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WEAKLY ASSOCIATIVE RELATION ALGEBRAS HOLD
THE KEY TO THE UNIVERSE

Abstract
Maddux observed a tantalisingly close connection between certain relation algebras and relevant logics $R$ and $RM$. He asks whether this connection amounts to full interpretability. Although unable to answer that question, we prove that a version of positive minimal relevant logic $B$ is fully interpretable in the variety of weakly associative relation algebras.

1. Two versions of $B$

The logic $B$, of whose two versions we shall speak presently, is the minimal relevant logic of Routley and Meyer (cf. e.g., [1]). We will consider $B$ as formulated in the language $\{\rightarrow, \land, \lor\}$, although noting of essence would change if we adopted some poorer or richer language, e.g., $\{\rightarrow, \land\}$, $\{\rightarrow, \land, \top\}$, $\{\rightarrow, \land, \lor\}$, for all these are conservative extensions of purely implicational $B$.

The two versions are the sequent version, where the basic logical entity is a sequent, which in the present context is just a pair of formulae $(\alpha, \beta)$, written $\alpha \leq \beta$ (with no contexts and no empty sides allowed). The formula version on the other hand, has formulae as basic entities.

Semantically, the two versions differ thus. The sequent version is complete with respect to ternary frames, which are structures $(K, R)$ with $K$ a nonempty set and $R$ a ternary relation on $K$. The fact that there are no restrictions whatever on $R$, distinguishes $B$ as basic relevant logic. However,
our concern here will primarily be with the formula version of B. This is complete with respect to structures one might call frames with right\textsuperscript{1} unit, namely, structures \((K, R, e)\) such that \((K, R)\) is a frame, \(e \in K\), and the following condition holds:

\[
Reab \text{ if and only if } a = b.
\]

Defining \(\alpha \leq \beta\) as \(e \models \alpha \rightarrow \beta\), we recover the sequent version from the formula version. And the recovery is conservative in the sense that a sequent \(\alpha \leq \beta\) holds in the sequent version of B if and only if it holds in the formula version. To prove it, first observe that for a frame \((K, R, e)\) with right unit \(e\), the structure \((K \setminus \{e\}, R|_{K \setminus \{e\}})\) is a frame. Conversely, for any frame \((K, R)\), put \(K' = K \cup \{e\}\), for some \(e \notin K\) and define \(R'\) as \(R \cup \{(e, a, a) : a \in U\}\). Then \((K', R', e)\) is a frame with right unit. The rest follows.

The story just told is folklore and there would be no reason at all to recall it, were it not for the augmenting construction above. For it turns out that similar constructions can do a little more.

2. Unital frames

First we spell out a Hilbert style presentation of B. Here are the axioms

1. \(\alpha \rightarrow \alpha\)
2. \((\alpha \land \beta) \rightarrow \alpha\)
3. \((\alpha \land \beta) \rightarrow \beta\)
4. \(((\alpha \rightarrow \beta) \land (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \land \gamma))\)
5. \(\alpha \rightarrow (\alpha \lor \beta)\)
6. \(\beta \rightarrow (\alpha \land \beta)\)
7. \(((\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma)) \rightarrow ((\alpha \lor \beta) \rightarrow \gamma)\)
8. \((\alpha \land (\beta \lor \gamma) \rightarrow ((\alpha \land \beta) \lor (\alpha \land \gamma))\)

and inference rules

\textsuperscript{1}Although \(e\) appears on the left in \(Rexx\), the relation is to be thought of as \(xe = x\), because of a peculiarity in the definition of truth condition for implication that we adopt here for consistency with relevant logic literature. Similar apparent incongruities will appear later, so be warned.
\[ \frac{\alpha \rightarrow \beta}{\beta} \quad \text{(Modus Ponens)} \]
\[ \frac{\alpha \beta}{\alpha \land \beta} \quad \text{(Adjunction)} \]
\[ \frac{\alpha \rightarrow \beta}{(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)} \quad \text{(Suffixing)} \]
\[ \frac{\alpha \rightarrow \beta}{(\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)} \quad \text{(Prefixing)} \]

Valuations are the usual maps from propositional variables to subsets of \( K \) extended to complex formulae via satisfiability conditions. These in turn are standard for conjunction and disjunction, and non-standard for implication, namely

\[ x \models \alpha \rightarrow \beta \quad \text{iff} \quad \forall y, z : Rxyz, \ y \models \alpha \Rightarrow z \models \beta \]

Then, \((K, R, e)\) verifies a formula \( \varphi \), if for all valuations \( v \) we have \( e \models \varphi \), so truth in a frame is determined by truth at the unit.

The system, as we already mentioned, is complete with respect to frames with right unit, but a slightly better completeness theorem is in fact implicit in [2]. We will call a frame \((K, R, e)\) \textit{unital}, if the following conditions hold

1. \( Reab \) if and only if \( a = b \)
2. \( Rbe \) if and only if \( a = b = e \)
3. \( Raeb \) if and only if \( a = b = e \)

\textbf{Theorem 1.} \textit{The logic B is complete with respect to finite unital frames.}

\textbf{Proof.} As right unital frames are a subclass of frames with right unit, they verify B. Let \( \varphi \) be a non-theorem and \((K, R, e)\) a frame with right unit such that \( e \notin v(\varphi) \) for some valuation \( v \). Let \( \Sigma \) be the set of subformulae of \( \varphi \). Define an equivalence relation \( \sim \) on \( K \) putting \( x \sim y \) if \( x \in v(\sigma) \iff y \in v(\sigma) \) holds for all \( \sigma \in \Sigma \). Consider a frame \((F, S, 1)\), where \( F = K/\sim \cup \{1\} \) for some arbitrarily chosen \( 1 \notin K/\sim \), and \( S \) is defined by putting \( Sxyz \) if \( Rabc \) holds for some \( a \in x, b \in y, c \in z \) for triples not involving 1 and then adding \( S1uu \) for all \( u \in F \). This forces (2) and (3). Define a valuation \( v' \) on \( F \) putting, for all variables \( p \)

\[ v'(p) = \begin{cases} 
\{[x]: x \in v(p)\} \cup \{1\}, & \text{if } e \in v(p) \\
\{[x]: x \in v(p)\}, & \text{if } e \notin v(p)
\end{cases} \]

where \([x]\) stands for the \( \sim \) equivalence class of \( x \). Notice that for \( p \in \Sigma \) we have \( x \in v(p) \iff [x] \subseteq v(p) \). We claim that the definitional property above
holds for every $\sigma \in \Sigma$. Inductive proof of this claim is left for the reader (see p. 210 of [2] for hints).

\[ \square \]

3. Well tempered frames

Let $K = (K, R)$ be a frame. Consider the following condition.

$$\forall a, c \in K \exists b \in K : Rabc$$

Intuitively, it says that the “polynomial” $Raxc$, with parameters $a$ and $c$ from $K$ has at least one solution for $x$ in $K$. If a frame satisfies the above condition, we will say that it has solution property. For unital frames, we need a slightly weaker condition. Namely, we will say that a unital frame $(K, R, e)$ has weak solution property, if it satisfies the condition below.

$$\forall a, c \in K \setminus \{e\} \exists b \in K \setminus \{e\} : Rabc$$

Thus, a unital frame has weak solution property iff its $e$-free substructure with the universe $K \setminus \{e\}$ has solution property. It is easy to see that a unital frame has solution property if and only if it has precisely one element; hence the weak property. We will now show how to get a unital frame with weak solution property out of any frame.

For a frame $K = (K, R)$, define an augmented frame to be a structure $K^+ = (K^+, R^+, e, d)$ satisfying the following properties.

- $K^+ = K \cup \{e, d\}$, with $e \neq d$ and $\{e, d\} \cap K = \emptyset$.
- $R^+ = R \cup \{(e, x, x) : x \in K^+\} \cup \{(x, d, y) : x, y \in K \cup \{d\}\}$.

Again, intuitively, $K^+$ arises from $K$ by augmenting it by two elements $e$ and $d$, and then making $e$ into a unit and $d$ into a “dummy” solution, that is, an object that fits the second place of relation $R$ in all situations except those involving the unit. Notice that the augmenting construction can be applied to a frame $(K, R, e)$ with unit $e$, by first diminishing $K$ to $K \setminus \{e\}$ and restricting $R$ accordingly, and then augmenting the resulting frame $(K \setminus \{e\}, R|_{K \setminus \{e\}})$. If $(K, R, e)$ is unital, the augmenting construction restores the original frame completely. The next lemma is immediate.

**Lemma 1.** For any (unital) frame $K$, the augmented frame $K^+$ is unital and has weak solution property.
We will call unital frames with weak solution property well tempered (because they are produced by diminishing and augmenting, of course).

Lemma 2. The logic $\mathsf{B}$ is complete with respect to finite well tempered frames.

Proof. Soundness is clear. Let $\alpha \notin \mathsf{B}$. Then there is a unital frame $K = (K, R, e)$ and a valuation $v$ such that $e \notin v(\alpha)$. We will produce a unital frame with weak solution property that falsifies $\alpha$. Consider the augmented frame $K^+ = (K^+, R^+, e, d)$ and the valuation $v'$ defined by setting $v'(p) = v(p)$ for any propositional variable $p$. By induction on complexity of $\varphi$ it then follows that $v'(\varphi) = v(\varphi)$, for any formula $\varphi$. We will only show the case for $\varphi = \sigma \rightarrow \tau$. By definition and induction hypothesis, we have $x \in v'(\sigma \rightarrow \tau)$ iff $\forall y, z \in K^+: R^+xyz, y \in v(\sigma) \Rightarrow z \in v(\tau)$. If $x \in K$, then $R^+xyz$ holds if either $y, z \in K$ and $Rxyz$, or $y = d$ and $z$ is arbitrary. If the former, then $x \in v(\sigma \rightarrow \tau)$ by definition. If the latter, then $y = d \notin v(\sigma)$ so again $x \in v(\sigma \rightarrow \tau)$ as needed. This shows $v'(\sigma \rightarrow \tau) \subseteq v(\sigma \rightarrow \tau)$. For the converse, suppose $x \in v(\sigma \rightarrow \tau)$. Then, we have $\forall y, z \in K: Rxyz, y \in v(\sigma) \Rightarrow z \in v(\tau)$. Suppose for contradiction there are $a, b \in K^+$ with $R^+xab$ and $a \in v(\sigma)$, but $b \notin v(\tau)$. If $a, b \in K$, this cannot happen, so at least one of $a, b$ must be $d$. Further, if $b = d \neq a$, then $R^+xab$ does not hold, so we have $a = d$. But then, $a = d \notin v(\sigma)$, so contradiction follows. This ends the proof of $v'(\sigma \rightarrow \tau) = v(\sigma \rightarrow \tau)$.

Now go back to our non-theorem $\alpha$. We have just shown that $v'(\alpha) = v(\alpha)$. Thus $e \notin v'(\alpha)$, so $K^+$ falsifies $\alpha$ proving the claim. \qed

4. Frames and atom structures

Let now $(K, R)$ be a frame. Let $K'$ be disjoint copy of $K$ and $e \notin K \cup K'$. Put $W = K \cup K' \cup \{e\}$. Take any bijection $f: K \rightarrow K'$ and define a unary operation $\cdot$ on $W$ by putting $x' = fx$ for $x \in K$, $x' = f^{-1}(x)$ for $x \in K'$, and $e' = e$. Clearly, so defined $\cdot$ satisfies $x' \cdot x = x$.

Now we are going to define a ternary relation $C$ on $W$. For a technical reason, it will be convenient to begin by defining another ternary relation $R' \subseteq K^3$ by putting $R'bc$ iff $R'bec$. Then, we put

- $C' = \{(x, z, y), (z, fy, x), (fx, x, fy), (y, f, extr), (fy, fx, fz): (x, y, z) \in R'\}$,
- $I = \{(x, e, x), (e, x, x), (fx, x, e), (x, fx, e): x \in W\}$,
- $C = R' \cup C' \cup I$. 
Lemma 3. The relation $C$ satisfies the following conditions.

1. $Cxyz$ implies $Cx'zy$,
2. $Cxyz$ implies $Czy'x$,
3. $Cxey$ iff $x = y$.

Proof. Observe that (3) always holds by definition, so we only need to deal with (1) and (2). Suppose $(x, y, z) \in C$. If $(x, y, z) \in R'$, then (1) and (2) hold trivially. We have two more cases to consider.

If $(x, y, z) \in I$, then either $x = z$ and $y = e$, or $x = e$ and $y = z$. The cases are completely analogous, so we only consider the former. Take $(x', y, z) = (x', x, e)$. If $x = e$, then $(e', e, e) = (f e, e, e) \in I \subset C$. If $x \in K$, then $(x', x, e) = (f x, x, e) \in I \subset C$. If $x \in K'$, then $x = fu$ for some $u \in K$ and we have $(x', x, e) = (f^{-1}(fu), fu, e) = (u, fu, e) \in I \subset C$.

For $(x, y, z) \in C'$ we have 25 further cases, but as the proofs are entirely straightforward, we only show one. Suppose $(x, y, z) = (f u, u, f v)$ for some $(u, v, w) \in R'$. Then, we have $(x', z, y) = (f^{-1}(f w), f v, u) = (w, f v, u) \in C' \subset C$. □

Following notation common in relation algebra literature we write $[x, y, z]$ for the set $\{(x', z, y), (z, y', x), (z', x, y'), (y, z', x'), (y', x', z')\}$. It is easy to show that for any triple $(a, b, c) \in [x, y, z]$ we have $(a, b, c) \in C$ iff $[x, y, z] \subseteq C$. Thus, $\{[x, y, z] : (x, y, z) \in C\}$ is a partition of the domain of $C$ and so $C$ is completely specified by the representatives of partition classes. We will refer to $[x, y, z]$ as a cycleset, and to its member triples as cycles. Thus, $C$ is the set of all cycles, and it partitioned into cyclesets. Let us call a relational structure $(U, C, \cdot, \{e\})$ satisfying all properties of Lemma 3, a unital nonassociative atom structure. For more on cycles and atom structures, etc., see [4]. The next lemma is not difficult to derive as a corollary of Lemma 3.

Lemma 4. For a frame $(K, R)$, the structure $(U, C, \cdot, \{e\})$ is a unital nonassociative atom structure.

Less trivially, the same can be shown for unital frames.

Lemma 5. For a unital frame $(K, R, e)$, the structure $(U, C, \cdot, \{e\})$ is a unital nonassociative atom structure.
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Proof. Construct \((U, C, \cdot, \{e\})\) taking \((K \setminus \{e\}, R|_K\setminus\{e\})\) as the initial frame. All we need to show is that \(C\) extends the original \(R'\) (i.e., \(R\) with the two first arguments swapped). As \((K, R, e)\) is unital we have \(R'xey\) iff \(x = y\) and \(R'exy\) iff \(x = y = e\) iff \(R'xye\). Thus, \(R' \subseteq C\) as claimed. \(\square\)

5. Nonassociative and weakly associative relation algebras

A nonassociative relation algebra (NA) is an algebra \((A, \land, \lor, \neg, 0, 1, e)\) with three binary operations \(\land, \lor, ;\), two unary operations \(\neg, \cdot\) and three constants 0, 1, \(e\) such that

- \((A, \land, \lor, \neg, 0, 1)\) is a Boolean algebra,
- \((A, ;, \cdot, e)\) is an involutive groupoid, whose operations are additive normal operators with respect to the Boolean part.
- \(x \land (y ; z) = 0\) iff \(y \land (x ; z) = 0\) iff \(z \land (y ; x) = 0\), known as Peirce’s or triangle laws, hold.

From now on we will usually omit the “multiplication” symbol \(;\) and adopt the convention that it binds strongest. The second strongest is \(\rightarrow\), followed by \(\land\) and \(\lor\), whose binding strength is equal. With these conventions we now expressed the conditions above by means of identities, comprising.

- some finite base of identities for Boolean algebras
- \(xe = x = ex, x^\cdot = x, (xy)^\cdot = y^\cdot x^\cdot\) (involutive groupoid identities)
- \((x \lor y)^\cdot = x^\cdot \lor y^\cdot, x(y \lor z) = xy \lor xz, (x \lor y)z = xz \lor yz\) (additivity of \(\cdot\) and \(;\))
- \(0^\cdot = 0, x0 = 0 = 0x, (\text{normality of } ^\cdot \text{ and } ;)\)
- \(x^\cdot ; (xy) \land y = 0\) (equivalent in presence of the others to triangle laws)

Therefore, the class NA of nonassociative relation algebras is a finitely based variety. These algebras were first defined in [3], where the variety of weakly associative relation algebras (WA) was also introduced. WA is defined relative to NA by the identity

\[(e \land x) ; (1 ; 1) = ((e \land x) ; 1) ; 1\]

The importance of WAs lies partly in the fact (cf. [3]) that they are exactly the subalgebras of relativisations of representable relation algebras. Our
interest in WAs here stems from the following two observations, which are immediate consequences of Theorem 2.2 in [3].

**Lemma 6.** Let $A$ be a weakly associative relation algebra, complete and atomic as a Boolean algebra and such that the identity element $e$ is an atom. Let $\text{At}(A)$ be the set of atoms of $\text{At}(A)$. Define $C \subseteq (\text{At}(A))^3$ by putting $(a, b, c) \in C$ iff $ab \geq c$. Then, the structure $(\text{At}(A), C, \cdot, \{e\})$ is a unital nonassociative atom structure.

**Lemma 7.** Let $U = (U, C, \cdot, \{e\})$ be a unital nonassociative atom structure. Define the algebra $U = (\wp(U), \cup, \cap, \cdot, -1, \emptyset, U, \text{id})$ first setting $\cup, \cap, -1$ and $\emptyset$ to have their usual set-theoretical meaning, and then putting $X \circ Y = \{u \in U : (x, y, u) \in C \text{ for some } x \in X, y \in Y\}$ and $X^{-1} = \{u \in U : u^{-} \in X\}$. Then $U$ is an atomic and complete weakly associative relation algebra whose identity element is an atom.

### 6. Positive logic of WAs

Define a term-operation $\rightarrow$ in WA, putting $x \rightarrow y = \neg(x^\circ \neg y)$. The next lemma is well-known, at least in the context of relation algebras and residuated lattices (see [6] for more on residuated lattices).

**Lemma 8.** The operation $\rightarrow$ is the right residual of $\circ$; that is, in any WA we have $xy \leq z$ if and only if $y \leq x \rightarrow z$.

**Proof.** We have $y \leq x \rightarrow z = \neg(x^\circ \neg z)$ iff $\neg y \geq x^\circ \neg z$ iff $x^\circ \neg z \wedge y = 0$. This, by triangle laws, holds if and only if $xy \wedge \neg z = 0$ iff $xy \leq z$ as claimed.

Surely, the left residual can be defined analogously, but since we wish to remain within the confines of the language of (positive) B, we have no use for it here. Consider the propositional language with the connectives $\wedge$, $\vee$ and $\rightarrow$. The formulae of this language are identified with terms, i.e., elements of the absolutely free algebra in the appropriate signature over a countable set of generators $X$. A valuation on a WA $A$ is any homomorphism from the term algebra to $A$ obtained by extending a map $v:X \rightarrow A$, via the conditions
1. \( v(\alpha \land \beta) = v(\alpha) \land v(\beta) \)
2. \( v(\alpha \lor \beta) = v(\alpha) \lor v(\beta) \)
3. \( v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta) = \neg(v(\alpha)^{\ast}; \neg v(\beta)) \)

As usual, we call the resulting homomorphism \( v \) as well. We say that \( v \) satisfies a formula \( \varphi \) in \( A \) if \( v(\varphi) \geq e \) holds. We say that \( \varphi \) is true or holds in \( A \) if any \( v \) satisfies \( \varphi \) in \( A \). We abbreviate this by \( A \models \varphi \). Finally, \( \varphi \) is true in \( WA \), if for all nonassociative relation algebras \( A \) we have \( A \models \varphi \).

**Lemma 9.** Let \( A \) be a WA. For any formulae \( \alpha \) and \( \beta \), we have \( A \models \alpha \rightarrow \beta \) iff \( A \models \alpha \leq \beta \) in the usual algebraic sense.

**Proof.** Let \( v(\alpha) = a \) and \( v(\beta) = b \) for some valuation \( v \) into \( A \). Then \( a \rightarrow b = \neg(a^{\ast}; \neg b) \). Thus, \( e \leq a \rightarrow b \) iff \( e \leq \neg(a^{\ast}; \neg b) \) iff \( e \neq e \geq a^{\ast}; \neg b \) iff \( a^{\ast}; \neg b \land e = 0 \). By triangle laws this holds if and only if \( xe \land \neg y = 0 \) iff \( x \leq y \) as claimed.

Thus, we can define the logic of WA to be the set of all theorems, i.e., all formulae \( \alpha \) such that \( A \models \alpha \) for any \( A \in WA \). We could define logic in a stronger sense, preferred by algebraic logicians, as the set of derivable inference rules, but we will not do so here. Let \( WAP \) stand for the set of theorems in the language \( \{ \rightarrow, \land, \lor \} \).

**Lemma 10.** Let \( \alpha \) be a theorem of \( B \). Then \( \alpha \) is a theorem of \( WAP \).

**Proof.** All the axioms and rules of \( B \) amount, by Lemmas 8 and 9 to facts about the lattice ordering and residuation. Consider suffixing. Suppose \( a \leq b \) holds. Then \( a(b \rightarrow c) \leq b(b \rightarrow c) \leq c \) and thus by residuation, we get \( b \rightarrow c \leq a \rightarrow c \) as needed. All the others follow with similar ease.

Before we proceed further, it will be useful to make one observation. Let \( (K, R, e) \) be a unital frame and \( (U, C, \bar{\cdot}, \{ e \}) \) the corresponding unital atom structure, as defined in Section 4. For \( A \) and \( B \) subsets of \( K \), we put \( A \rightarrow B = \{ c \in K: (\forall x, y \in K) Rcx, x \in A \Rightarrow y \in B \} \). This is just the truth condition for implication of \( B \) expressed in a funny way. Now for the same \( A \) and \( B \) consider \( A \rightarrow B \) in the algebra \( U \) of Lemma 7.

**Lemma 11.** Let \( A, B \subseteq K \). Then \( A \rightarrow B = (A \rightarrow B)_{|K} \).

**Proof.** Consider \( A \rightarrow B = \neg(A^{\ast}; \neg B) \). This is the set-theoretical complement of \( \{ u \in U: (\exists a \in A, b \notin B) a^{\ast}; b \geq u \} \). Since \( a, b, \) and \( u \) are
atoms, $a^*; b \geq u$ amounts to $a^*; b \cap u \neq \emptyset$, which by triangle laws yields $au \cap b \neq \emptyset$. This means, by definition, that $Caub$ holds. So, $A^*; \neg B = \{ u \in U : (\exists a \in A, b \notin B) Caub \}$. Its complement is $\{ u \in U : (\forall a \in A, b \notin B) \neg Caub \}$. From this, by contraposition and some fiddling, we get that $\neg(A^*; \neg B) = \{ u \in U : (\forall a, b \in U) a \in A, Caub \Rightarrow b \in B \},$ but $A, B \subseteq K$, so it is equal to $\{ c \in K : (\forall a, b \in K) a \in A, Caub \Rightarrow b \in B \}$. But now all $a, b$ and $c$ belong to $K$, and by construction, the relation $C$ on $K$ is just $R$ with the first two arguments swapped. Therefore, the restriction of $\neg(A^*; \neg B)$ to $K$ is $\{ c \in K : (\forall a, b \in K) a \in A, Cacb \Rightarrow b \in B \}$, that is precisely $A \rightarrow^B B$, as claimed.

So far we have not used the weak solution property in any way. It will come into play in the next lemma.

**Lemma 12.** Let $A, B \subseteq K$. Then $A \rightarrow^B B = A \rightarrow B$.

**Proof.** By Lemma 11, we have $A \rightarrow^B B = (A \rightarrow B)|_K$. It thus suffices to show that $A \rightarrow B \subseteq K$. We have

$$A \rightarrow B = \neg(A^*; \neg B)$$

$$= \neg(A^*; ((\neg B \cap K) \cup (\neg B \cap \neg K)))$$

$$= \neg(A^*; (\neg B \cap K) \cup A^*; (\neg B \cap \neg K))$$

$$= \neg(A^*; (\neg B \cap K) \cup A^*; \neg K)$$

where the last equality follows from $B \subseteq K$.

Now we will show that $A^*; \neg K \supseteq \neg K$. Notice first that if $e \in A^*$, the claim follows. Assume $e \notin A^*$. Take an arbitrary $x \in \neg K$. Then $x = u^*$, for some $u \in K \setminus \{ e \}$. Take an $a \in A \subseteq K \setminus \{ e \}$. By weak solution property, there is a $z \in K \setminus \{ e \}$ with $Czau$ (recall that $C$ swaps the first two arguments of $R$). It follows that $Ca^*z^*u^*$ holds, and moreover $a^* \in A^*$, $z^* \in \neg K$ and $u^* = x$. Therefore, $a^*; z^* \geq x$. Since $x$ was arbitrary, the claim follows.

Thus, we finally obtain

$$A^*; (\neg B \cap K) \cup A^*; \neg K \supseteq \neg K$$

$$\neg(A^*; (\neg B \cap K) \cup A^*; \neg K) \subseteq K$$

$$A \rightarrow B \subseteq K$$

as required. \qed
Lemma 13. Let $\alpha$ be a theorem of WAP. Then $\alpha$ is a theorem of B.

Proof. Contrapositive, as usual. Suppose $\alpha$ is not a theorem of B. By Lemma 2, there is a unital frame $(K, R, e)$, possessing weak solution property, and a valuation $v$ such that $e \notin v(\alpha)$. Construct a unital atom structure $(U, C, ', \{ e \})$ as in Lemma 5. Define a valuation $v'$ by putting $v'(p) = v(p)$. We will show by induction on complexity of $\varphi$ that $v'(\varphi) = v(\varphi)$ for any formula $\varphi$.

For propositional variables it is clear. The induction step for $\land$ and $\lor$ is immediate. Suppose $\alpha = \beta \rightarrow \gamma$. Now $v'(\beta \rightarrow \gamma) = v'(\beta) \rightarrow v'(\gamma)$, which by inductive assumption is equal to $v(\beta) \rightarrow v(\gamma)$, with $v(\beta), v(\gamma) \subseteq K$. By Lemma 12, we have $v(\beta) \rightarrow v(\gamma) = v(\beta) \rightarrow v(\gamma)$ and from this—recalling that $\rightarrow_B$ is an abbreviation of the truth condition for $\rightarrow$ appropriate for the original frame—we obtain $v(\beta) \rightarrow_B v(\gamma) = v(\beta \rightarrow \gamma)$. Thus, $v'(\beta \rightarrow \gamma) = v(\beta \rightarrow \gamma)$ as needed.

Now taking the original valuation $v$ with $e \notin v(\alpha)$, we conclude that $e \notin v'(\alpha)$ either. Thus, $\alpha$ is not a theorem of WAP.

Combining Lemmas 10 and 13 we obtain

Theorem 2. The logic WAP is precisely B.

7. Concluding remarks: Maddux’s question

The whole exercise was motivated by R. Maddux’s talk at a recent conference (cf. [5]), where he observed that two well-known relevant logics, R and RM can be given sound relation algebra interpretations. He then asked whether these interpretations are also complete. It can be easily shown that for certain rich languages they are not. For instance if both the truth constant $t$ and negation are present, then relation algebras (or WAs, for that matter) prove $\varphi \rightarrow (t \lor \neg t)$, which is not a theorem of either R or RM. Without the truth constant however; even with negation present, this quick answer does not work, and Maddux’s question concerned the language without the truth constant.
On some reflection it became clear that negation might be another obstacle. For one thing, it is not clear what negation “should be”. Two candidates suggest themselves, depending on the point of view. On the one hand, we may follow the residuated lattice approach and define negation with respect to a dualising element $d$, putting $\sim x = x \rightarrow d$. One natural choice for $d$ is the Boolean complement of the identity element, i.e., $d = \neg e$. Then, we obtain $\sim x = \neg (x \cdot ; d) = \neg (x \cdot ; e) = \neg (x \cdot )$. This is the choice made by Maddux. On the other hand, since relation algebras (or WAs) have Boolean algebra reducts, we also have Boolean negation freely available. Adding Boolean negation to relevant logics is not unprecedented either (cf. [2]).

Unfortunately, either choice had its shortcomings re Maddux. With no proof and no counterexample forthcoming I felt I had no choice but to go the climber’s way and establish a base camp somewhere, which the present paper attempts to do.

One final word on the Key to the Universe. It is not so much the logic $B$, as the correspondence between its relational models and certain models of $\lambda$-calculus with intersection types (and combinators in Curry’s Illative Combinatory Logic). But this is quite another story.

References


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