PLAUSIBLE REASONING EXPRESSED BY
\textit{p}-CONSEQUENCE

Abstract
In this paper we present a formal way of describing plausible (or non-deductive) reasoning. Ajdukiewicz’s distinction between deductive and non-deductive reasoning [1] provides our theoretical framework. Our formal approach is given by so called \textit{operation of p-consequence}, which has been described earlier e.g. in [2]. At every stage of our work we try to show that Ajdukiewicz’s framework is relevant for our investigations. In the last paragraph axiomatization of “plausible” counterpart of Lukasiewicz’s many valued logics is given.

1.

Assume that a propositional language $\mathcal{L} = (L, f_1, \ldots, f_n)$ is given. For the purposes of this paper it is assumed that $\mathcal{L}$ is fixed.

Definition 1. By \textit{p-consequence operation on the language} $\mathcal{L}$ we understand any operation that fulfills the following two conditions (for every $X, Y \subseteq L$):

(i) $X \subseteq Z(X)$ (reflexivity)

(ii) $X \subseteq Y$ implies $Z(X) \subseteq Z(Y)$ (monotonicity).

It is well known that if we add to the definition of \textit{p-consequence} an idempotency condition: $Z(Z(X)) \subseteq Z(X)$ then we obtain an ordinary
notion of logical consequence. The latter is a formal counterpart of deductive reasoning that is specific for formal sciences. In fact, idempotency of operation can be expressed by the statement - conclusions of conclusions are conclusions, as well. In other words, immediate consequences of given premises have the same (true) value as their premises (see [1], §40), so two-steps reasoning has the same quality.

In plausible reasoning things look different – if a sentence $B$ is a conclusion of the set $\{A_1, A_2, \ldots, A_n\}$ then its value can be less than the value of the worst of premises ([1], §44). So, if we add $B$ to $\{A_1, A_2, \ldots, A_n\}$, we automatically diminish the value of the set of premises.

**Definition 2.** By $p$-inference (for the language $\mathcal{L}$) we shall mean finite and non empty sequence $(\langle \alpha_1, x_1 \rangle, \langle \alpha_2, x_2 \rangle, \ldots, \langle \alpha_k, x_k \rangle)$, where $\alpha_i$ is a formula and $x_i \in \{*, 1\}$ for $i = 1, 2, \ldots, k$.

A symbol $x_i$ occurring in a pair, indicates a degree of certainty of formula $\alpha_i$. Thus, if a pair is of the form $\langle \beta, 1 \rangle$, then the formula is understood as “absolutely” true or “accepted with a maximal degree”. When a formula $\gamma$ is followed by the symbol $*$, then one can read this as “$\gamma$ is plausible” or “$\gamma$ is not rejected”.

In $p$-inference $((\alpha_1, x_1), (\alpha_2, x_2), \ldots, (\alpha_k, x_k))$ a subsequence $((\alpha_1, x_1), (\alpha_2, x_2), \ldots, (\alpha_{k-1}, x_{k-1}))$ is a string of premises and the last pair $\langle \alpha_k, x_k \rangle$ (we remind that $k \geq 1$) is a conclusion. $p$-rule is simply any set of $p$-inferences.

Now, we can put the definition of $p$-proof:

**Definition 3.** By $p$-proof of formula $\alpha$ from the set of formulas $X$, based on the set of $p$-rules $\mathcal{R}$ we understand a sequence $(a_1, \ldots, a_m) \in (L \times \{*, 1\})^m$ such that:

1. $a_m = (\alpha, x)$ for some $x \in \{*, 1\}$;
2. for every $i \in \{1, \ldots, k\}$:
   2.1. $a_i = (\alpha_i, 1)$ and $\alpha_i \in X$ or
   2.2. $(b_1, \ldots, b_k, a_i) \in \bigcup \mathcal{R}$ for some $b_1, \ldots, b_k \in \{a_1, \ldots, a_{i-1}\}$.

The condition 2.1. in the above definition means that a formula taken from the set of initial premises is treated as proven with maximal degree of certainty.
We put \( X \vdash \alpha \) iff there exists \( p \)-proof of \( \alpha \) from the set \( X \) based on \( \mathcal{R} \).

We say that \( p \)-consequence \( Z \) is finitary iff \( Z(X) = \bigcup \{ Z(X_f) : X_f \text{ is finite subset of } X \} \).

**Theorem 1.** (See [2]) For every finitary \( p \)-consequence \( Z \) there exists a set of \( p \)-rules \( \mathcal{R} \) for which
\[
X \vdash \mathcal{R} \alpha \iff \alpha \in Z(X) \quad \text{for every } X \subseteq L, \alpha \in L
\]
holds. \( \square \)

It is easy to see that the opposite statement holds true, i.e. every operation \( Z_R \) defined by \( \alpha \in Z_R(X) \) iff \( X \vdash \mathcal{R} \alpha \) is a finitary \( p \)-consequence.

When we have \( X \vdash \mathcal{R} \alpha \) it is not possible to recognize whether \( \alpha \) has been proved with 1 or with \( * \). Similarly, like in the case of operation of \( p \)-consequence we lost an information about a strength of deductivity of \( \alpha \), that is we do not know if \( \alpha \) is deductive or plausible conclusion from \( X \).

So, let us introduce the other relation:

**Definition 4.** \( X \Longrightarrow \mathcal{R} (\alpha, x) \) iff there exists \( p \)-proof \((a_1, \ldots, a_k, (\alpha, x))\) from the set \( X \) based on \( \mathcal{R} \).

We leave without proof the following characterization of \( \Longrightarrow \) relation:

**Theorem 2.** For every families of \( p \)-rules \( \mathcal{R}, \mathcal{S} \), sets of formulas \( X, Y \):

i) \( \alpha \in X \Rightarrow (X \Longrightarrow \mathcal{R} (\alpha, 1)) \)

ii) \((X \subseteq Y \& X \Longrightarrow \mathcal{R} \alpha) \Rightarrow (Y \Longrightarrow \mathcal{R} \alpha) \)

iii) \((X \Longrightarrow \mathcal{R} (\beta_1, 1), \ldots, (\beta_k, 1)) \& \{ \beta_1, \ldots, \beta_k \} \Longrightarrow \mathcal{R} \alpha) \Rightarrow (X \Longrightarrow \mathcal{R} \alpha) \)

iv) \( X \Longrightarrow \mathcal{R} \alpha \) iff \( (X_f \Longrightarrow \mathcal{R} \alpha) \) for some \( X_f \in \text{Fin}(X) \)

v) \( \mathcal{R} \subseteq \mathcal{S} \& (X \Longrightarrow \mathcal{R} \alpha) \Rightarrow (X \Longrightarrow \mathcal{S} \alpha) \). \( \square \)

What is important - point iii) is a counterpart of idempotency condition. In the case of \( \Longrightarrow \) relation there does not occur any loss of information, which makes iii) holds true.

**Definition 5.** \( p \)-rule \( r \) is *derivable* from the set of \( p \)-rules \( \mathcal{R} \) (symb. \( r \in \text{Der}(\mathcal{R}) \)) iff for every \((a_1, \ldots, a_n) \in r \) there exist: \( p \)-inference \((b_1, \ldots, b_m)\), such that:

\( (a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_k), \ (k < m), \ a_n = b_m \) and for every \( i \in \{k+1, \ldots, m\} \) the following holds:

there exists \( (c_1, \ldots, c_j) \in \bigcup \mathcal{R} \) such that \( \{c_1, \ldots, c_{j-1}\} \subseteq \{b_1, \ldots, b_{i-1}\} \)

and \( c_j = b_i \).

\( p \)-rule \( r \) is weakly derivable from the set \( \mathcal{R} \) (symb. \( r \in \overline{\text{Der}}(\mathcal{R}) \)) iff for every \( (a_1, \ldots, a_n) \in r, X \subseteq L \), condition \( X \Rightarrow_R a_1, \ldots, a_{n-1} \) implies \( X \Rightarrow^r a_n \).

\textbf{Fact 1.}

i) \( \mathcal{R} \subseteq \text{Der}(\mathcal{R}) \subseteq \overline{\text{Der}}(\mathcal{R}) \).

ii) \( \Rightarrow^r = \Rightarrow_{\text{Der}(\mathcal{R})} = \Rightarrow_{\overline{\text{Der}}(\mathcal{R})} \).

iii) \( \mathcal{R} \subseteq \mathcal{S} \Rightarrow \text{Der}(\mathcal{R}) \subseteq \text{Der}(\mathcal{S}) \& \overline{\text{Der}}(\mathcal{R}) \subseteq \overline{\text{Der}}(\mathcal{S}) \).

iv) \( \text{Der}(\text{Der}(\mathcal{R})) \subseteq \text{Der}(\mathcal{R}) \& \overline{\text{Der}}(\overline{\text{Der}}(\mathcal{R})) \subseteq \overline{\text{Der}}(\mathcal{R}). \) \( \square \)

It can be shown that the inclusions from i) can be strict.

\textbf{2.}

\( p \)-consequence can be characterized in a semantical way.

Let \( p \)-matrix for the language \( \mathcal{L} \) be any structure \( \mathfrak{M} = (M, F_1, \ldots, F_n, D_1, D_*) \), where \( M = (M, F_1, \ldots, F_n) \) is an algebra similar to \( \mathcal{L} \) and \( D_1 \subseteq D_* \subseteq M \).

\( (2.1) \) \( \alpha \in Z_{\mathfrak{M}}(X) \) iff \( \forall h \in \text{hom}(\mathcal{L}, M)[h(X) \subseteq D_1 \Rightarrow h\alpha \in D_*] \).

Let \( \mathcal{M} \) be a class of \( p \)-matrices. Let us define: \( Z_{\mathcal{M}}(X) = \bigcap_{\mathfrak{M} \in \mathcal{M}} Z_{\mathfrak{M}}(X) \).

It is easy to show that \( Z_{\mathcal{M}} \) is a structural \( p \)-consequence operation. By structurality of \( Z \) we mean that for every endomorphism \( e \) of \( \mathcal{L} \): \( eZ(X) \subseteq Z(eX) \) (where \( X \) is any set of formulas).

\( p \)-matrix is a structure which differs from an ordinary matrix by two distinguished sets of values. The smaller one, \( D_1 \), is the set of the values which corresponds to "absolute truth". It is a counterpart of the set of distinguished values in logical matrix. The set \( D_* \), additionally, contains these values which can be interpreted as "values of plausibility".
**Theorem 3.** [2] For any structural $p$-consequence $Z$ there exists a class of $p$-matrices $M(Z)$ that fulfills: $Z(X) = \bigcap_{M \in M(Z)} Z_M(X)$ for any $X \subseteq L$. □

$p$-rule $r$ is valid for $Z_M$, where $M = \{ M_t : t \in T \}$ is a class of $p$-matrices of the form $M_t = (M_t, F_{t,1}, \ldots, F_{t,n}, D'_t, D'_*)$ iff for every: $t \in T$, $h \in \text{Hom}(L, M_t)$, $(a_1, \ldots, a_n) \in r$ ($n \geq 1$) : the condition $\overrightarrow{h}(A_1(n-1)) \subseteq D'_t$ & $h(A_*(n-1)) \subseteq D'_*$ implies $h(pr_1(a_n)) \in D'_t$. (pr_2(a_n)),

where

$A_1(k) = \{ pr_1(a_i) : 1 \leq i \leq k \}$

and

$A_*(k) = \{ pr_1(a_i) : 1 \leq i \leq k \}$

The above definition indicates the relationship between semantic ($p$-matrices) and syntax ($p$-rules). Moreover it explains a 1/* notation in the lower indexes of sets - $D_1$ and $D_*$.

We put $R(M)$ for the set of all $p$-rules valid for $Z_M$.

**Theorem 4.** [4] For any class of $p$-matrices $M = \{(M_t, F_{t,1}, \ldots, F_{t,n}, D'_t, D'_*) : t \in T \}$ and any $X \subseteq L$, $\alpha \in L$:

(i) $X \models_{R(M)} \alpha \Rightarrow \alpha \in Z_M(X)$.

(ii) Moreover, if $Z_M$ is finitary: $X \models_{R(M)} \alpha \iff \alpha \in Z_M(X)$.

By the definitions – every class of $p$-matrices defines structural $p$-consequence (by the operation of intersection). However, there is more general way that allows for describing of any $p$-consequence.

Let us put $Val := \{0, \frac{1}{2}, 1\}^L$. Define for $V \subseteq Val$:

$\alpha \in Z_V(X)$ iff $\forall v \in V(\overrightarrow{v}(X) \subseteq \{1\} \Rightarrow v(\alpha) \in \{\frac{1}{2}, 1\})$. 

Moreover, for $p$-consequence $Z$ let $V_Z = \{v_X\}_{X \subseteq L}$, where

$$v_X(\beta) = \begin{cases} 1, & \text{iff } \beta \in X; \\ \frac{1}{2}, & \text{when } \beta \in Z(X) - X \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to prove the following:

**Theorem 5.** For every $p$-consequence $Z$: $Z = Z_{V_Z}$. □

Given a set of formulas $X$ and a formula $\beta$, a valuation $v_X : L \rightarrow \{0, \frac{1}{2}, 1\}$ indicates a degree of plausibility of formula $\beta$. Naturally, if $v_X(\beta) = 1$, then $\beta$ is strongly accepted on the basis of $X$, when $v_X(\beta)$ equals $\frac{1}{2}$, then $\beta$ is plausible only, $\beta$ is rejected iff $v_X(\beta) = 0$.

3.

Consider a language $L = (L, \rightarrow, \vee, \wedge, \neg)$ of type $(2, 2, 2, 1)$. Moreover, for $0 \leq k < n$ let $M_n^k := (\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \rightarrow, \vee, \wedge, \neg, \{1\})$, where $M_n = (\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \rightarrow, \vee, \wedge, \neg, \{1\})$ is $n$-valued Łukasiewicz matrix, i.e. $x \rightarrow y = \min\{1, 1 - x + y\}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Moreover, assume that a formula $\Phi_n^k(p)$ fulfilling for every $h \in \text{Hom}(L, M_n^k)$:

1). $h(\Phi_n^k(p)) = 1$ iff $hp \geq \frac{k}{n-1}$,

2). $h(\Phi_n^k(p)) - \frac{n-1-k}{n-1} \leq hp$

is given.

Let $R_n^k$ be the following set of $p$-rules (where $Ax_n$ is the set of axioms adequate to $n$-valued Łukasiewicz logic $L_n$ (see [3])):
\begin{align*}
A_1 \quad & \frac{\langle \alpha, 1 \rangle}{\alpha \in \text{Ax}_n} \\
A_2 \quad & \frac{((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \rightarrow \alpha, *}, \\
\text{p(MP)}_1 \quad & \frac{\langle \alpha \rightarrow \beta, 1 \rangle, \langle \alpha, 1 \rangle}{\langle \beta, 1 \rangle} \quad \text{p(MP)}_2 \quad \frac{\langle \alpha \rightarrow \beta, 1 \rangle, \langle \alpha, * \rangle}{\langle \beta, * \rangle} \\
\text{p(MP)}_3 \quad & \frac{\langle \alpha \rightarrow \beta, * \rangle, \langle \alpha, 1 \rangle}{\langle \beta, * \rangle}.
\end{align*}

where
\begin{align*}
r \quad & \frac{\langle \alpha_1, x_1 \rangle, \ldots, \langle \alpha_{m-1}, x_{m-1} \rangle}{\langle \alpha_m, x_m \rangle}
\end{align*}

means \( r = \{(\alpha_1, x_1), \ldots, (\alpha_{m-1}, x_{m-1}), (\alpha_m, x_m) : \alpha_1, \ldots, \alpha_m \in L\} \).

For simplicity, we will write \( \Phi^n_k(x) \) for \( h(\Phi^n_k(p)) \) where \( hp = x \).

An interpretation of the above schemas can be stated as follows: when \( \alpha \) is an axiom of Lukasiewicz logic, then \( \alpha \) is strongly provable (axiom \( A_1 \)). In MP rules a conclusion can be derived, only if in the premises at most one * appears. Finally MP is inherited from Lukasiewicz logic, whenever there is no *.

**Soundness theorem for \( R^n_k \):** For any \( X \subseteq L, \alpha \in L : X \models_{R^n_k} \alpha \Rightarrow \alpha \in Z_{\Phi^n_k}(X) \).

**Proof:** According to Theorem 4 it is enough to show that for every \( p \)-rule \( r \in R^n_k \), \( r \) is \( p \)-rule valid for \( p \)-consequence \( Z_{\Phi^n_k} \), that is we show:

(a) for every \((a_1, \ldots, a_n) \in r, h \in \text{Hom}(L, \mathcal{M}^*_n)\) such that \( \tilde{h}(A_1(n-1)) \subseteq D_1 \), and \( \tilde{h}(A_*(n-1)) \subseteq D_* \), we have \( h(pr_1(a_n)) \in D_{pr_2(a_n)} \).
For \( r = A_1 \) the condition (a) is obvious. So, assume that:
\[
h(((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \rightarrow \alpha) < \frac{k}{n-1},
\]
that is:
\[
\frac{n-1-k}{n-1} + ha < h(((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \leq 1,
\]
which contradicts the second condition of \( \Phi^n_k(p) \).

The validity of \( p \)-rules \( p(MP)_1, p(MP)_2, p(MP)_3 \) can be checked in a straightforward way. □

**Completeness theorem for \( R^n_k \):** For any \( X \subseteq L \), \( \alpha \in L : \alpha \in Z_{\mathcal{M}^{n,k}_L}(X) \Rightarrow X \parallel R^n_k \alpha \).

**Proof:** Let for every \( h \in Hom(L, \mathcal{M}^{n,k}_L) \), \( \vec{h}(X) \subseteq 1 \Rightarrow ha \geq \frac{k}{n-1} \). At first we will show that \( h(((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \lor \alpha) = 1 \) if \( \vec{h}(X) \subseteq \{1\} \).

Assume for contradiction, that \( \vec{h}(X) \subseteq \{1\} \) and \( ((x \rightarrow_{n-2} \neg x) \land \Phi^n_k(x)) \lor x < 1 \), where \( x = ha \). Then by the fact that \( x < 1 \) and \( \Phi^n_k(x) = 1 \) (condition 1) and the assumptions \( 1 > x \rightarrow_{n-2} \neg x = (n-1)(1-x) \), thus \( x = 1 \), a contradiction. Hence, there exists a proof (from the set \( X \) and in the logic \( L_n \) of the form \( (\alpha_1, \ldots, \alpha_r, (\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \lor \alpha \), which can be transformed into \( p \)-proof \( ((\alpha_1, 1), \ldots, (\alpha_r, 1), ((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \lor \alpha, 1)) \) from the set \( X \) based on \( R^n_k \).

Let us consider a formula:
\[
\Xi := (((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \rightarrow \alpha) \rightarrow (((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^n_k(\alpha)) \lor \alpha) \rightarrow \alpha),
\]
for which the following holds true - \( h \Xi = 1 \) if \( \vec{h}(X) \subseteq \{1\} \). Assume for a contradiction, that \( \vec{h}(X) \subseteq \{1\} \) and \( \Xi < 1 \), therefore
\[
1 > (((x \rightarrow_{n-2} \neg x) \land \Phi^n_k(x)) \rightarrow x) \rightarrow (1 \rightarrow x) = ((n-1)(1-x) \rightarrow x) \rightarrow x
\]
(the condition 1 for \( \Phi^n_k \) implies that \( \Phi^n_k(x) = 1 \),

that is
\[
x < 1 - (n-1)(1-x) + x, \text{ so } (n-1)(1-x) < 1, \text{ but it is possible only under condition } x = 1. \text{ Then it is easy to check that } h \Xi = 1.\]
Similarly as above, there exists \( p \)-proof \( ((\beta_1, 1), \ldots, (\beta_s, 1), (\Xi, 1)) \). Finally, the sequence
\[
((\beta_1, 1), \ldots, (\beta_s, 1), (\Xi, 1), ((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^p_k(\alpha)) \rightarrow \alpha, *)(A_2),
((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^p_k(\alpha)) \lor \alpha) \rightarrow \alpha, *)(p(MP)_{2}), (\alpha_1, 1), \ldots, (\alpha_r, 1),
((\alpha \rightarrow_{n-2} \neg \alpha) \land \Phi^p_k(\alpha)) \lor \alpha, 1), (\alpha, *)(p(MP)_{3}))
\]
forms the required \( p \)-proof. □

To show that \( p \)-consequence \( Z_{MP}^n \) has for every \( n \in \mathbb{N}, \ k \in \{1, \ldots, n-2\} \)
an adequate set of \( p \)-rules, it is enough to show that there exists a formula \( \Phi^p_k(p) \) fulfilling the assumed requirements (for \( k = n-1, A_1 \) together with \( p(MP)_1 \) is an adequate axiomatization).

For such \( k \) a function
\[
I_k(x) = \begin{cases} 
1, & \text{for } x \geq \frac{k}{n-1} \\
0, & \text{otherwise} 
\end{cases}
\]
is definable ([4], p.82). It is easy to see that \( \Phi^p_k(x) := x \lor I_k(x) \) fulfills the conditions needed.

So, the set of \( p \)-rules, adequate w.r.t. \( Z_{MP}^n \) can be listed as the sum of the following:
\[
A_1 = \{((\alpha, 1)) : \alpha \in \text{Ax}_n\}
A_2 = \{(((\alpha \rightarrow_{n-2} \neg \alpha) \land (\alpha \lor I_k(\alpha) ) \rightarrow \alpha, *)) : \alpha \in L\}
p(MP)_1 \cup p(MP)_2 \cup p(MP)_3.
\]

In [2] we have given the adequate \( p \)-axiomatization of \( Z_{MP}^2 \), hence the above is a generalization of that result.

References


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