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A NOTE ON THE NON-INVOLUTIVE ROUTLEY STAR

Abstract

In this note, we define a series of logics included in R-Mingle and without the axiom of elimination of double negation.

1. Introduction

As is well-known, "the Routley-Star" is the operator by which negation is explained in standard semantics for relevant logics (see [6]). Not less well-known is the fact that negation in these logics is involutive in the sense that the double negation axioms $A \to \neg\neg A$ (dn1) and $\neg\neg A \to A$ (dn2) are valid.

Now, in [9], R. Sylvan (formerly, Routley) and V. Plumwood define the logic $\mathrm{B_M}$ and some of its extensions in two and a half really significant pages.

When negation is present, the logic $\mathrm{B_M}$ is in fact the *basic logic* in (Routley and Meyer) ternary relational semantics in the same sense that $\mathrm{B_+}$ (see [5]) is the basic positive (i.e., without negation) logic in the same semantics.

In $\mathrm{B_M}$, neither dn1 nor dn2 hold. And in this note we are interested in extensions of $\mathrm{B_M}$ without dn2 when negation is represented with the Routley Star. In particular, its aim is to study the logic $\mathrm{RMO_{lcNI}}$. This logic is the result of adding to the positive fragment $\mathrm{R_+}$ of relevance logic R, the mingle axiom (see [1]) $A \to (A \to A)$, the LC axiom $(A \to B) \vee (B \to A)$, (see [3]) the weak contraposition axiom $(A \to \neg B) \to (B \to \neg A)$ and the principle of "tertium non datur" $A \vee \neg A$.

It is shown that the axiom dn2 is not provable in $\mathrm{RMO}_{\mathrm{lcNI}}$ (i.e., $\mathrm{RMO}_{\mathrm{lc}+}$ with a non-involutive negation). On the other hand, a Routley-Meyer semantics is provided for $\mathrm{RMO}_{\mathrm{lcNI}}$, although, it is to be remarked, this semantics is in fact present in (or is easily derived from) [7] and/or [9].

In §5 we shall briefly discuss a strong extension of R_+ with the non-constructive "reductio" axioms but without dn2 that seems not to be representable in the present semantical framework.

Knowledge of Routley-Meyer semantics for relevant logics is presupposed.

2. The logic B_M and its semantics

Routley and Meyer's basic positive logic B_+ (see [5] or [7]) can be axiomatized as follows:

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules:

Modus ponens (MP): $(\vdash A \to B \;\&\vdash A) \Rightarrow\; \vdash B$

Adjunction (Adj): $(\vdash A \;\&\vdash B) \Rightarrow\; \vdash A \wedge B$

Suffixing (Suf): $\vdash (A \to B) \Rightarrow\; \vdash (B \to C) \to (A \to C)$

Prefixing (Pref): $\vdash (B \to C) \Rightarrow\; \vdash (A \to B) \to (A \to C)$

Then, Sylvan and Plumwood's logic B_M is the result of adding to B_+ the axioms

A7. $\neg(A \wedge B) \to (\neg A \vee \neg B)$

A8. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

and the rule

Contraposition (con): $\vdash A \to B \Rightarrow\; \vdash \neg B \to \neg A$

Next, we define the semantics.

DEFINITION 1. A B_{M}-*model* is a structure $\langle K,\ O,\ R,\ *,\ \vDash\rangle$ where O is a non-empty subset of K, R is a ternary relation on K, and $*$ a unary operation on K subject to the following definitions and postulates for all a, b, c, $d \in K$:

$$\begin{aligned}&\text{d1. } a \leq b =_{df} (\exists x \in O)Rxab\\ &\text{P1. } a \leq a\\ &\text{P2. } (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd\\ &\text{P3. } a \leq b = b* \leq a*\end{aligned}$$

Finally, $\vDash$ is a (valuation) relation from K to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables p, wff A, B and $a \in K$

(i). $(a \leq b\ \ \&\ \ a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$
(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv). $a \vDash A \to B$ iff for all b, $c \in K$ $(Rabc\ \&\ b \vDash A) \Rightarrow c \vDash B$
(v). $a \vDash \neg A$ iff $a* \nvDash A$

A formula A is $\mathrm{B_M}$ *valid* ($\vDash_{\mathrm{B_M}} A$) iff $a \vDash A$ for all $a \in O$ in all $\mathrm{B_M}$-models.

Next, we sketch a proof of the soundness and completeness theorems (it is still more summarily sketched in [9]).

In order to prove soundness, the two following lemmas are (significant and) useful (see, e.g., [7]).

LEMMA 1. *For any wff A and a, $b \in K$, $(a \leq b\ \ \&\ \ a \vDash A) \Rightarrow b \vDash A$.*

PROOF. Induction on the length of A. The conditional case is proved with P2, and the negation case with P3. □

LEMMA 2. *For any wff A, B, $\vDash_{B_M} A \to B$ iff $a \vDash A \Rightarrow a \vDash B$ for all $a \in K$ in all B_M-models.*

PROOF. By lemma 1 and P1 (with d1). □

Then, by using lemmas 1, 2, it is easily proved (see, e.g., [7]):

THEOREM 1. [Soundness of $\mathrm{B_M}$] *If $\vdash_{B_M} A$, then $\vDash_{B_M} A$.*

Regarding completeness:

DEFINITION 2. The B_{M}-*canonical model* is the structure $\langle K^C, O^C, R^C, *^C, \vDash^C\rangle$ where K^C is the set of all prime theories, O^C is the set of all regular prime theories and R^C, $*^C$ and $\vDash^C$ are defined as follows. R^C: for any $a, b, c \in K^C$, $R^C abc$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$ for any wff A, B. $*^C$: for any $a \in K^C$, $a*^C = \{A \mid \neg A \notin a\}$. $\vDash^C$: for any $a \in K^C$, $a \vDash^C A$ iff $A \in a$.

A *theory* is a set of formulas closed under adjunction and $\mathrm{B_M}$-entailment; and the terms "prime" and "regular" are understood in the standard sense (see, e.g., [7]).

Then, the three essential lemmas are (cf., e.g., [7]):

LEMMA 3. *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

PROOF. (a) From left to right: it is immediate. (b) Given that any theory is closed by $\mathrm{B_M}$-entailment, the proof consists in extending $\mathrm{B_M}$ to a (regular) prime theory x such that $R^C xaa$, and, so, $R^C xab$. $\square$

LEMMA 4. *$*^C$ is an operation on K^C.*

PROOF. By A7, A8 and con. $\square$

LEMMA 5. *If A is not a theorem of B_M, then A fails to belong to some regular, prime theory.*

PROOF. By a "maximizing" argument (see, e.g., [7]). $\square$

Then, the $\mathrm{B_M}$-canonical model is immediately shown to be a model and, moreover, by lemma 5, we have:

THEOREM 2. [Completeness of $\mathrm{B_M}$] *If $\vDash_{B_M} A$, then $\vdash_{B_M} A$.*

3. The logic $\mathbf{RMO_{lcNI}}$

The positive fragment of Relevance Logic R, R_+ can be axiomatized as follows (cf., e.g., [1]): A1-A6 plus:

$$\text{A9. } (A \to B) \to [(B \to C) \to (A \to C)]$$
$$\text{A10. } [A \to (A \to B)] \to (A \to B)$$

$$\text{A11.}\ A \to [(A \to B) \to B]$$

The rules of derivation are MP and Adj. Then, the logic RMO_{+} (R_{+} plus the mingle axiom) is R_{+} plus

$$\text{A12.}\ A \to (A \to A)$$

Next, $\mathrm{RMO}_{\mathrm{lc}+}$ is the result of adding the RMO_{+} the LC axiom

$$\text{A13.}\ (A \to B) \vee (B \to A)$$

Finally, $\mathrm{RMO}_{\mathrm{lcNI}}$ is axiomatized by adding to $\mathrm{RMO}_{\mathrm{lc}+}$ the following axioms: A7 and

A14. $(A \to B) \to (\neg B \to \neg A)$
A15. $A \to \neg\neg A$
A16. $A \vee \neg A$

Some theorems and rules of inference of $\mathrm{RMO}_{\mathrm{lcNI}}$ are (a proof is sketched to the right of each one of them):

T1. $(A \to \neg B) \to (B \to \neg A)$	A14, A15
T2. $(\neg A \vee \neg B) \to \neg(A \wedge B)$	A14
T3. $\neg(A \vee B) \to (\neg A \wedge \neg B)$	A14, T1
T4. $(A \to \neg A) \to \neg A$	By R_{+} and T1
T5. $(A \to B) \to [(A \to \neg B) \to \neg A]$	A14, T15
T6. $(A \to \neg B) \to [(A \to B) \to \neg A]$	By R_{+}, T5
T7. $(A \to \neg B) \to \neg(A \wedge B)$	T5
T8. $(A \to B) \to \neg(A \wedge \neg B)$	T6
T9. $\neg(A \wedge \neg A)$	T8
T10. $\vdash \neg A \to A \Rightarrow\ \vdash A$	A16
T11. $(\vdash \neg A \to B\ \ \&\ \ \vdash A \to B) \Rightarrow\ \vdash B$	A14, T11

We note the following:

REMARK 1. (a) $\mathrm{B_M}$ is, of course, included in $\mathrm{RMO}_{\mathrm{lcNI}}$: A8 is provable (cf. T3). (b) $\mathrm{RMO}_{\mathrm{lc}+}$ plus A7, A14 and A15 is a sublogic of Dummett's LC (see [3]). (c) $\mathrm{RMO}_{\mathrm{lcNI}}$ is not, of course, included in R, but it is included in R-Mingle (cf., e.g., [1]).

Now, we prove the following:

PROPOSITION 1. *The strong double negation axiom is not a theorem of RMO_{lcNI}.*

PROOF. By MaGIC, the matrix generator developed by J. Slaney (see [8]). □

Therefore, notice that, for example, the following are not derivable in $\mathrm{RMO_{lcNI}}$: (a) the non-constructive reductio axioms as, e.g., $(\neg A \to \neg B) \to [(\neg A \to B) \to A]$, $(\neg A \to B) \to [(\neg A \to \neg B) \to A]$, $(\neg A \to A) \to A$. (b) The non-constructive contraposition axioms as e.g., $(\neg A \to \neg B) \to (B \to A)$, $(\neg A \to B) \to (\neg B \to A)$, $B \to [(\neg A \to \neg B) \to A]$, $\neg B \to [(\neg A \to B) \to A]$. Moreover, we note the following:

REMARK 2. If either dn2 or any of theses listed in (a) and (b) above is added to $\mathrm{RMO_{lcNI}}$, the resulting logic is equivalent to R-Mingle.

Next, we provide a semantics for $\mathrm{RMO_{lcNI}}$.

4. Semantics for $\mathbf{RMO_{lcNI}}$

DEFINITION 3. An $\mathrm{RMO_{lcNI}}$ -model is defined, similarly, as a $\mathrm{B_M}$-model except that the following definition and postulates are added:

d2. $R^2abcd =_{df} (\exists x \in K)(Rabx \ \ \& \ \ Rxcd)$

P4. $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \ \& \ \ Rbxd)$

P5. $Rabc \Rightarrow R^2abbc$

P6. $Rabc \Rightarrow Rbac$

P7. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$

P8. $(a \in O \ \ \& \ \ Rabc \ \ \& \ \ Rade) \Rightarrow (b \leq e \text{ or } d \leq c)$

P9. $Rabc \Rightarrow Rac * b*$

P10. $a \leq a **$

P11. $a \in O \Rightarrow a* \leq a$

As in the case of $\mathrm{B_M}$, A formula A is $\mathrm{RMO_{lcNI}}$ *valid* ($\vDash_{\mathrm{RMO_{lcNI}}} A$) iff $a \vDash A$ for all $a \in O$ in all $\mathrm{RMO_{lcNI}}$ -models.

Now, given the soundness and completeness of $\mathrm{B_M}$, it is clear that those of $\mathrm{RMO_{lcNI}}$ follow immediately from the following lemma:

LEMMA 6. *Given the logic B_M and B_M-semantics, postulates P4, P5, P6, P7, P8, P9, P10 and P11 are the corresponding postulates (c.p) to, respectively, A9, A10, A11, A12, A13, A14, A15 and A16.*

PROOF. The $\mathrm{RMO_{lcNI}}$-canonical model is defined in a similar way to which the $\mathrm{B_M}$-model was, its items being now referred, of course, to $\mathrm{RMO_{lcNI}}$-theories. Then, we have to prove that, given the logic $\mathrm{B_M}$ and $\mathrm{B_M}$-semantics, each axiom is proved $\mathrm{RMO_{lcNI}}$-valid with the c.p, and this one is proved $\mathrm{RMO_{lcNI}}$-canonically valid with the corresponding axiom. Now, that this is the case for P4 (A9), P5 (A10), P6 (A11), P9 (A14), P10 (A15) is proved in (or can easily be derived from) [7]. So, let us prove that P7, P8 and P11 are the c.p to A12, A13 and A16, respectively. We begin by proving:

(1). *P8 is the c.p to A13:* (a) Suppose that for wff A, B, $a \nvDash A \to B$, $a \nvDash B \to A$ for $a \in O$ in some model. Then, $b \vDash A$, $d \vDash B$, $c \nvDash B$, $e \nvDash A$ for b, c, d, $e \in K$ such that $Rabc$ and $Rade$. By P8, either $b \leq e$ or $d \leq c$. So, by lemma 1, either $e \vDash A$ or $c \vDash B$, a contradiction. (b) Suppose for $a \in O^C$ and b, c, d, $e \in K^C$ such that $R^C abc$ and $R^C ade$, that there are wff A, B such that $A \in b$, $B \in d$, $A \notin e$, $B \notin c$. As a is regular, $(A \to B) \vee (B \to A) \in a$ by A13; as a is prime, $A \to B \in a$ or $B \to A \in a$. So, $B \in c$ ($R^C abc$, $A \in b$) or $A \in e$ ($R^C ade$, $B \in d$), a contradiction.

(2). *P10 is the c.p to A16*: (a) Suppose that for some wff A and $a \in O$ in some model, $a \nvDash A \vee \neg A$. Then $a \nvDash A$ and $a \vDash \neg A$, i.e., $a* \vDash A$. But by P10 and lemma 1, $a \vDash A$, a contradiction. (b) Let $a \in O^C$ and suppose $A \in a*$, i.e., $\neg A \notin a$. As a is regular and prime, $A \in a$ or $\neg A \in a$ by A16. So, $A \in a$, as was to be proved.

The proof that P7 is the c.p to A12 is similar to that of (1) above and is left to the reader. □

Now, before stating the completeness theorem, we note the following proposition in connection with the proof of lemma 6.

PROPOSITION 2. *Given the logic B_+ and B_+ semantics, postulates P4, P5, P6, P7 and P8 are the c.p to A9, A10, A11, A12 and A13, respectively.*

Proof. Regarding P4, P5 and P6, the proof can be found (or is easily derived from) [7]. As for P7 and P8, the proof has implicitly been given above. □

Finally we state:

Theorem 3. [Soundness and completeness of $\mathrm{RMO_{lcNI}}$]
$\vdash_{\mathrm{RMO_{lcNI}}} A$ iff $\vDash_{\mathrm{RMO_{lcNI}}} A$.

The proof of this theorem has been sketched above.

5. The logic $\mathbf{R_{MNI}}$

The logic $\mathrm{R_M}$, i.e., $\mathrm{R_+}$ plus the minimal negation definable with the Routley star (cf. §2) is the result of adding A7, A8 and con to $\mathrm{R_+}$. Or, equivalently, the result of adding A9, A10 and A11 to $\mathrm{B_M}$ (of course, rules Suf and Pref are not then independent). The logic $\mathrm{R_{MNI}}$ (i.e., $\mathrm{R_M}$ plus a non-involutive negation) is axiomatized by adding to $\mathrm{R_M}$ the constructive contraposition axiom T1 $(A \to \neg B) \to (B \to \neg A)$ and the (non-constructive) reductio axiom A17 $(\neg A \to \neg B) \to [(\neg A \to B) \to A]$. It is clear that T1-T11 (of $\mathrm{RMO_{lcNI}}$) are theorems of $\mathrm{R_{MNI}}$. Moreover, A15 is immediate by T1, and A16, by A17. So, $\mathrm{R_{MNI}}$ is a strong extension of $\mathrm{R_+}$ that can intuitively be described as having (a) the principles of non-contradiction and of excluded middle (T9, A16), (b) the constructive contraposition axioms (A14, T1) and introduction of double negation (A15), (c) the De Morgan laws (A7, A8, T2, T3), (d) the constructive reductio axioms (T4-T8) and (c) the non-constructive reductio axioms: A17 as well as $(\neg A \to B) \to [(A \to B) \to B]$ and $(\neg A \to A) \to A$. However, it is proved:

Proposition 3. *Thesis dn2 (so, the non-constructive contraposition axioms —cf. proposition 1—) is not derivable in R_{MNI}.*

Proof. By MaGIC. □

Now, in [4], corresponding postulates are provided for each one of the non-constructive reductio axioms in the context of Routley and Meyer's basic positive logic B (cf., e.g., [7]) plus the contraposition axiom A14. Unfortunately, these postulates are not adequate if P12 $a** \leq a$ (i.e., if dn2) is not present. Therefore, it seems not possible to provide adequate models for $\mathrm{R_{MNI}}$ in the present semantical framework.

To end this note, we remark that $\mathrm{RMO}_{\mathrm{lcNI}}$ and $\mathrm{R}_{\mathrm{MNI}}$ are, of course, independent logics: $\mathrm{R}_{\mathrm{MNI}}$ is included in relevance logic R, but $\mathrm{RMO}_{\mathrm{lcNI}}$ is not (cf. Remark 1); and $\mathrm{R}_{\mathrm{MNI}}$ is not included in $\mathrm{RMO}_{\mathrm{lcNI}}$ (cf. Proposition 1).

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References

[1] A. R. Anderson, N. D. Jr. Belnap, **Entailment. The Logic of Relevance and Necessity**, Vol I. Princeton University Press (1975).

[2] R. T. Brady (ed.), **Relevant logics and their rivals**, Vol. II, Ashgate (2003).

[3] M. Dummett, *A propositional calculus with a denumerable matrix*, **The Journal of Symbolic Logic**, 24 (1959), pp. 97-106.

[4] G. Robles, J. M. Méndez, *The logic B and the reductio axioms*, **Bulletin of the Section of Logic**, 33/2 (2004), pp. 87-94.

[5] R. Routley, R. K. Meyer, *Semantics of Entailment III*, **Journal of Philosophical Logic**, 1 (1972), pp. 192-208.

[6] R. Routley, R. K. Meyer, *Semantics of Entailment I*, in Leblanc, H. (ed.), **Truth, Syntax and Modality**, North Holland (1973), pp. 199-243.

[7] R. Routley, R. K. Meyer, V. Plumwood, R. T. Brady , **Relevant Logics and their Rivals**, Vol. 1, Atascadero, CA, Ridgeview Publishing Co. (1982).

[8] J. Slaney, **MaGIC, Matrix Generator for Implication Connectives: Version 2.1, Notes and Guide**, Canberra, Australian National University (1995). URL: http:// users.rsise.anu.edu.au/~jks

[9] R. Sylvan, V. Plumwood, *Non-normal relevant logics*, in Brady (ed.), **Relevant logics and their rivals**, Vol.II, Ashgate (2003).

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