Abstract
A notion of so-called analytic equivalence is considered, in the form of Suszko’s connective of propositional identity. Two axiomatic strengthenings of the Sentential Calculus with Identity of Suszko involving that connective are presented. Both realize the following principle due to Wójcicki: two sentences whose logical forms in sentential language are logically equivalent and have the same sentential variables, express the same proposition.

The paper is devoted to some axiomatic strengthenings of the SCI logic of Suszko. These logics can be viewed as possible formalizations (in the framework of Suszko) of some important notion of the identity of propositions expressed by sentences (or situations described by sentences). This notion was introduced by Wójcicki in [13] (however mentioned earlier in [11, 12] and later independently considered by Vanderveken, cf. for example [8, 9] and Bilat [3, 4]) by means of the principle: two sentences whose logical forms in sentential language are logically equivalent and have the same sentential variables, describe the same situation (that is, according to our understanding, express the same proposition). Wójcicki’s principle (hereafter called WP) on the one hand, is a particularization of the principle considered by Barwise and Perry [1]: two sentences whose logical forms are logically equivalent and have the same extralogical constants, describe the same situation, on the other hand, is a weakening of some criterion of identity of propositions known in Polish literature as Wittgenstein’s principle (cf. [10]): two sentences whose logical forms are logically equivalent express the same proposition. Nowadays Wittgenstein’s principle is rather rejected as leading, in a presence of some other principles, by so-called
slingshot argument (cf. for example [2, 14]) to the paradoxical axiom of Frege: any two sentences having the same logical value describe the same situation. The title of the paper is tied with the title of [7] from which the idea of formalization is adopted. The connective of analytic equivalence which is a counterpart of analytic classical implication considered in [7], has to be conceived in the framework of Suszko as the connective of sentential identity.

1. Preliminaries

In the sequel we will work with the sentential language of the form $L = (L, \neg, \land, \lor, \rightarrow, \iff, \equiv)$, where $\neg, \land, \lor, \rightarrow$ are standard connectives and $\equiv$ is the binary connective interpreted as sentential identity, in the sense that when a formula $\alpha \equiv \beta$ is valid then the same proposition is expressed by both formulae, $\alpha$ and $\beta$. We will also use the symbol $L_{cl}$ for the subset of $L$ composed out of all the formulas in which $\equiv$ does not occur. The basic non-Fregean sentential logic, well known under the name of SCI (sentential calculus with identity), is a consequence relation $\vdash_{SCl}$ defined on the language $L$ by means of the rule Modus Ponens: $\alpha, \alpha \rightarrow \beta / \beta$ and the following axioms:

1. (Cl) a group of axioms defining the classical sentential logic,

   - $\alpha \equiv \alpha$,
   - $\neg \alpha \equiv \neg \beta$,
   - $\alpha \equiv \beta \land \gamma \equiv \delta \Rightarrow (\alpha \land \gamma \equiv \beta \land \delta)$,
   - $\alpha \equiv \beta \Rightarrow (\alpha \iff \beta)$.

The logic $\vdash_{SCl}$ is really basic for comparison of the propositions expressed by formulas, due to its following property: for any $\alpha, \beta \in L$, $\alpha$ is the same formula as $\beta$ whenever $\vdash_{SCl} \alpha \equiv \beta$. In other words, the SCI logic provides the most or fully grained criterion for identity of propositions: different formulas express different propositions.

In the sequel, the following valuational semantics for $\vdash_{SCl}$ will be used (originally the semantics for $\vdash_{SCl}$ provided by Bloom and Suszko [5, 6] is a bundle of some kind of matrices called SCI-models). Given any algebra $A = (A, \neg, \land, \lor, \rightarrow, \iff, \circ)$ similar to the language $L$ and any homomorphism
h : \mathcal{L} \rightarrow \mathcal{A}

consider the class $\text{Val}(\mathcal{A}, h)$ of all the mappings $v : \mathcal{L} \rightarrow \{0, 1\}$ fulfilling for any $\alpha, \beta \in \mathcal{L}$ the following conditions:

$v(\neg \alpha) = 1$ iff $v(\alpha) = 0$,
$v(\alpha \land \beta) = 1$ iff $v(\alpha) = v(\beta) = 1$,
$v(\alpha \lor \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
$v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
$v(\alpha \equiv \beta) = 1$ iff $v(\alpha) = v(\beta)$,
$v(\alpha \equiv \beta) = 1$ iff $v(\alpha) = v(\beta) \& h(\alpha) = h(\beta)$.

It is clear that any such $v$ from $\text{Val}(\mathcal{A}, h)$ is uniquely determined by its values on sentential variables and by the homomorphism $h$. Now, the semantics for $\vdash_{SCI}$ is the class $\text{Val}_{SCI} = \bigcup \{\text{Val}(\mathcal{A}, h) : \mathcal{A} \text{- algebra similar to } \mathcal{L} \& h \in \text{Hom}(\mathcal{L}, \mathcal{A})\}$ in the sense that it defines the consequence relation $\models \subseteq P(\mathcal{L}) \times \mathcal{L}$ in the standard way (that is $X \models \alpha$ iff for each $v \in \text{Val}_{SCI}$, $v(\alpha) = 1$ whenever $v(X) \subseteq \{1\}$) such that $\models = \vdash_{SCI}$. The proof of soundness theorem: $\vdash_{SCI} \subseteq \models$, is of standard matter. The proof of completeness theorem: $\models \subseteq \vdash_{SCI}$, runs along the quite standard lines by application of the Lindenbaum lemma for the logic $\vdash_{SCI}$ and the canonical valuation $v_X \in \text{Val}_{SCI}$ uniquely determined by a maximal theory $X$ of $\vdash_{SCI}$ in the following way. The values on the sentential variables are: $v_X(p) = 1$ iff $p \in X$, for any variable $p$. The quotient algebra $\mathcal{L}/\approx$ (where a congruence relation $\approx$ is defined on $\mathcal{L}$ as follows: $\alpha \approx \beta$ iff $\alpha \equiv \beta \in X$) plays a role of the algebra $\mathcal{A}$ and $h : \mathcal{L} \rightarrow \mathcal{L}/\approx$ is the canonical homomorphism: $h(\alpha) = [\alpha]_{\approx}$, any $\alpha \in \mathcal{L}$.

2. A strengthening of $SCI$ realizing $WP$ in a natural way

Let $\models_1$ be a consequence relation defined on $\mathcal{L}$ by the class $\text{Val}_1 \subseteq \text{Val}_{SCI}$ of all the valuations $v \in \text{Val}_{SCI}$ determined by homomorphisms $h : (\mathcal{L}, \neg, \land, \lor, \rightarrow, \leftrightarrow, \equiv) \rightarrow (\mathcal{A}, id, \cup, \cup, \cup, \cup)$, where $\mathcal{(A, \cup)}$ is any similitude and $id$ is the identity function of $\mathcal{A}$, more precisely, $v \in \text{Val}_1$ iff $v \in \text{Val}(\mathcal{A}, h)$ for some similitude $\mathcal{A} = (\mathcal{A}, id, \cup, \cup, \cup, \cup)$ and a homomorphism $h : \mathcal{L} \rightarrow \mathcal{A}$. First we will show that the axiomatic counterpart $\vdash_1$ of $\models_1$ is an axiomatic strengthening of $SCI$ defined as follows:
where \( \beta \) follows the theses of
\[
(Ax7) \ ((\alpha || \beta) \land (\gamma || \delta)) \rightarrow ((\alpha \equiv \gamma) || (\beta \equiv \delta)),
\]
\[
(Ax8) \ (\alpha \equiv \beta) || \alpha,
\]
\[
(Ax9) \ (\alpha \equiv \beta) || (\beta \equiv \alpha),
\]
\[
(Ax10) \ (\alpha \equiv (\beta \equiv \gamma)) || ((\alpha \equiv \beta) \equiv \gamma).
\]
\[
(Ax11) \ (\alpha \equiv \beta) \rightarrow ((\alpha \leftrightarrow \beta) \land (\alpha || \beta)),
\]
\[
(Ax12) \ ((\alpha \leftrightarrow \beta) \land (\alpha || \beta)) \rightarrow (\alpha \equiv \beta),
\]
where \( || \) is a new binary connective of the form: \( \alpha || \beta =_{\text{def}} (\alpha \equiv \alpha) \equiv (\beta \equiv \beta) \). Notice that this connective is important due to its semantic behaviour: for any \( v \in Val \):
\[
v(\alpha || \beta) = 1 \ \text{iff} \ h(\alpha) = h(\beta),
\]
where the homomorphism \( h \) determines this \( v \).

The proof of soundness theorem is straightforward. In proving the completeness the following theses of \( \vdash_1 \) will be useful:
\[
(t1) \ \alpha || \alpha,
\]
\[
(t2) \ (\alpha || \beta) \rightarrow (\beta || \alpha),
\]
\[
(t3) \ ((\alpha || \beta) \land (\beta || \gamma)) \rightarrow (\alpha || \gamma).
\]
Obviously, \( t1 \) is a consequence of \( Ax1 \) while \( t2 \) and \( t3 \) follow from the SCI theses: \( (\alpha \equiv \beta) \rightarrow (\beta \equiv \alpha) \) and \( ((\alpha \equiv \beta) \land (\beta \equiv \gamma)) \rightarrow (\alpha \equiv \gamma) \), respectively. The formulae \( t1 \), \( t2 \) and \( t3 \) together with Modus Ponens are responsible for the fact that given any maximal theory \( X \) of the logic \( \vdash_1 \), the relation \( \approx \) defined on \( L \) by means of the clause: \( \alpha \approx \beta \iff \alpha || \beta \in X \), is an equivalence relation. Moreover, one can show that it is in fact a congruence relation of the language. This follows from the property of the theory \( X : \alpha \land \beta \in X \iff \alpha, \beta \in X \), the theses \( t2 \), \( t3 \) and the axioms: \( Ax5, Ax7, Ax6 \). The quotient algebra \( (L/\approx, \neg, \land, \lor, \rightarrow, \leftrightarrow, \equiv) \), due to the axioms \( Ax5, Ax7 \), has the following operations: \( \neg[\alpha] = [\alpha] \) and for any \( f \in \{ \land, \lor, \rightarrow, \leftrightarrow, \equiv \} : [\alpha]/f[\beta] = [\alpha \equiv \beta] \). Hence in fact the algebra is of the form: \( (L/\approx, id, \equiv, \equiv, \equiv, \equiv) \), where obviously \( [\alpha \equiv \beta] = [\alpha \equiv \beta] \). In order to show that \( (L/\approx, \equiv) \) is a semilattice apply \( Ax8 \) to prove the idempotence of the operation \( \equiv \) as well as \( Ax9 \), \( Ax10 \) to prove that \( \equiv \) is commutative and associative, respectively. Finally, consider the canonical homomorphism \( h_0 : L \longrightarrow L/\approx \), that is for any \( \alpha \in L \), \( h_0(\alpha) = [\alpha] \). Then we have the following
Lemma. Given any maximal theory \( X \) of \( \vdash_1 \) for any formulae \( \alpha, \beta \):

\[
\begin{align*}
\neg \alpha & \in X \text{ iff } \alpha \notin X, \\
\wedge \alpha, \beta & \in X \text{ iff } \alpha, \beta \in X, \\
\vee \alpha, \beta & \in X \text{ iff } \alpha \notin X \text{ or } \beta \in X, \\
\rightarrow \alpha \rightarrow \beta & \in X \text{ iff } \alpha \notin X \text{ or } \beta \in X, \\
\leftrightarrow \alpha \leftrightarrow \beta & \in X \text{ iff } (\alpha \in X \text{ iff } \beta \in X), \\
\equiv \alpha \equiv \beta & \in X \text{ iff } (\alpha \in X \text{ iff } \beta \in X) \text{ and } h_0(\alpha) = h_0(\beta).
\end{align*}
\]

Proof. The proof of conditions \((\neg) - (\leftrightarrow)\) is the same as for the classical logic. To show the condition \((\equiv)\) one can use the axioms \((Ax11), (Ax12), Modus Ponens\) and the conditions \((\wedge), (\rightarrow)\) of the lemma.

Now, on the basis of the lemma, given a maximal theory \( X \) of the logic \( \vdash_1 \) one can show that the valuation \( v_X \in Val_1 \) determined by the following values on the sentential variables: \( v_X(p) = 1 \) iff \( p \in X \), and the homomorphism \( h_0 \), is the characteristic function of \( X \) in \( L \), that is for any \( \alpha \in L \), \( v_X(\alpha) = 1 \iff \alpha \in X \). This leads directly to completeness theorem: \( \models_1 \subseteq \vdash_1 \).

The following theorem in a simple way characterizes the logic \( \vdash_1 \):

Theorem 1. For any \( \alpha, \beta \in L : \models_1 \alpha \equiv \beta \iff \models_1 \alpha \leftrightarrow \beta \) and \( V(\alpha) = V(\beta) \), where for each formula \( \gamma \), \( V(\gamma) \) is the set of all sentential variables occurring in \( \gamma \).

Proof. \((\Rightarrow)\): Assume that \( \models_1 \alpha \equiv \beta \) and take any \( v \in Val_1 \). From the assumption it follows that \( v(\alpha \equiv \beta) = 1 \), therefore \( v(\alpha) = v(\beta) \) and, consequently, \( v(\alpha \leftrightarrow \beta) = 1 \), which results in \( \models_1 \alpha \leftrightarrow \beta \). In order to show that \( V(\alpha) = V(\beta) \), first observe that one can treat the symbol \( V \) as naming the homomorphism \( V : (L, \neg, \wedge, \vee, \rightarrow, \equiv) \longrightarrow (P_{fin}(Var), id, \cup, \cup, \cup, \cup, \cup) \) (where \( Var \) is the set of all sentential variables of \( L \) and \( P_{fin}(Var) \) is the family of all finite subsets of \( Var \)) uniquely determined by the following values on sentential variables: \( V(p) = \{p\}, \) any \( p \in Var \). Now, taking into account any valuation \( v' \in Val_1 \) determined by the homomorphism \( V \), according to the assumption it can be obtained that \( v'(\alpha \equiv \beta) = 1 \), so \( V(\alpha) = V(\beta) \).

\((\Leftarrow)\): Suppose that \( (1) \models_1 \alpha \leftrightarrow \beta \) and \( (2) V(\alpha) = V(\beta) \). Consider any \( v \in Val_1 \). Obviously such \( v \) is determined by some homomorphism \( h : (L, \neg, \wedge, \vee, \rightarrow, \equiv) \longrightarrow (A, id, \cup, \cup, \cup, \cup, \cup) \), where \( (A, \cup) \) is a similattice. Then one can show inductively on the length of a formula that for any \( \gamma \in L \), \( h(\gamma) = \)}
$h(p_1) \cup \ldots \cup h(p_n)$, where \{p_1, \ldots, p_n\} = V(\gamma)$. Hence and from (2) it follows that $h(\alpha) = h(\beta)$. Moreover, from (1): $v(\alpha) = v(\beta)$, which together yields: $v(\alpha \equiv \beta) = 1$, so consequently, $\models_1 \alpha \equiv \beta$.

To the end of the section let us point out that a part of WP - the expression "logically equivalent", is pretty ambiguous, so it can be understood in several ways. As it is obvious, due to the Theorem 1, the logic $\vdash_1$ realizes only one way of understanding, maybe the most expected or natural. For "logically equivalent" refers here to the very logic used to establish the identity of propositions. However, the expression may be understood more trivially: two formulas are logically equivalent iff their equivalence is a classical tautology - moreover, a tautology in a complete classical sense, i.e. comprising only standard connectives. Next section will deal with an appropriate logic.

3. A trivial strengthening of SCI realizing WP

Let us start with the very definition of the second axiomatic strengthening of SCI: for any $X \subseteq L$ and $\gamma \in L$: $X \vdash_2 \gamma$ iff $X \cup \{\alpha \equiv \beta : \alpha, \beta \in L_{cl} \land \vdash_{cl} \alpha \leftrightarrow \beta \land V(\alpha) = V(\beta)\} \vdash_{SCI} \gamma$, where $\vdash_{cl}$ is the classical logic. The most important property of the logic $\vdash_2$ is presented in the following

**Theorem 2.** For any formulas $\alpha, \beta \in L_{cl}$, $\vdash_2 \alpha \equiv \beta$ iff $\vdash_{cl} \alpha \leftrightarrow \beta \land V(\alpha) = V(\beta)$.

**Proof.** ($\Leftarrow$): Obvious.

($\Rightarrow$): Assume that $\vdash_2 \alpha \equiv \beta$ for some formulas $\alpha, \beta$ in which the sentential identity $\equiv$ does not occur. So from the definition of $\vdash_2$ we have: $AX \vdash_{SCI} \alpha \equiv \beta$, where $AX = \{\gamma \equiv \delta : \gamma, \delta \in L_{cl}, \vdash_{cl} \gamma \leftrightarrow \delta, V(\gamma) = V(\delta)\}$. So

(1) $AX \models_1 \alpha \equiv \beta$,

where $\models_1$ is the consequence relation determined by the semantics $Val_{SCI}$. In order to show that $\vdash_{cl} \alpha \leftrightarrow \beta$ suppose that it does not hold. Then there exists a classical valuation $v$ such that

(2) $v(\alpha) \not= v(\beta)$.

So extend it to the valuation $v' \in Val_{SCI}$ in the way that $v'(p) = v(p)$ for any $p \in Var$, and $v'$ is determined by the homomorphism $h : L \rightarrow A$, where $A$ is any fixed 1-element algebra similar to $L$. Then,

(3) for any formulas $\gamma, \delta \in L_{cl}$, $v'(\gamma \equiv \delta) = 1$ iff $v(\gamma) = v(\delta)$. 
Hence, $\tilde{v}(AX) = \{1\}$. Thus $v'(\alpha \equiv \beta) = 1$ due to (1), which, according to (3) yields $v(\alpha) = v(\beta)$, a contradiction with (2).

In order to prove that $V(\alpha) = V(\beta)$ consider any valuation $w \in Val_{SCI}$ determined by the homomorphism $V : (L, \neg, \land, \lor, \rightarrow, \leftrightarrow, \equiv) \rightarrow (P_{fin}(Var), id, \cup, \cup, \cup, \cup)$, i.e. for any formulas $\gamma, \delta \in L$, $w(\gamma \equiv \delta) = 1$ iff $w(\gamma) = w(\delta)$ and $V(\gamma) = V(\delta)$. Thus we obtain that $\tilde{w}(AX) = \{1\}$.

Hence and from (1) it follows that $w(\alpha \equiv \beta) = 1$ and consequently, $V(\alpha) = V(\beta)$.

As it could be expected, the former strengthening is stronger than the latter one:

**Theorem 3.** $\vdash_2 \subseteq \vdash_1$.

**Proof.** One has to show that every rule from the inferential base defining the logic $\vdash_2$ is a rule of $\vdash_1$. To that aim it is sufficient to show that $\vdash_1 \alpha \equiv \beta$ for any $\alpha, \beta \in L_{cl}$ such that $\vdash_{cl} \alpha \leftrightarrow \beta$ and $V(\alpha) = V(\beta)$. So, while it is obvious that for any $\alpha, \beta \in L_{cl}$:

$\vdash_1 \alpha \leftrightarrow \beta$ whenever $\vdash_{cl} \alpha \leftrightarrow \beta$, we obtain from Theorem 1 that $\vdash_1 \alpha \equiv \beta$ whenever $\vdash_{cl} \alpha \leftrightarrow \beta$ and $V(\alpha) = V(\beta)$.

**References**


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