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Marek Nasieniewski
Andrzej Pietruszczak

THE WEAKEST REGULAR MODAL LOGIC DEFINING JAŚKOWSKI'S LOGIC D_2

Abstract

Jaśkowski's logic $\mathbf{D_2}$ was formulated with the help of the modal logic **S5**, [8]–[10]. It was shown (see [12], [16]) that to define $\mathbf{D_2}$ one could use normal modal logics weaker than **S5**. In the present paper we indicate a certain regular modal logic which also defines $\mathbf{D_2}$. This logic is the weakest regular modal logic defining $\mathbf{D_2}$.

Key words: the discussive logic $\mathbf{D_2}$, normal and regular modal logics for $\mathbf{D_2}$.

1. The discussive logic $\mathbf{D_2}$

In this paper we are not discussing the idea of Jaśkowski's logic $\mathbf{D_2}$. The paper concerns only some "technical" results.[1]

In Appendix A we recall some chosen facts concerning modal logic. Lemmas A.1, A.2, etc., as well as an explanation of the standard notations ('$\Diamond$', '**S5**', '**PL**', etc.) can be also found in Appendix.

Let $\mathrm{For_m}$ be the set of all modal formulae, while $\mathrm{For^d}$ be the set of all formulae of the discussive language with logical constants: '$\neg$', '$\vee$', '$\wedge^{\mathrm{d}}$', '$\to^{\mathrm{d}}$', and '$\leftrightarrow^{\mathrm{d}}$'. *Jaśkowski's transformation* is the function $-^\bullet$ from $\mathrm{For^d}$ into $\mathrm{For_m}$ such that:

1. $(a)^\bullet = a$, for any propositional letter a,

[1] As regards intuitions concerning the logic $\mathbf{D_2}$ see the original Jaśkowski's paper [8], [9] and for example [15], [18].

2. for any $A, B \in \mathrm{For}^{\mathrm{d}}$:
   (a) $(\neg A)^\bullet = \ulcorner \neg A^\bullet \urcorner$,
   (b) $(A \vee B)^\bullet = \ulcorner A^\bullet \vee B^\bullet \urcorner$,
   (c) $(A \wedge B)^\bullet = \ulcorner A^\bullet \wedge \Diamond B^\bullet \urcorner$,
   (d) $(A \to^{\mathrm{d}} B)^\bullet = \ulcorner \Diamond A^\bullet \to B^\bullet \urcorner$,
   (e) $(A \leftrightarrow^{\mathrm{d}} B)^\bullet = \ulcorner (\Diamond A^\bullet \to B^\bullet) \wedge \Diamond(\Diamond B^\bullet \to A^\bullet) \urcorner$.

Jaśkowski used notation '$\mathbf{D_2}$' referring to a logic, i.e. a set of formulae. Thus, $\mathbf{D_2}$ is defined as follows:

DEFINITION 1.1. $\mathbf{D_2} := \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5} \}$.

2. Other normal logics defining $\mathbf{D_2}$

DEFINITION 2.1. Let $\boldsymbol{L}$ be any modal logic.

(i) We say that $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^\bullet \urcorner \in \boldsymbol{L} \}$.
(ii) Let $\mathsf{S5}_\Diamond$ be the set of all modal logics which have the same theses beginning with '$\Diamond$' as $\mathbf{S5}$, i.e., $\boldsymbol{L} \in \mathsf{S5}_\Diamond$ iff $\forall_{A \in \mathrm{For_m}} (\ulcorner \Diamond A \urcorner \in \boldsymbol{L} \iff \ulcorner \Diamond A \urcorner \in \mathbf{S5})$.[2]
(iii) Let $\mathsf{NS5}_\Diamond$ be the set of all normal logics from $\mathsf{S5}_\Diamond$.

Of course, we have the following

LEMMA 2.1. *If* $\boldsymbol{L} \in \mathsf{S5}_\Diamond$, *then* $\boldsymbol{L}$ *defines* $\mathbf{D_2}$.[3]

In [12] it was shown that $\mathbf{S4}$ and $\mathbf{S5}$ have the same members beginning with '$\Diamond$' – consequently one can use weaker modal logics to define $\mathbf{D_2}$. In [16] the smallest normal modal logic (denoted by $\mathbf{S5^M}$) possessing this property was indicated.

In [16] $\mathbf{S5^M}$ is defined as the smallest normal logic containing (D),

$$\Diamond\Box(\Diamond\Box p \to \Box p) \tag{ML5}$$

$$\Diamond\Box(\Box p \to p) \tag{MLT}$$

[2] Notice that the set M-$\mathbf{S5} := \{A \in \mathrm{For_m} : \ulcorner \Diamond A \urcorner \in \mathbf{S5}\}$ from [13] (in [17] this set is called 'M-counterpart of $\mathbf{S5}$' and in [2] it is denoted by '$\mathbf{J}$') does not belong to the family $\mathsf{S5}_\Diamond$. The set $\mathbf{J}$ is not a modal logic in the sense of this paper, because it is not closed under (MP), i.e. *modus ponens* for '$\to$'. For example, '$\Diamond p \to p$', '$(\Diamond p \to p) \to (\Diamond p \to \Box p)$' $\in \mathbf{J}$, but '$\Diamond p \to \Box p$' $\notin \mathbf{J}$. Of course, $\forall_{\boldsymbol{L} \in \mathsf{S5}_\Diamond} \forall_{A \in \mathrm{For_m}} (A \in \mathbf{J} \iff \ulcorner \Diamond A \urcorner \in \boldsymbol{L})$.

[3] Further we will show that for classical modal logics the reverse statement also holds.

and closed under the following rule:

$$\text{if } \ulcorner\Diamond\Diamond A\urcorner \in \mathbf{S5^M} \text{ then } \ulcorner\Diamond A\urcorner \in \mathbf{S5^M}. \qquad (\mathrm{RM}^2_1)$$

FACT 2.1 ([16]). $\mathbf{S5^M}$ *is the smallest logic in* $\mathsf{NS5}_\Diamond$.

COROLLARY 2.1. *For any modal logic* $\boldsymbol{L}$: *if* $\mathbf{S5^M} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}$, *then* $\boldsymbol{L} \in \mathsf{S5}_\Diamond$.

We observe that one can drop two out of the three axioms of the original formulation of $\mathbf{S5^M}$. In Fact 3.2 we prove that (D) and (ML5) belong to any regular modal logic which contains (MLT) and is closed under (RM^2_1).

FACT 2.2. $\mathbf{S5^M}$ *is the smallest normal logic which contains* (MLT) *and is closed under* (RM^2_1).

This observation has a direct connection with the results from [3], [5] and [14].

In [16] $\mathbf{T^*}$ denotes the smallest normal logic containing (T) and (ML5), which is closed under the rule (RM^2_1). Of course, $\mathbf{S5^M} \subsetneq \mathbf{T^*} \subsetneq \mathbf{S5}$; so $\mathbf{T^*}$ also defines $\mathbf{D_2}$ (see Lemma 2.1 and Fact 2.1). In [3] it was observed that (ML5) can be dropped from the axiomatization of $\mathbf{T^*}$ given in [16].

FACT 2.3 ([3]). $\mathbf{T^*}$ *is the smallest normal logic which contains* (T) *and is closed under* (RM^2_1).

Besides, it was proved in [3], [5] that one can define the logic $\mathbf{T^*}$ without the rule (RM^2_1), using instead – as an additional axiom – the following formula ("semi-4"):

$$\Box p \to \Diamond\Box\Box p \qquad (4_s)$$

FACT 2.4 ([3], [5]). $\mathbf{T^*}$ *is the smallest normal logic which contains* (T) *and* (4_s).

In [5] the logic $\mathbf{S5^M}$ was axiomatized without the rule (RM^2_1).

FACT 2.5 ([5]). $\mathbf{S5^M}$ *is the smallest normal logic which contains* (MLT) *and* (4_s).

Additionally, in [14] another axiomatization of the logic $\mathbf{S5^M}$ without the rule (RM^2_1) was given.

FACT 2.6 ([14]). $\mathbf{S5^M}$ *is the smallest normal logic which contains* $(\mathtt{4_s})$ *and the converse of* $(\mathtt{5})$*:*

$$\Box p \to \Diamond\Box p \tag{$\mathtt{5_c}$}$$

The formula $(\mathtt{5_c})$ was already used before for example in [16], where $\mathbf{D^*}$ is meant as the smallest normal logic containing $(\mathtt{D})$ and $(\mathtt{5_c})$.[4]

Besides, we have the upward analogon of the result from Fact 2.1.

FACT 2.7 ([4]) *Let* $\boldsymbol{L} \in \mathsf{NS5}_\Diamond$*. Then* $\boldsymbol{L} \subseteq \mathbf{S5}$.

LEMMA 2.2. *If* $\boldsymbol{L} \in \mathsf{S5}_\Diamond$ *and* $\boldsymbol{L}$ *is closed under* (RPN)*, then* $\boldsymbol{L} \subseteq \mathbf{S5}$.

PROOF: Assume that $A \in \boldsymbol{L}$. Then $\ulcorner\Diamond\Box A\urcorner \in \boldsymbol{L}$ and $\ulcorner\Diamond\Box A\urcorner \in \mathbf{S5}$. Therefore, since $(\mathtt{5}), (\mathtt{T}) \in \mathbf{S5}$, by (MP) also $A \in \mathbf{S5}$. ⊣

PROOF OF FACT 2.7: By Fact 2.1, $(\mathtt{MLT}) \in \boldsymbol{L}$. Hence, by Lemma A.1, $\boldsymbol{L}$ is closed under (RPN). Finally, we use Lemma 2.2.[5] ⊣

3. The weakest regular logic defining $\mathbf{D_2}$

We consider a regular version of the logic $\mathbf{S5^M}$. We prove that while defining logic $\mathbf{D_2}$ one can use a weaker modal logic than $\mathbf{S5^M}$.

DEFINITION 3.1. $\mathbf{rS5^M}$ is the smallest regular logic which contains $(\mathtt{MLT})$ and is closed under the rule (RM^2_1).

FACT 3.1. *The logic* $\mathbf{rS5^M}$ *is not normal. In other words,* $\mathbf{rS5^M}$ *has no thesis of the form* $\ulcorner\Box B\urcorner$*. Consequently,* $\mathbf{rS5^M} \subsetneq \mathbf{S5^M}$.

PROOF: Let v be any valuation from $\mathrm{For_m}$ into $\{0,1\}$ such that it preserves classical truth conditions for classical constants, $v(\Box A) = 0$ and $v(\Diamond A) = 1$, for any $A \in \mathrm{For_m}$. Notice that for any thesis of $\mathbf{rS5^M}$ we have $v(A) = 1$.

Indeed, we can consider $\mathbf{rS5^M}$ as being axiomatized by $\mathbf{PL}$, $\mathrm{Sub}[(\mathtt{df}\Diamond), (\mathtt{K}), (\mathtt{MLT})]$, (MP), (RM) and (RM^2_1). Thus $A \in \mathbf{rS5^M}$ iff there exists a sequence $A_1, \ldots, A_n = A$ in which for any $i \leqslant n$ either $A_i \in \mathbf{PL} \cup \mathrm{Sub}[(\mathtt{df}\Diamond),$

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[4] Since $(\mathtt{D}) \in \mathbf{K5_c}$ (see [6], Exercise 4.33), hence $\mathbf{D^*} := \mathbf{KD5_c} = \mathbf{K5_c}$. Notice that $(\mathtt{D})$ belongs to any regular modal logic containing $(\mathtt{5_c})$.

[5] We have another proof of Fact 2.7 without the use of Fact 2.1 (cf. [4] and [12]): '$\Diamond(p \to p)$' $\in \mathbf{S5}$, so '$\Diamond(p \to p)$' $\in \boldsymbol{L}$ and $(\mathtt{D}) \in \boldsymbol{L}$, by $(\mathrm{R}^{\Diamond\Box})$. Hence, by Lemma A.4, $\boldsymbol{L}$ is closed under (RPN). By Lemma 2.2 we obtain the thesis of Fact.

$(\mathtt{K}), (\mathtt{MLT})]$, or there are $j, k < i$ such that $A_k = \ulcorner A_j \to A_i \urcorner$, or for some $j < i$ and $B, C \in \mathrm{For_m}$ we have $A_j = \ulcorner B \to C \urcorner$ and $A_i = \ulcorner \Box B \to \Box C \urcorner$, or for some $j < i$ and $B \in \mathrm{For_m}$ we have $A_j = \ulcorner \Diamond\Diamond B \urcorner$ and $A_i = \ulcorner \Diamond B \urcorner$.

It is easy to prove by induction on the length of the proof, relative to the chosen axiomatization, that: if $A \in \mathbf{rS5^M}$, then $v(A) = 1$. ⊣

As it was noted in Section 2, we prove that formulae ($\mathtt{D}$) and ($\mathtt{ML5}$) are theses of $\mathbf{rS5^M}$. In the case of ($\mathtt{D}$) this fact follows from Lemma A.2, since $\mathbf{rS5^M}$ contains an axiom of the form $\ulcorner \Diamond B \urcorner$. In the case of ($\mathtt{ML5}$) we will prove that ($\mathtt{5_c}$), ($\mathtt{4_s}$), and the formula (†) from p. 202 belong to $\mathbf{rS5^M}$.

Firstly, the formula

$$\Box\Diamond p \to \Diamond p \qquad (\mathtt{5_c^\Diamond})$$

belongs to $\mathbf{rS5^M}$. Indeed:

1. $\Box(\Box\Diamond p \to \Diamond p) \to \Diamond(\Box\Diamond p \to \Diamond p)$ — ($\mathtt{D}$): $p/(\Box\Diamond p \to \Diamond p)$
2. $\Diamond\Box(\Box\Diamond p \to \Diamond p) \to \Diamond\Diamond(\Box\Diamond p \to \Diamond p)$ — by 1 and (RM)
3. $\Diamond\Box(\Box\Diamond p \to \Diamond p)$ — ($\mathtt{MLT}$): $p/\Diamond p$
4. $\Diamond\Diamond(\Box\Diamond p \to \Diamond p)$ — by 2, 3 and (MP)
5. $\Diamond(\Box\Diamond p \to \Diamond p)$ — by 4 and (RM_1^2)
6. $\Diamond(\Box\Diamond p \to \Diamond p) \to \Diamond\Diamond(\Diamond p \to p)$ — by ($\mathrm{R}^{\Diamond\Box}$), ($\mathrm{RM}^\Diamond$) and **PL**
7. $\Diamond\Diamond(\Diamond p \to p)$ — by 5, 6 and (MP)
8. $\Diamond(\Diamond p \to p)$ — by 7 and (RM_1^2)
9. $\Box\Diamond p \to \Diamond p$ — by 8, ($\mathrm{R}^{\Diamond\Box}$), (US) and **PL**

Thus, by the standard duality result, also $(\mathtt{5_c}) \in \mathbf{rS5^M}$.

Secondly, the formula

$$\Box\Diamond\Diamond p \to \Diamond p \qquad (\mathtt{4_s^\Diamond})$$

belongs to $\mathbf{rS5^M}$. Indeed:

1. $\Box\Diamond\Diamond p \to \Diamond\Diamond p$ — ($\mathtt{5_c^\Diamond}$): $p/\Diamond p$
2. $\Box\Box\Diamond\Diamond p \to \Box\Diamond\Diamond p$ — by 1 and (RM)
3. $\Box\Box\Diamond\Diamond p \to \Diamond\Diamond p$ — by 1, 2 and **PL**
4. $(\Box\Box\Diamond\Diamond p \to \Diamond\Diamond p) \to \Diamond(\Box\Diamond\Diamond p \to \Diamond p)$ — ($\mathrm{R}^{\Diamond\Box}$) and **PL**
5. $\Diamond(\Box\Diamond\Diamond p \to \Diamond p) \to \Diamond\Diamond(\Diamond\Diamond p \to p)$ — ($\mathrm{R}^{\Diamond\Box}$), **PL** and ($\mathrm{RM}^\Diamond$)
6. $\Diamond\Diamond(\Diamond\Diamond p \to p)$ — by 3, 4, 5 and $2 \times$ (MP)
7. $\Diamond(\Diamond\Diamond p \to p)$ — by 6 and (RM_1^2)
8. $\Box\Diamond\Diamond p \to \Diamond p$ — by 7, ($\mathrm{R}^{\Diamond\Box}$), (US) and **PL**

So also $(\mathtt{4_s}) \in \mathbf{rS5^M}$.

Thirdly, the following formula (†) also belongs to $\mathbf{rS5^M}$.

$$\Diamond\Diamond(\Diamond\Diamond\Box p \to \Box\Box p) \tag{†}$$

Indeed:

1. $\Box\Diamond\Diamond\Box p \to \Diamond\Box p$ — $(\mathtt{4}^{\Diamond}_{\mathtt{s}})$: $p/\Box p$
2. $\Box\Box\Diamond\Diamond\Box p \to \Box\Diamond\Box p$ — 1 and (RM)
3. $\Box\Diamond\Box p \to \Diamond\Box p$ — $(\mathtt{5}^{\Diamond}_{\mathtt{c}})$: $p/\Box p$
4. $\Box\Box\Diamond\Diamond\Box p \to \Diamond\Box p$ — 2, 3 and **PL**
5. $\Box p \to \Diamond\Box\Box p$ — $(\mathtt{4}_{\mathtt{s}})$
6. $\Diamond\Box p \to \Diamond\Diamond\Box\Box p$ — 5 and $(\mathrm{RM}^{\Diamond})$
7. $\Box\Box\Diamond\Diamond\Box p \to \Diamond\Diamond\Box\Box p$ — 4, 6 and **PL**
8. $\Diamond\Diamond(\Diamond\Diamond\Box p \to \Box\Box p)$ — 7, $2 \times (\mathrm{R}^{\Diamond\Box})$, (US) and **PL**

Now, we can prove that (ML5) $\in \mathbf{rS5^M}$:

1. $\neg\Diamond\Diamond\Box(\Diamond\Box p \to \Box p) \to \Box\Box\Diamond(\Diamond\Box p \wedge \neg\Box p)$ — **PL**, (REP), $(\mathtt{df}\Diamond)$ and $(\mathtt{df}\Box)$
2. $\Diamond(\Diamond\Box p \wedge \neg\Box p) \to (\Diamond\Diamond\Box p \wedge \Diamond\neg\Box p)$ — **PL** and $(\mathrm{RM}^{\Diamond})$
3. $\Box\Box\Diamond(\Diamond\Box p \wedge \neg\Box p) \to (\Box\Box\Diamond\Diamond\Box p \wedge \neg\Diamond\Diamond\Box\Box p)$ — by 2, (RM), (R), $(\mathtt{df}\Diamond)$ and $(\mathtt{df}\Box)$
4. $(\Box\Box\Diamond\Diamond\Box p \wedge \neg\Diamond\Diamond\Box\Box p) \to \neg(\Box\Box\Diamond\Diamond\Box p \to \Diamond\Diamond\Box\Box p)$ — **PL**
5. $(\Box\Box\Diamond\Diamond\Box p \to \Diamond\Diamond\Box\Box p) \to \Diamond\Diamond\Box(\Diamond\Box p \to \Box p)$ — 1, 3, 4 and **PL**
6. $(\dagger) \to \Diamond\Diamond\Box(\Diamond\Box p \to \Box p)$ — 5, $2 \times (\mathrm{R}^{\Diamond\Box})$, (REP) and **PL**
7. $\Diamond\Diamond\Box(\Diamond\Box p \to \Box p)$ — 6, (†) and (MP)
8. $\Diamond\Box(\Diamond\Box p \to \Box p)$ — by 7 and (RM^{2}_{1})

By the above considerations we obtain:

FACT 3.2. (D), (ML5) $\in \mathbf{rS5^M}$.

Let $\mathbf{RS5}_{\Diamond}$ be the set of all regular logics from $\mathbf{S5}_{\Diamond}$. Now we prove the main theorem of the paper:

THEOREM 3.1. $\mathbf{rS5^M} \in \mathbf{RS5}_{\Diamond}$.

PROOF: Of course, if $\ulcorner\Diamond A\urcorner \in \mathbf{rS5^M}$, then $\ulcorner\Diamond A\urcorner \in \mathbf{S5}$. For the reverse implication we consider **S5** as axiomatized by **PL**, Sub[$(\mathtt{df}\Diamond)$, (K), (T), (5)], (MP) and (RN). Then $A \in \mathbf{S5}$ iff there exists a sequence $A_1, \ldots, A_n = A$ in which for any $i \leqslant n$, either $A_i \in \mathbf{PL} \cup \mathrm{Sub}[(\mathtt{df}\Diamond), (\mathtt{K}), (\mathtt{T}), (\mathtt{5})]$, or there is $j < i$ such that $A_i = \ulcorner\Box A_j\urcorner$, or there are $j, k < i$ such that $A_k = \ulcorner A_j \to$

$A_i\urcorner$. We prove by the induction on the length of the proof, relative to the chosen axiomatization, that: if $A \in \mathbf{S5}$, then $\ulcorner\Diamond\Box A\urcorner \in \mathbf{rS5^M}$.

Consider the case $n = 1$: we have that $A_1 \in \mathbf{PL} \cup \mathrm{Sub}[(\mathtt{df}\Diamond), (\mathtt{K}), (\mathtt{T}), (\mathtt{5})]$. Notice that $\mathbf{rS5^M}$ is closed under the rule (RPN): $A/\Diamond\Box A$ (see Lemma A.1) and $\mathbf{PL} \cup \mathrm{Sub}[(\mathtt{df}\Diamond), (\mathtt{K})] \subseteq \mathbf{rS5^M}$. Moreover, $(\mathtt{MLT}) \in \mathbf{rS5^M}$ and $(\mathtt{ML5}) \in \mathbf{rS5^M}$, by Fact 3.2. So $\ulcorner\Diamond\Box A_1\urcorner \in \mathbf{rS5^M}$,

Assume that $n > 1$: if $A \in \mathbf{PL} \cup \mathrm{Sub}[(\mathtt{df}\Diamond), (\mathtt{K}), (\mathtt{T}), (\mathtt{5})]$, then we repeat the above reasoning.

If $A = \ulcorner\Box B\urcorner$ for some $B \in \mathbf{S5}$, then by the inductive hypothesis we have $\ulcorner\Diamond\Box B\urcorner \in \mathbf{rS5^M}$ and, moreover we obtain:

1.	$\Diamond\Box B \to \Diamond\Diamond\Box\Box B$	$(4_{\mathtt{s}})$ and $(\mathrm{RM}^\Diamond)$
2.	$\Diamond\Diamond\Box\Box B$	the inductive hypothesis, 1 and (MP)
3.	$\Diamond\Box\Box B$	2 and (RM^2_1)

If $B = \ulcorner C \to A\urcorner$, for some $B, C \in \mathbf{S5}$, then by the inductive hypothesis we have $\ulcorner\Diamond\Box C\urcorner$, $\ulcorner\Diamond\Box B\urcorner \in \mathbf{rS5^M}$ and, moreover we obtain:

1.	$\Box(C \to A) \to \Diamond\Box\Box(C \to A)$	$(4_{\mathtt{s}})$: $p/(C \to A)$
2.	$\Diamond\Box(C \to A) \to \Diamond\Diamond\Box\Box(C \to A)$	1 and (RM)
3.	$\Diamond\Diamond\Box\Box(C \to A)$	the inductive hypothesis, 2 and (MP)
4.	$\Diamond\Box\Box(C \to A)$	3 and (RM^2_1)
5.	$\Diamond\Box(\Box C \to \Box A)$	4, $(\mathtt{K})$, (RM), $(\mathrm{RM}^\Diamond)$ and (MP)
6.	$\Diamond(\Diamond\Box C \to \Diamond\Box A)$	5, $(\mathtt{K}^\Diamond)$, $(\mathrm{RM}^\Diamond)$ and (MP)
7.	$\Box\Diamond\Box C \to \Diamond\Diamond\Box A$	6, $(\mathrm{R}^{\Diamond\Box})$ and **PL**
8.	$\Diamond\Box\Diamond\Box C \to \Diamond\Diamond\Diamond\Box A$	7 and $(\mathrm{RM}^\Diamond)$
9.	$\Diamond\Box\Diamond\Box C$	the inductive hypothesis and (RPN)
10.	$\Diamond\Diamond\Diamond\Box A$	9 and (MP)
11.	$\Diamond\Box A$	10 and $2 \times (\mathrm{RM}^2_1)$

This ends the inductive proof. So, for $\ulcorner\Diamond A\urcorner \in \mathbf{S5}$, we have that $\ulcorner\Diamond\Box\Diamond A\urcorner \in \mathbf{rS5^M}$. By Lemma A.3 we obtain that $\ulcorner\Diamond\Diamond\Diamond A\urcorner \in \mathbf{rS5^M}$. Thus, $\ulcorner\Diamond A\urcorner \in \mathbf{rS5^M}$, by $(\mathtt{RM}^2_1)$. ⊣

LEMMA 3.1. *Let* $\boldsymbol{L} \in \mathbf{S5}_\Diamond$. *Then*

(i) $\boldsymbol{L}$ *is closed under the rule* $(\mathtt{RM}^2_1)$,

(ii) $(\mathtt{MLT})$ *belongs to* $\boldsymbol{L}$.

PROOF: (i) Assume that $\ulcorner\Diamond\Diamond A\urcorner \in \boldsymbol{L}$. Then $\ulcorner\Diamond\Diamond A\urcorner \in \mathbf{S5}$. Since $(4^\Diamond) \in \mathbf{S5}$, so also $\ulcorner\Diamond A\urcorner \in \mathbf{S5}$. Hence $\ulcorner\Diamond A\urcorner \in \boldsymbol{L}$.

(ii) $(\mathtt{T}), (\mathtt{D}) \in \mathbf{S5}$, so $(\mathtt{MLT}) \in \mathbf{S5}$, by Lemma A.4. So, $(\mathtt{MLT}) \in \boldsymbol{L}$. ⊣

By Theorem 3.1 and Lemma 3.1 we obtain:

THEOREM 3.2. $\mathbf{rS5^M}$ *is the smallest logic in* $\mathsf{RS5}_\Diamond$.

Finally, we prove that $\mathbf{rS5^M}$ is the smallest regular logic defining $\mathbf{D_2}$. To this end we define a function $-^\circ$ from $\mathrm{For_m}$ into $\mathrm{For^d}$ which "un-modalizes" every formula from $\mathrm{For_m}$:[6]

1. $(a)^\circ = a$, for any propositional letter a,
2. for any $A, B \in \mathrm{For_m}$:
   (a) $(\neg A)^\circ = \ulcorner \neg A^\circ \urcorner$,
   (b) $(A \vee B)^\circ = \ulcorner A^\circ \vee B^\circ \urcorner$,
   (c) $(A \wedge B)^\circ = \ulcorner \neg(\neg A^\circ \vee \neg B^\circ) \urcorner$,
   (d) $(A \to B)^\circ = \ulcorner \neg A^\circ \vee B^\circ \urcorner$,
   (e) $(A \leftrightarrow B)^\circ = \ulcorner \neg(\neg(\neg A^\circ \vee B^\circ) \vee \neg(\neg B^\circ \vee A^\circ)) \urcorner$,
   (f) $(\Diamond A)^\circ = \ulcorner (p \vee \neg p) \wedge^{\mathrm{d}} A^\circ \urcorner$,
   (g) $(\Box A)^\circ = \ulcorner \neg A^\circ \to^{\mathrm{d}} \neg(p \vee \neg p) \urcorner$.

We immediately have:

LEMMA 3.2. *For any* $A \in \mathrm{For_m}$: $\ulcorner A \leftrightarrow A^{\circ\bullet} \urcorner$ *is a thesis of all classical modal logics.*[7]

We can define a discussive logic for any modal logic.

DEFINITION 3.2. Let $\boldsymbol{L}$ be any modal logic. The *discussive logic for* $\boldsymbol{L}$ is the following subset of $\mathrm{For^d}$:

$$\mathbf{D}^{\boldsymbol{L}} := \{ A \in \mathrm{For^d} : \ulcorner \Diamond A^\bullet \urcorner \in \boldsymbol{L} \} .$$

By definitions, $\boldsymbol{L}$ defines $\mathbf{D_2}$ iff $\mathbf{D}^{\boldsymbol{L}} = \mathbf{D_2}$. We have $\mathbf{D_2} := \mathbf{D}^{\mathbf{S5}} = \mathbf{D}^{\mathbf{S5^M}} = \mathbf{D}^{\mathbf{rS5^M}}$, by Lemma 2.1, Fact 2.1, and Theorem 3.1.

LEMMA 3.3. *For any classical modal logics* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *the following conditions are equivalent*:

(a) $\mathbf{D}^{\boldsymbol{L}_1} \subseteq \mathbf{D}^{\boldsymbol{L}_2}$.
(b) *Every thesis of* $\boldsymbol{L}_1$ *beginning with '*$\Diamond$*' is a thesis of* $\boldsymbol{L}_2$.

[6] The function $-^\circ$ is a certain version of the function i_2 from [13].

[7] Taking into account Footnote 2, notice that $[\mathbf{D_2}]^\bullet \subseteq \mathbf{J}$ and $[\mathbf{J}]^\circ \subseteq \mathbf{D_2}$ (cf. Lemma 6 in [13]).

Proof: "(a) $\Rightarrow$ (b)" Let $\ulcorner\Diamond A\urcorner \in \boldsymbol{L}_1$. Then, by Lemma 3.2, **PL**, and (REP), we have that $\ulcorner\Diamond A^{\circ\bullet}\urcorner \in \boldsymbol{L}_1$. Hence $A^{\circ} \in \mathbf{D}^{\boldsymbol{L}_1}$, by Definition 3.2. So $A^{\circ} \in \mathbf{D}^{\boldsymbol{L}_2}$, by the assumption. Thus, $\ulcorner\Diamond A^{\circ\bullet}\urcorner \in \boldsymbol{L}_2$ and $\ulcorner\Diamond A\urcorner \in \boldsymbol{L}_2$, by Lemma 3.2, **PL**, and (REP).

"(b) $\Rightarrow$ (a)" Obvious. $\dashv$

From the above lemma we obtain:

Corollary 3.1. *For any classical modal logic $\boldsymbol{L}$: $\boldsymbol{L}$ defines $\mathbf{D_2}$ iff $\boldsymbol{L} \in \mathsf{S5}_{\Diamond}$.*

By the above corollary and Theorem 3.2 we obtain:

Theorem 3.3. $\mathbf{rS5^M}$ *is the smallest regular logic defining* $\mathbf{D_2}$.

By Lemmas 2.2, A.1 and 3.1, and Corollary 3.1 we obtain:

Fact 3.3. *If $\boldsymbol{L}$ is a regular logic defining $\mathbf{D_2}$, i.e. $\boldsymbol{L} \in \mathsf{RS5}_{\Diamond}$, then $\boldsymbol{L} \subseteq \mathbf{S5}$.*

Let us mention that using similar methods, by Fact 2.1 we obtain:

Fact 3.4. $\mathbf{S5^M}$ *is the smallest normal logic defining* $\mathbf{D_2}$.

A. Basic facts from modal logic

As in [6] modal formulae are formed in a standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', …; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); modal operators: the necessity sign '$\Box$' and the possibility sign '$\Diamond$'; and brackets. By $\mathrm{For_m}$ we denote the set of modal formulae, and – as in [6] – let **PL** be the set of modal formulae which are instances of classical tautologies.

As in [1], [7], a set $\boldsymbol{L}$ of modal formulae is a (*modal*) *logic* iff

- $\mathbf{PL} \subseteq \boldsymbol{L}$,
- for any $A, C \in \mathrm{For_m}$, $\boldsymbol{L}$ contains the following formula

$$C[{}^{\neg\Box\neg A}/_{\Diamond A}] \leftrightarrow C\,, \qquad (\mathtt{rep}^{\Box})$$

  where $C[{}^{A}/_{B}]$ is any formula that results from C by replacing one or more occurrences of A, in C, by B, i.e. using $(\mathtt{rep}^{\Box})$ we are *replacing* one or more occurrences of '$\neg\Box\neg$' by '$\Diamond$'.[8]

[8]In [1], [7] the symbol '$\Diamond$' is only an abbreviation of '$\neg\Box\neg$'. In the present paper

- $\boldsymbol{L}$ is closed under the following three rules: *modus ponens* for '$\rightarrow$':

$$\text{if } A \text{ and } \ulcorner A \rightarrow B \urcorner \text{ are members of } \boldsymbol{L}, \text{ so is } B. \tag{MP}$$

uniform substitution:

$$\text{if } A \in \boldsymbol{L} \text{ then } \mathrm{s}\,A \in \boldsymbol{L}, \tag{US}$$

where $\mathrm{s}\,A$ is the result of uniform substitution of formulae for propositional letters in A.

For any $\Gamma \subseteq \mathrm{For_m}$ we put $\mathrm{Sub}[\Gamma] := \{\mathrm{s}\,A \in \mathrm{For_m} : A \in \Gamma$ and s is an uniform substitution of formulae for propositional letters$\}$.

All members of the set $\boldsymbol{L}$ are called *theses* of the logic $\boldsymbol{L}$. By ($\mathtt{rep}^{\Box}$), every modal logic has the following thesis:

$$\Diamond p \leftrightarrow \neg \Box \neg p \tag{$\mathtt{df}\,\Diamond$}$$

A modal logic $\boldsymbol{L}$ is *classical* iff $\boldsymbol{L}$ is closed under the following rule for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \leftrightarrow B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner \Box A \leftrightarrow \Box B \urcorner \in \boldsymbol{L}, \tag{RE}$$

Every classical logic $\boldsymbol{L}$ is closed under rule of replacement, i.e. for any $A, B, C \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \leftrightarrow B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner C \leftrightarrow C[^{A}/_{B}] \urcorner \in \boldsymbol{L}. \tag{REP}$$

It is known (cf. e.g. [6]) that while defining classical logics one uses ($\mathtt{df}\,\Diamond$) instead of ($\mathtt{rep}^{\Box}$), i.e. treats them as subsets of $\mathrm{For_m}$ which include **PL** and ($\mathtt{df}\,\Diamond$) and which are closed under rules (MP), (US) and (RE). We also have an analogous situation in the case of monotonic, regular, and normal modal logics defined further.

Every classical modal logic has the following thesis:

$$\Box p \leftrightarrow \neg \Diamond \neg p \tag{$\mathtt{df}\,\Box$}$$

A modal logic $\boldsymbol{L}$ is *monotonic* iff $\boldsymbol{L}$ is closed under the monotonicity rule, i.e. for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \rightarrow B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner \Box A \rightarrow \Box B \urcorner \in \boldsymbol{L}, \tag{RM}$$

'$\Diamond$' is a primary symbol, thus we have to admit an axiom of the form ($\mathtt{rep}^{\Box}$). Theses of this form are equivalent to the usage of '$\Diamond$' as the abbreviation of '$\neg\Box\neg$'.

Every monotonic logic $\boldsymbol{L}$ is classical and it is closed under the dual form of (RM), i.e. for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \to B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner \Diamond A \to \Diamond B \urcorner \in \boldsymbol{L}. \tag{RM$^\Diamond$}$$

LEMMA A.1. *If a monotonic logic $\boldsymbol{L}$ has a thesis of the form $\ulcorner \Diamond B \urcorner$ (resp. $\ulcorner \Box B \urcorner$, $\ulcorner \Diamond \Box B \urcorner$), then*

(i) *$\boldsymbol{L}$ is closed under the following rules, respectively*:

$$\textit{if } A \in \boldsymbol{L} \textit{ then } \ulcorner \Diamond A \urcorner \in \boldsymbol{L}, \tag{RP}$$

$$\textit{if } A \in \boldsymbol{L} \textit{ then } \ulcorner \Box A \urcorner \in \boldsymbol{L}. \tag{RN}$$

$$\textit{if } A \in \boldsymbol{L} \textit{ then } \ulcorner \Diamond \Box A \urcorner \in \boldsymbol{L}. \tag{RPN}$$

(ii) *$\boldsymbol{L}$ has the following theses, respectively*:

$$\Diamond \top \tag{P}$$

$$\Box \top \tag{N}$$

$$\Diamond \Box \top \tag{PN}$$

where $\top$ is any classical tautology.

A logic $\boldsymbol{L}$ is *regular* iff $\boldsymbol{L}$ is monotonic and has the following thesis:

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

A logic $\boldsymbol{L}$ is regular iff $\boldsymbol{L}$ is closed under the *regularity rule*, i.e. for any $A, B, C \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \wedge B \to C \urcorner \in \boldsymbol{L} \text{ then } \ulcorner \Box A \wedge \Box B \to \Box C \urcorner \in \boldsymbol{L}. \tag{RR}$$

Every regular modal logic has the following theses

$$\Box(p \to q) \to (\Diamond p \to \Diamond q) \tag{K$^\Diamond$}$$

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \tag{R}$$

$$\Diamond(p \to q) \leftrightarrow (\Box p \to \Diamond q) \tag{R$^{\Diamond\Box}$}$$

By Lemma A.1(ii) and ($\mathtt{R}^{\Diamond\Box}$) we obtain.

LEMMA A.2. *For any regular logic: it has at least one thesis of the form $\ulcorner \Diamond B \urcorner$ iff it contains*

$$\Box p \to \Diamond p \tag{D}$$

By the rules ($\mathrm{RM}^{\Diamond}$) and (US) we obtain:

LEMMA A.3. *Every monotonic logic with thesis* (D) *contains also*

$$(\Diamond\Box)^k p \to (\Diamond)^{2k} p$$

A modal logic is *normal* iff it contains (K) and is closed under the rule of necessitation (RN) iff it is regular and contains (N).

K is the smallest normal modal logic. For other modal logics we make use of the following formulae:

$$\Box p \to p \tag{T}$$

$$\Box p \to \Box\Box p \tag{4}$$

$$\Diamond\Box p \to \Box p \tag{5}$$

Using names of the above formulae, to simplify naming of normal logics we write the *Lemmon code* $\mathbf{KA}_1 \ldots \mathbf{A}_n$ to denote the smallest normal logic containing the formulae $(\mathtt{A}_1), \ldots, (\mathtt{A}_n)$ (see [1], [6]). Thus, for example, **KT4** and **KT5** are the smallest normal modal logics which include: $\{(\mathtt{T}), (\mathtt{4})\}$, and $\{(\mathtt{T}), (\mathtt{5})\}$, respectively. We standardly put $\mathbf{T} := \mathbf{KT}$, $\mathbf{S4} := \mathbf{KT4}$ and $\mathbf{S5} := \mathbf{KT5}$. As it is known, $\mathbf{T} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$.

LEMMA A.4. *Let* $\boldsymbol{L}$ *be any normal logic such that* $(\mathtt{D}) \in \boldsymbol{L}$*. Then* $\boldsymbol{L}$ *is closed under the rule* (RPN)*.*

PROOF: Assume that $A \in \boldsymbol{L}$. Then $\ulcorner\Box\Box A\urcorner \in \boldsymbol{L}$, by (RN). By (D) and (US), $\ulcorner\Box\Box A \to \Diamond\Box A\urcorner \in \boldsymbol{L}$; so we use (MP). ⊣

Of course, the following formulae:

$$\Diamond\Diamond p \to \Diamond p \tag{$4^{\Diamond}$}$$

$$\Diamond p \to \Box\Diamond p \tag{$5^{\Diamond}$}$$

are respectively members of **K4** and **K5**, and so of **S5**.

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Department of Logic
Nicolaus Copernicus University of Toruń
e-mail: {mnasien,pietrusz}@umk.pl