We introduce an abstract class of algebras that is a proper reduct of both Pinter’s substitution algebras and cylindric algebras. We show that the class of representable algebras have the amalgamation property.

Pinter’s substitution algebras were introduced by Pinter [13], [14]. Such algebras are obtained from cylindric algebras by dropping the diagonal elements and adding unary operations of substitutions. These substitutions, which are term definable in cylindric algebras, reflect the metalogical operation of substituting in formulas one variable for another, such that the substitution is free. Pinter proved that locally finite algebras are representable, and it seems that not much has been done for such algebras since. We start by defining the concrete versions of Pinter’s algebras. Let $\alpha$ be an ordinal. Let $U$ be a set. Then we define for $i, j < \alpha$ and $X \subseteq {}^\alpha U$:

\[
c_i X = \{ s \in {}^\alpha U : \exists t \in X, t(j) = t(i) \text{ for all } i \neq j \},
\]
\[
s_j^i X = \{ s \in {}^\alpha U : s \circ [i|j] \in X \}.
\]

Here $[i|j]$ is the replacement on $\alpha$ that takes $i$ to $j$ and leaves every other thing fixed. The extra non-boolean operations we deal with are as specified above. For set $X$, let $\mathfrak{B}(X) = (\wp(X), \cup, \cap, \setminus, \emptyset, X)$ be the full boolean set algebra with universe $\wp(X)$. Let $S$ be the operation of forming subalgebras, and $P$ be that of forming products. Then

\[
\text{RSC}_\alpha = SP\{(\mathfrak{B}(^{\alpha}U), c_i, s_j^i)_{i,j<\alpha} : U \text{ is a set }\}.
\]
From now on ordinals considered are infinite. It is not hard to show that $\mathbf{RSC}_\alpha$ is a variety and that is not finite schema axiomatizable. We shall need the notion of superamalgamation due to Maksimova [12].

**Definition 1.**

(1) Let $W$ be a class of algebras having a Boolean reduct. $W$ has the super amalgamation property, or $\text{SUP AP}$ for short, if

(i) for all $A_0, A_1$ and $A_2 \in W$, and all monomorphisms $i_1$ and $i_2$ of $A_0$ into $A_1$, $A_2$, respectively, there exists $A \in W$, a monomorphism $m_1$ from $A_1$ into $A$ and a monomorphism $m_2$ from $A_2$ into $A$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and

(ii) $(\forall x \in A_1)(\forall y \in A_2)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \land i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$.

In this case we say that $A$ is a super amalgam of $A_1$ and $A_2$ over $A_0$, via $m_1$ and $m_2$, or even simply a super amalgam.

(2) A class $K$ contained in $W$ has the (super) amalgamation property with respect to $W$, if the (super) amalgam for algebras in $K$ can be found in $W$.

Recall that $W$ has the amalgamation property or $\text{AP}$ for short, if $W$ satisfies (i). $W$ has the strong $\text{AP}$, or $\text{SAP}$ for short if in addition to (i), we have $m_1 \circ i_1(A_0) = m_1(A_1) \cap m_2(A_2)$. We note that $\text{SUP AP}$ is stronger than $\text{SAP}$ for boolean algebras with extra operations, and for varieties of cylindric algebras, it is strictly stronger, a deep result of Sagi and Shelah [16].

Our main result is:

**Theorem 2.** Let $\alpha \geq \omega$. Then $\mathbf{RSC}_\alpha$ has the $\text{AP}$.

Theorem 2 contrasts the cylindric algebra case [15]. We next define an abstract class that is only an approximation of $\mathbf{RSC}_\alpha$.

**Definition 3.** An algebra in $\mathbf{SC}_\alpha$ is of the form

$\mathfrak{A} = (A, +, \cdot, -, 0, 1, c_i, s_j^i)_{i,j \in \alpha}$

where $(A, +, \cdot, -, 0, 1)$ is a boolean algebra $c_i, s_j^i$ are unary operations on $\mathfrak{A}$ $(i, j < \alpha)$ satisfying the following equations for all $i, j, k, l \in \alpha$

1. $c_i0 = 0, x \leq c_ix, c_i(x \cdot c_iy) = c_ix \cdot c_iy, \text{ and } c_ic_jx = c_jc_ix$,
2. $s^i_jx = x$,
3. $s^j_i x = c_i x$,
4. $s^i_j x = s^j_i x$ whenever $i \neq j$,
5. $c_i s^j_i x = c_j s^j_i x$, whenever $k \notin \{i, j\}$,
6. $s^k_i x = s^j_i x$ whenever $i \neq j$,
7. $s^j_i x = s^k_i x$, whenever $i \neq j$,
8. $s^k_i x = s^j_i x$, whenever $|\{i, j, k, l\}| = 4$.

A key concept in our investigations is that of neat reducts, which we borrow from cylindric algebras [2]:

**Definition 4.** Let $1 < \alpha < \beta$. Let $A = \langle A, +, \cdot, -, c_i, s^j_i \rangle_{i, j \in \beta}$ be a SC$_\beta$. Then the neat $\alpha$ reduct of $A$, in symbols $R_{n}A$, is the SC$_\alpha$ $\langle A, +, \cdot, -, c_i, s^j_i \rangle_{i, j \in \alpha}$. The neat $\alpha$ reduct of $A$, in symbols $L_{n}A$, is the subalgebra of $R_{n}A$ with universe $\bigcup_{\alpha} A = \{x \in A : \Delta x \subseteq \alpha\}$. Here $\Delta x$, the dimension set of $x$, is the set $\{i \in | \text{c}_i x \neq x\}$. For $L \subseteq \text{SC}_\beta$, $L_{n}A = \{L_{n}A : \text{c}_i \in L\}$.

Neat reducts play an essential role in the representation theory of SC’s and related structures like cylindric algebras [6]. Indeed below we shall show that SC$_\alpha$’s enjoy a neat embedding Theorem, namely for any ordinal $\alpha$, $R_{SC_{\alpha}} = S\text{C}_{\alpha}SC_{\alpha + \omega}$. $\omega$ cannot be truncated to finite ordinals for $\alpha > 2$. For $\alpha > 2$ and $\omega > n > 0$ there are non representable algebras in $S\text{C}_{\alpha}SC_{\alpha + n}$. We note that the notion of amalgamation property for representable algebras is closely related to that of neat embeddings [10], [5]. Our next Theorem is the key Theorem in this paper. It is fully proved in [8].

**Theorem 5.** Let $\kappa$ be any ordinal $> 1$. Let $K = \{A \in \text{SC}_{\kappa + \omega} : A = \Sigma A^{\alpha} \bigcap L_{n}A\}$. Then $K$ has SUPAP.

**Proof.** [8]. The proof is similar to that in [7] and [4].

We recall the notion of dimension restricted free algebras.

**Definition 6.** Let $\delta$ be a cardinal. Let $\alpha$ be an ordinal. Let $\alpha \otimes \delta$ be the absolutely free algebra on $\delta$ generators and of type SC$_\alpha$. For an algebra $\mathcal{A}$,
we write $R \in \text{Con}_A$ if $R$ is a congruence relation on $A$. Let $\rho \in ^\delta \varphi(\alpha)$. Let $L$ be a class having the same similarity type as $\text{SC}_\alpha$. Let

$$C_{\delta}^{(\rho)} L = \bigcap \{ R : R \in \text{Con}_\alpha \mathcal{X}_{\delta, \alpha} \mathfrak{X}_\delta / R \in SL, c_k^{\alpha \eta} / R = \eta / R \text{ for each } \eta < \delta \text{ and each } k \in \alpha \setminus \rho \}$$

and

$$\mathfrak{X}_\delta ^\rho L = \mathfrak{X}_\beta / C_{\delta}^{(\rho)} L.$$

The ordinal $\alpha$ does not figure out in $C_{\delta}^{(\rho)} L$ and $\mathfrak{X}_\delta ^\rho L$ though it is involved in their definition. However, $\alpha$ will be clear from context so that no confusion is likely to ensue. For algebras $\mathfrak{A}$, $\mathfrak{B}$ and $X \subseteq \mathfrak{A}$, $\text{Hom}(\mathfrak{A}, \mathfrak{B})$ stands for the class of all homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and $\mathfrak{S}_g X$ when $\mathfrak{A}$ is clear from context, denotes the subalgebra of $\mathfrak{A}$ generated by $X$.

**Definition 7.** Assume that $\delta$ is a cardinal, $L \subseteq \text{SC}_\alpha$, $\mathfrak{A} \in L$, $x = \langle x_\eta : \eta < \beta \rangle \in \delta \mathfrak{A}$ and $\rho \in ^\delta \varphi(\alpha)$. We say that the sequence $x L$ freely generates $\mathfrak{A}$ under the dimension restricting function $\rho$, or simply $x$ freely generates $\mathfrak{A}$ under $\rho$, if the following two conditions hold:

(i) $\mathfrak{A} = \mathfrak{S}_g x$ and $\Delta^\mathfrak{A} x_\eta \subseteq \rho \eta$ for all $\eta < \delta$.

(ii) Whenever $\mathfrak{B} \in L$, $y = \langle y_\eta, \eta < \delta \rangle \in \beta \mathfrak{B}$ and $\Delta^\mathfrak{B} y_\eta \subseteq \rho \eta$ for every $\eta < \delta$, then there is a unique $h \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$ such that $h \circ x = y$.

For a class $L$, $PL$ stands for the operation of forming products of members of $L$. The following Theorem can be easily distilled from the literature.

**Theorem 8.** Assume that $\delta$ is a cardinal, $L \subseteq \text{SC}_\alpha$, $\mathfrak{A} \in L$, $x = \langle x_\eta : \eta < \beta \rangle \in \delta \mathfrak{A}$ and $\rho \in ^\delta \varphi(\alpha)$. Then the following hold:

(i) $\mathfrak{X}_\delta ^\rho L \in \text{SC}_\alpha$ and $x = \langle \eta / C_{\delta}^{(\rho)} L : \eta < \delta \rangle$ $\text{SPL}$- freely generates $\mathfrak{A}$ under $\rho$.

(ii) In order that $\mathfrak{A} \cong \mathfrak{X}_\delta ^\rho L$ it is necessary and sufficient that there exists a sequence $x \in ^\delta \mathfrak{A}$ which $L$ freely generates $\mathfrak{A}$ under $\rho$.

We now prove our Main theorem:

**Proof.** We first prove a Neat Embedding Theorem for infinite $\alpha$. Call $\mathfrak{A} \in \text{SC}_\alpha$ locally finite if $\Delta x$ is finite for every $x \in \mathfrak{A}$. We shall use the fact that
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locally finite algebras are representable. The proof of this can be recovered easily from [1]. Denote the class of all locally finite algebras of dimension $\alpha$ by $\mathbf{LFSC}_\alpha$. Then we have $\mathbf{LFSC}_\alpha \subseteq \mathbf{RSC}_\alpha$. We now show that for any $\beta > \alpha$, the class $L = \mathcal{R}_\alpha \mathcal{SC}_\beta$ is a variety. The proof is similar to [2] 2.6.32. Clearly $L$ is closed under forming subalgebras. If $\mathfrak{B} = \{ \mathfrak{B}_i : i \in I \}$ is a system of $\mathcal{SC}_\beta$s. Then $P_{i\in I}\mathcal{R}_\alpha \mathfrak{B}_i = \mathcal{R}_\alpha(P_{i\in I}\mathfrak{B}_i)$. Now let $\mathfrak{A} \subseteq \mathcal{R}_\alpha \mathfrak{B}$ and $I$ be an ideal of $\mathfrak{A}$. Let $J = \mathfrak{I}^\beta \mathfrak{B}$. Then $J \cap A = I$. Then it is easy to check that $x/I \mapsto x/J$ is an isomorphism from $A/I$ into $\mathcal{R}_\alpha(\mathfrak{B}/J)$. It is easy to see that $\mathbf{RSC}_\alpha \subseteq \mathcal{R}_\alpha \mathcal{SC}_\alpha + \omega$. The other inclusion is harder. It follows from the fact that $\mathcal{R}_\alpha \mathcal{SC}_\alpha + \omega = \mathcal{S}_\mathcal{U}_\mathcal{P} \mathbf{RSC}_\alpha \subseteq \mathcal{S}_\mathcal{U}_\mathcal{P} \mathbf{RSC}_\alpha = \mathbf{RSC}_\alpha$.

Now let $\mathcal{C}$, $\mathfrak{A}$, $\mathfrak{B}$ be in $\mathbf{RSC}_\alpha$. Let $f : \mathcal{C} \to \mathfrak{A}$ and $g : \mathcal{C} \to \mathfrak{B}$ be monomorphisms. We want to find an amalgam. Then by the first part of the proof there exist $\mathfrak{A}^+$, $\mathfrak{B}^+$, $\mathcal{C}^+ \in \mathcal{C}\mathcal{A}_{\alpha + \omega}$, $e_A : \mathfrak{A} \to \mathcal{R}_\alpha \mathfrak{A}^+$, $e_B : \mathfrak{B} \to \mathcal{R}_\alpha \mathfrak{B}^+$, and $e_C : \mathcal{C} \to \mathcal{R}_\alpha \mathcal{C}^+$. We can assume that $\mathcal{G}^B + e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathcal{C}^+$. Let $f(C)^+ = \mathcal{G}^B + e_A(f(C))$ and $g(C)^+ = \mathcal{G}^B + e_B(g(C))$. We claim that there exist $\bar{f} : \mathcal{C}^+ \to f(C)^+$ and $\bar{g} : \mathcal{C}^+ \to g(C)^+$ such that $(e_A \upharpoonright f(C)) \circ f = \bar{f} \circ e_C$ and $(e_B \upharpoonright g(C)) \circ \bar{g} = \bar{g} \circ e_C$. Indeed, let $\mu = |C|$. Let $x$ be a bijection from $\mu$ onto $C$. Let $y$ be a bijection from $\mu$ onto $f(C)$, such that $f(x_j) = y_j$ for all $j < \mu$. Let $\beta = \alpha + \omega$. Let $\rho = \{ \Delta(\mathfrak{A}) x_j : j < \mu \}$, $\mathfrak{F} = \mathfrak{F}^\beta + \mathfrak{S}_\beta$, $g_\xi = \xi/\mathfrak{C}(\rho) \mathcal{S}_\beta$ for all $\xi < \mu$ and $\mathfrak{G} = \mathcal{G}^\beta_{\mathcal{U}_\mathcal{S}_\alpha} \mathfrak{F}^{\nu} \{ g_\xi : \xi < \mu \}$. Then $\mathfrak{G} \subseteq \mathcal{R}_\alpha \mathfrak{F}$, and $\mathfrak{G}$ generates $\mathfrak{F}$. By properties of dimension restricted free algebras, there exist $h \in \text{Hom}(\mathfrak{F}, \mathcal{C}^+)$ and $h' \in \text{Hom}(\mathfrak{F}, f(C)^+)$ such that $h(g_\xi) = e_C(x_\xi)$ and $h'(g_\xi) = e_A \upharpoonright (f(C))(y_\xi)$ for all $\xi < \mu$. Because $\mathcal{C}$ generates $\mathcal{C}^+$ and $f(C)$ generates $f(C)^+$ it follows that $h$ and $h'$ are both onto. We now have $e_C \circ f^{-1} \circ (e_A \upharpoonright f(C))^{-1} \circ (h' \upharpoonright \mathfrak{F}) = h \upharpoonright \mathfrak{F}$.

Therefore $Ker h' \cap \mathfrak{F} = Ker h \cap \mathfrak{F}$. Let $I = Ker h'$. For an ideal $X \subseteq \mathfrak{F}$, $\mathcal{G}^\beta_X X$ denotes the ideal generated by $X$. Then we claim that $\mathcal{G}^\beta \{ I \cap \mathfrak{F} \} = I$. Clearly $\mathcal{G}^\beta \{ I \cap \mathfrak{F} \} \subseteq I$. Now let $x \in I$. Then since $\mathfrak{F}$ generates $\mathfrak{F}$, we have $\Delta x \cap \alpha$ is finite. Then $c_{\Delta x \cap \alpha} x \in \mathcal{R}_\alpha \mathfrak{F} = \mathcal{R}_\alpha \mathcal{G}^\beta_{\mathfrak{F}} = \mathcal{G}^\beta_{\mathcal{U}_\mathcal{S}_\alpha \mathfrak{F}} = \mathcal{G}^\beta \mathfrak{F} = F$. Therefore $c_{\Delta x \cap \alpha} x \in I \cap F$. But $x \subseteq c_{\Delta x \cap \alpha} x$, hence $x \in \mathcal{G}^\beta (I \cap \mathfrak{F})$. It follows that $Ker h' = Ker h$. Let $y \in \mathcal{C}^+$, then there exists $x \in \mathfrak{F}$ such that $y = f(x)$. Define $\bar{f}(y) = h'(x)$. The map is well defined and is as required. Similarly there exists $\bar{g} : \mathcal{C}^+ \to g(C)^+$ such that
$(e_B \upharpoonright g(C)) \circ g = g \circ e_C$. Now $K$ (in Theorem 3) has $SUPAP$, hence there is a $D^+$ in $K$ and $k : A^+ \rightarrow D^+$ and $h : B^+ \rightarrow D^+$ such that $k \circ f = h \circ g$. Let $D = \mathfrak{N}_\alpha D^+$. Then $k \circ e_A : A \rightarrow \mathfrak{N}_\alpha D$ and $h \circ e_B : B \rightarrow \mathfrak{N}_\alpha D$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$.


References


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