence of formulas $\chi_1, \ldots, \chi_n \in \text{Form}_L$ from $\{ \varphi \in \text{Form}_L : \Box \varphi \in T \}$ such that $\vdash (\chi_1 \land \cdots \land \chi_n) \Rightarrow \psi \in QL_L$ and that $\vdash \Box \chi_1, \ldots, \Box \chi_n \in T$.

Thanks to (grQl) we get $\vdash (\Box \chi_1 \land \cdots \land \Box \chi_n) \Rightarrow \Box \psi \in L$, so $\vdash (\Box \chi_1 \land \cdots \land \Box \chi_n) \Rightarrow \Box \psi \in T$. Since $\vdash \Box \chi_1, \ldots, \Box \chi_n \in T$, then also $\Box \chi_1 \land \cdots \land \Box \chi_n \in T$, so given (mp) we get $\Box \psi \in T$. ■

4. Semantics

Our semantics is a natural mix of semantics for (classical) first-order logic and of semantics for $C_1$, $D_1$ and $E_1$ presented by A. Pietruszczak in [4]. (BF) corresponds to the fact of building models with constant domains.

**Definition 4.1.** A frame for QC1 (in short: QC1-frame) is any quadruple $\mathcal{S} = (W, w_r, L, D)$, where:

(a) $W$ is a non-empty set (of “worlds”),

(b) $w_r$ is a distinct element of $W$ (an “actual” or a “real” world),

(c) $L$ is a (possibly empty) subset of $W$ (alternatives to $w_r$),
(d) $D$ is a non-empty such that $W \cap D = \emptyset$. ■

**Definition 4.2.** A frame for $Q\mathcal{D}1$ is a quadruple $\mathfrak{F} = \langle W, w, L, D \rangle$ as defined in Definition 4.1 with a condition that $L \neq \emptyset$.

**Definition 4.3.** A frame for $Q\mathcal{E}1$ is a quadruple $\mathfrak{F} = \langle W, w, L, D \rangle$ as defined in Definition 4.1 with a condition that $L = W$.

Frames will be denoted by $\mathfrak{F}$.

An assignment of variables in a frame $\mathfrak{F}$ is any function $v : \text{Var} \rightarrow D$.

An assignment $v'$ is an $x_i$-variant of $v$, when $v'$ differs from $v$ at most on variable $x_i$.

An interpretation of extralogical symbols of $\mathcal{L}$ in $\mathfrak{F}$ is a function $I$ such that:
- for every $n$-ary predicate letter $P$, $I(P)$ is a subset of $D^n \times W$,
- for every individual constant $c$, $I(c)$ is an element of $D$.

We intend to treat all terms as rigid designators, hence a denotation of a term $t \in \text{Term}_L$ in a frame $\mathfrak{F}$ with respect to an assignment $v$ (in symbols: $I_v$) is defined as:

$$I_v(t) = \begin{cases} v(t) & \text{if } t \text{ is a variable} \\ I(t) & \text{if } t \text{ is an individual constant.} \end{cases}$$

**Definition 4.4.** Let $\mathcal{L}$ be a first-order modal language and let $\mathcal{L}$ be any of the QL-regular logics. An $\mathcal{L}$-model is any triple $\langle \mathfrak{F}, I, V \rangle$, where $\mathfrak{F}$ is an $\mathcal{L}$-frame, $I$ is an interpretation function and $V$ is a function from $\text{Form}_L \times D^{\text{Var}} \times W$ to $\{0, 1\}$, called valuation of formulas of $\mathcal{L}$ (in symbols: $V^w_v(\varphi)$) and satisfying the following conditions.

For any $n$-ary predicate $P$, for every formulas $\varphi, \psi \in \text{Form}_L$, assignment $v \in D^{\text{Var}}$ and world $w \in W$:

- $\langle VP \rangle$ $V^w_v(Pt_1, \ldots, t_n) = 1$ iff $\langle I_v(t_1), \ldots, I_v(t_n) \rangle \in I(P)$,
- $\langle V= \rangle$ $V^w_v(t_1 \equiv t_2) = 1$ iff $I_v(t_1) = I_v(t_2)$,
- $\langle V\neg \rangle$ $V^w_v(\neg \varphi) = 1$ iff $V^w_v(\varphi) = 0$,
- $\langle V\land \rangle$ $V^w_v(\varphi \land \psi) = 1$ iff $V^w_v(\varphi) = 1$ and $V^w_v(\psi) = 1$,
- $\langle V\lor \rangle$ $V^w_v(\varphi \lor \psi) = 1$ iff $V^w_v(\varphi) = 1$ or $V^w_v(\psi) = 1$,
- $\langle V\Rightarrow \rangle$ $V^w_v(\varphi \Rightarrow \psi) = 1$ iff $V^w_v(\varphi) = 0$ or $V^w_v(\psi) = 1$,
- $\langle V\Leftarrow \rangle$ $V^w_v(\varphi \Leftarrow \psi) = 1$ iff $V^w_v(\varphi \Rightarrow \psi) = 1$ and $V^w_v(\psi \Rightarrow \varphi) = 1$,
- $\langle V\forall \rangle$ $V^w_v(\forall x_i \varphi) = 1$ iff for every $x_i$-variant $v'$ of $v$: $V^w_{v'}(\varphi) = 1$,
(V∃) \( V_v^w(\exists x_i \varphi) = 1 \) iff for some \( x_i \)-variant \( v' \) of \( v \): \( V_{v'}^w(\varphi) = 1 \).

Moreover, for the real world \( w_r \) one of the following conditions is satisfied:

(a) \( \) either for every \( \varphi \in \text{Form}_L \) and \( v \in \mathcal{D}_\text{Var} \) holds
\[
(V_{w_r}^w \Box) V_{w_r}^w(\Box \varphi) = 1 \text{ iff for every } w \in L, V_v^w(\varphi) = 1,
\]
\[
(V_{w_r}^w \Diamond) V_{w_r}^w(\Diamond \varphi) = 1 \text{ iff for some } w \in L, V_v^w(\varphi) = 1,
\]

(b) \( \) either for every \( \varphi \in \text{Form}_L \) and \( v \in \mathcal{D}_\text{Var} \) we have:
\[
V_{w_r}^v(\Box \varphi) = 0 \text{ and } V_{w_r}^v(\Diamond \varphi) = 1.
\]

In case of (a) the world \( w_r \) is called normal, and in case of (b) non-normal.

Formula \( \varphi \in \text{Form}_L \) is satisfiable in an \( L \)-model \( M \) iff for some assignment \( v \), \( V_{w_r}^v(\varphi) = 1 \). Formula \( \varphi \) is valid in an \( L \)-model \( M \) (in symbols: \( M |\models \varphi \)) iff for every \( v \), \( V_v^w(\varphi) = 1 \). Finally, a formula \( \varphi \) is a tautology of \( \text{logic } L \) (in symbols: \( |\models \varphi \)) iff it is valid in every \( L \)-model.

In fact, in Definition 4.1 the set \( W \) can be limited to \( L \cup \{ w_r \} \). This observation will be used for construction of canonical models later on.

**FACT 4.5.** Let \( M = (W, w_r, L, D, I, V) \) and \( M' = (W', w_r, L, D', I', V') \) be \( L \)-models, where \( W' := L \cup \{ w_r \} \) and with \( I \) and \( V \) being extensions of, respectively, \( I' \) and \( V' \). Then for every \( \varphi \in \text{Form}_L \): \( M |\models \varphi \) wtw \( M' |\models \varphi \).

**DEFINITION 4.6.** If \( T_A \) is an \( L \)-theory, then an \( L \)-model \( M \) is a model for \( T_A \) (in symbols: \( M |\models T_A \)) if for every \( \varphi \in A \), \( M |\models \varphi \).

Let us notice that in case of theses of \( \text{QL} \), worlds in \( L \)-models do not play any part and do not differ at all. More precisely:

**FACT 4.7.** If \( \varphi \in \mathcal{Q}_L \), then for every world \( w \) of every \( L \)-model we have that for any assignment \( v \), \( V_v^w(\varphi) = 1 \).

**Lemma 4.8.** Let \( L \). Then \( |\models \varphi \).

**Proof.** Cf. Lemma 2.4. We will verify only the case for \( \text{(rrQL)} \). Let \( V_v^w((\varphi \land \psi) \Rightarrow \chi) = 1 \) and \( V_v^w(\Box \varphi \land \Box \psi) = 1 \). Then \( V_v^w(\Box \varphi) = 1 \) and \( V_v^w(\Box \psi) = 1 \), hence for every \( w \in L \), \( V_v^w(\varphi) = 1 \) and \( V_v^w(\psi) = 1 \), so for every \( w \in L \), \( V_v^w(\varphi \land \psi) = 1 \). Now, due to \( \tau(\varphi \land \psi) \Rightarrow \chi \in \mathcal{Q}_L \) and Fact 4.7, we have that for every \( w \in L \), \( V_v^w(\chi) = 1 \), so \( V_v^w(\Box \chi) = 1 \).
**Theorem 4.9.** Let $L$ be any of the logics under consideration. Let $\mathcal{T}_A$ be an $L$-theory. Let $T_A$ be an $L$-theory. Let $\varphi \in \mathcal{T}_A$ and $L$-model $M$ is a model for $T_A$. If $\varphi \in T_A$ and $L$-model $M$ is a model for $T_A$, then $M \models \varphi$.

**Proof.** Since $\varphi \in T_A$, then $\bar{A} \vdash_L \varphi$. Therefore for some $\psi_1, \ldots, \psi_n \in A$ we have that $\bar{\psi}_1 \land \cdots \land \bar{\psi}_n \Rightarrow \varphi \in L$. By Lemma 4.8, $\models_L \bar{\psi}_1 \land \cdots \land \bar{\psi}_n \Rightarrow \varphi$. Now because $M$ is a model for $T_A$, for every $\psi \in A$, $M \models \psi$. In particular $M \models \bar{\psi}_1, \ldots, M \models \bar{\psi}_n$, so also $M \models \psi_1, \ldots, M \models \psi_n$ and $M \models \bar{\psi}_1 \land \cdots \land \bar{\psi}_n$. Thus $M \models \varphi$. ■

5. Completeness

Let $\mathbf{Term}_L$ be a set of all terms of a language $L$ of a theory $T$. Let $t_i \sim t_j$ if $\models_L t_i \equiv t_j \in T$. Obviously, $\sim$ is an equivalence relation. Moreover, if $t_1 \sim t_1', t_n \sim t_n'$, then for every $n$-ary predicate letter $P$ of $L$

$$\models_L Pt_1 \cdots t_n \sim Pt_1' \cdots t_n' \in T.$$ 

Let $|t| = \{ t' \in \mathbf{Term}_L : t \sim t' \}$ and let $|\mathbf{Term}_L| = \{ |t| : t \in \mathbf{Term}_L \}$. Clearly, $|\mathbf{Term}_L| \neq \emptyset$ and

$$|t_1| = |t_2| \text{ if } t_1 \equiv t_2 \in T.$$ 

**Definition 5.1.** Let $L$ be any of the QL-regular logics. Let $T$ be an $L$-theory built in $L$. A canonical model for $T$, $\mathcal{R} = (W_T, w_T, L_T, D_T, I_T, V_T)$, is defined as follows:

(a) $w_T = T^h$,
(b) $L_T = \{ T' : T' \text{ is a complete } QL-\text{theory and } \{ \varphi \in \mathbf{Form}_L : \bar{\square} \varphi \in w_T \} \subseteq T' \}$,
(c) $W_T = \{ w_T \} \cup L_T$,
(d) $D_T = |\mathbf{Term}_L|$, where $L_C$ is a language of the Henkin theory $T^h$,
(e) canonical assignment is a function $k$ such that for any $x_i \in \mathbf{Var}$, $k = |x_i|$, 
(f) for any $c, t_1, \ldots, t_n \in \mathbf{Term}_L$,

$$I_T(c) = |c|$$

$$\langle |t_1|, \ldots, |t_n|, w \rangle \in I_T(P) \text{ if } \models_L Pt_1 \cdots t_n \in w,$$
(g) finally $V_T$ is a characteristic function of elements of $W_T$, i.e. for any $\varphi, v, w$:

$$V_T^w(\varphi) = \begin{cases} 1 & \text{if } \varphi \in w \\ 0 & \text{in opposite case.} \end{cases}$$

\[\blacksquare\]

**Lemma 5.2.** A canonical model for an $L$-theory $T$ is an $L$-model.

**Proof.** Let $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in w_T\} = \emptyset$. We will show that $w_T$ is non-normal. We know that for every $\varphi \in \text{Form}_{\mathcal{L}_C}$, $\square \varphi \notin w_T$, so given $V_T$ we have that $V_T^w(\square \varphi) = 0$ for any assignment $v$. Obviously, all formulas of the form (def $\Diamond$) belong to $w_T$, hence $V_T^w(\Diamond \varphi \leftrightarrow \neg \square \neg \varphi) = 1$. Therefore, with respect to $V_T^w(\neg \square \varphi) = 1$ and conditions (V$^\neg$), (V$\Rightarrow$), satisfied by $V_T$: $V_T^w(\Diamond \varphi) = 1$.

Now, let $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in w_T\} \neq \emptyset$. We will show that the world $w_T$ is normal, i.e. $V_T$ satisfies $(V^w, \square)$ and $(V^w, \Diamond)$ from Definition 4.4.

Let $\varphi = \square \psi$. Then $V_T^w(\square \psi) = 1$ iff $\varphi \in w_T$ (by the definition of $V_T$) iff for every $w \in L_T$, $\psi \in w$ (by the definition of $L_T$ and Lemma 3.4) iff for every $w \in L_T$, $V_T^w(\psi) = 1$ (by the definition of $V_T$).

Let $\varphi = \Diamond \psi$. Then $V_T^w(\Diamond \psi, w_T) = 1$ iff $\Diamond \psi \in w_T$ (by the definition of $V_T$) iff for some $w \in L_T$, $\psi \in w$ (by the definition $L_T$ and Lemma 3.4) iff for some $w \in L_T$, $V_T^w(\psi) = 1$ (by definition of $V_T$).

However, in case of QD1-models and QE1-models we have also to show that $L_T \neq \emptyset$ and that $L_T = W_T$, respectively.

For QD1-models let us assume that $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in w_T\} = \emptyset$. Then $L_T = \{T' : T'$ is complete QL-theory$\}$, hence it is not empty.

Now let $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in w_T\} \neq \emptyset$. Then there is such $\varphi_0 \in \text{Form}_{\mathcal{L}_C}$ that $\square \varphi_0 \in w_T$. Thus $\Diamond \varphi_0 \in w_T$, since $\square \varphi_0 \Rightarrow \Diamond \varphi_0 \in w_T$. So, given Lemma 3.4 (b), $\psi \in T'$ for some complete QL-theory $T'$ such that $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in T'\} \subseteq T'$ and clearly $L_T \neq \emptyset$.

For QE1-models observe that $\{\varphi \in \text{Form}_{\mathcal{L}_C} : \square \varphi \in w_T\} \subseteq w_T$ due to (T), hence $w_T \in L_T$.

\[\blacksquare\]

**Lemma 5.3.** Let $T$ be an $L$-theory. For every proposition $\varphi$ from language of the theory $w_T$ (= $T^h$) we have that in a canonical model $\mathcal{R}$ of $T$:

$\mathcal{R} \models \varphi$ iff $\varphi \in w_T$. 

\[\blacksquare\]
Proof. Since $\varphi$ is a sentence, then for all assignments $v_1$ and $v_2$ we have $V_{v_1}^w(\varphi) = V_{v_2}^w(\varphi)$. Thus it is enough to show that

$$V_{T \mathcal{K}}^w(\varphi) = 1$$

iff $\varphi \in w_T$.

This, however, is an immediate consequence of Definition 5.1 and Lemma 5.2.

Lemma 5.4. Every canonical model of an $L$-theory $T$ is a model for $T$.

Proof. Let $\varphi \in T$, hence $\varphi \in T$ and $\varphi \in T^h$. Given Lemma 5.3 $\mathcal{R} \models \varphi$, so also $\mathcal{R} \models \varphi$.

Theorem 5.5. An $L$-theory $T$ is consistent iff there is an $L$-structure $\mathcal{M}$ such that it is a model for $T$.

Proof. Let us assume that $T$ is consistent. Let $\mathcal{R}$ be a canonical model for $T$. Then, by Lemmas 5.2 and 5.4, $\mathcal{R}$ is an $L$-structure and a model for $T$. Now, if there is a structure $\mathcal{M}$ which is a model for $T$, then, given Theorem 4.9, $T$ is consistent.

References


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