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José M. Méndez, Gemma Robles and Francisco Salto

# THE BASIC CONSTRUCTIVE LOGIC FOR NEGATION-CONSISTENCY DEFINED WITH A PROPOSITIONAL FALSITY CONSTANT

**Abstract**

The logic $B_{K+}$ is Routley and Meyer's basic positive logic $B_+$ plus the K rule. The logic $B_{Kc4}$ is a negation extension of $B_{K+}$ in which consistency can be understood in the standard sense, i.e. as the absence of any contradiction. The logic $B_{Kc4}$ is a weak logic, but we prove that a definitionally equivalent logic formulated with a falsity constant can be defined.

## 1. Introduction

As it is known, minimal negation arose in the context of intuitionistic logic. The idea is to add a falsity constant $F$ to positive intuitionistic logic $J_+$ and to define negation as follows:

$\neg A =_{df} A \rightarrow F$

As no axioms for $F$ are introduced, it is $J_+$ which, so to speak, takes charge of defining the intrinsic negation in $J_+$. The result is minimal intuitionistic logic $J_m$ (see, e.g. [3]).

Of course, the concept can be generalized. Thus, a "minimal negation" for a given positive logic $L_+$ is the negation we get when it is introduced by means of a falsity constant (and without any axioms for $F$), as above. Obviously, the more powerful the positive logic is, the stronger the negation defined in it will get.

Now, if $L_+$ is not a decidedly weak logic, it is not difficult to find an equivalent logic formulated with a negation connective. Thus, for example, minimal intuitionistic logic $J_m$ and minimal relevance logic $R_m$ can be axiomatized by adding

(i). $(A \to \neg B) \to (B \to \neg A)$

and

(ii). $(A \to \neg A) \to \neg A$

to $J_+$ and to positive Relevance Logic $R_+$, respectively (see [2] and [3]).

Well, this is not so easy a task in the case of weak positive logics. Consider, for example, the logic $B_{+,F}$ defined in [10]. The logic $B_{+,F}$ is the result of introducing a minimal negation in Routley and Meyer's basic positive logic $B_+$. The question is: which extension, if any, of $B_+$ with a negation connective is equivalent to $B_{+,F}$? To discuss this topic with the mere due attention will take us too far from the aim of the present paper.

Now, the logic $B_{Kc1}$ defined in [8] is the basic constructive logic adequate to consistency understood as the absence of the negation of a theorem. That is, and *grosso modo*, consistency in theories whose underlying logic is $B_{Kc1}$ cannot be understood as, say, negation-consistency, but exactly as the absence of the negation of a theorem. In $B_{Kc1}$, negation is introduced with a negation connective. The logic $B_{Kc1}$ is a weak logic, but in [9], the logic $B_{Kc1_F}$, in which negation is introduced by means of a falsity constant, is shown to be definitionally equivalent to $B_{Kc1}$.

The aim of this paper is to define the logic $B_{Kc4_F}$. The logic $B_{Kc4}$ (the logics $B_{Kc1}$- $B_{Kc3}$ are defined in [8]) is the basic constructive logic in the ternary relational semantics without a set of designated points adequate (in the sense explained above) to negation-consistency as understood in the following definition:

Definition 1. Let $a$ be a theory (a theory is a set of formulas closed under adjunction and provable entailment, cf. §5). Then, $a$ is *inconsistent* iff for some wff $A$, $A \wedge \neg A \in a$. A theory is *consistent* iff it is not inconsistent.

The logic $B_{Kc4}$ is *basic* because it is the minimal logic (in the semantics referred to above) for negation-consistency as understood in definition 1, and it is *constructive* because it is endowed with a (weak) type of intuitionistic negation.

Negation is introduced in $B_{Kc4}$ with a negation connective. In [6], it is shown how to extend $B_{Kc4}$, consistency still having to be understood as negation-consistency, within the spectrum delimited by minimal intuitionistic logic.

The logic $B_{Kc4}$ is not a strong logic, but we shall prove that the logic $B_{Kc4_F}$, in which negation is defined via a falsity constant, is definitionally equivalent to $B_{Kc4}$. By using the results in [10], it would not be difficult to define the extensions of $B_{Kc4}$ (in which negation is introduced with a negation connective) considered in [6] with a propositional falsity constant. But this point will not be pursued here.

The structure of the paper is as follows. In §2, the logic $B_{K+}$ is recalled. It is the result of adding the K rule to Routley and Meyer's basic positive logic $B_+$. In §3, the logic $B_{Kc4_F}$ is introduced, in §4 semantics for $B_{Kc4_F}$ is defined, and in §5, completeness in respect of this semantics is proved. In §6, the axiomatization of $B_{Kc4}$ is recalled and some of its theorems are proved. Finally, in §7, the definitional equivalence of $B_{Kc4}$ and $B_{Kc4_F}$ is proved,

## 2. The logic $B_{K+}$

$B_{K+}$ is axiomatized with:

A1. $A \to A$
A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of derivation are:

Modus ponens (MP): $(\vdash A \to B \ \& \vdash A) \Rightarrow \ \vdash B$
Adjunction (Adj): $(\vdash A \ \& \vdash B) \Rightarrow \ \vdash A \wedge B$
Suffixing (Suf): $\vdash A \to B \Rightarrow \ \vdash (B \to C) \to (A \to C)$
Prefixing (Pref): $\vdash (B \to C) \Rightarrow \ \vdash (A \to B) \to (A \to C)$
K: $\vdash A \Rightarrow \ \vdash B \to A$

Therefore, $B_{K+}$ is $B_+$ with the addition of the K rule.

We now define the semantics for $B_{K+}$. A $B_{K+}$ *model* is a triple $\langle K, R, \models\rangle$ where $K$ is a non-empty set and $R$ a ternary relation on $K$ subject to the following definitions and postulates for all $a$, $b$, $c$, $d \in K$ with quantifiers ranging over $K$:

d1. $a \leq b =_{df} \exists x\ Rxab$
d2. $R^2abcd =_{df} \exists x\,(Rabx\ \&\ Rxcd)$
P1. $a \leq a$
P2. $(a \leq b\ \&\ Rbcd) \Rightarrow Racd$

Finally, $\models$ is a valuation relation from $K$ to the sentences of the positive language satisfying the following conditions for all propositional variables $p$, wffs $A$, $B$ and $a \in K$:

(i). $(a \models p\ \&\ a \leq b) \Rightarrow b \models p$
(ii). $a \models A \vee B$ iff $a \models A$ or $a \models B$
(iii). $a \models A \wedge B$ iff $a \models A$ and $a \models B$
(iv). $a \models A \to B$ iff for all $b$, $c \in K$ $(Rabc\ \&\ b \models A) \Rightarrow c \models B$

A formula $A$ is $B_{K+}$ *valid* ($\models_{B_{K+}} A$) iff $a \models A$ for all $a \in K$ in all models.

As it is known, there is a set of "designated points" in the standard semantics for relevance logics (see e.g. [11]). It is in respect of this set that d1 is introduced and wff are evaluated. The absence of this set in $B_{K+}$ semantics (and the corresponding changes in d1 and the definition of validity) are the only but crucial differences between $B_+$ models and $B_{K+}$ models.

The logic $B_{K+}$ is, as it is shown in [7], *the basic positive logic* in the ternary relational semantics when there is not a set of designated points and validity is defined in respect of all points of $K$. That is, $B_{K+}$ is the basic positive logic in the semantics just referred to in the same sense as Routley and Meyer's $B_+$ is the basic positive logic in general ternary relational semantics.

It is proved in [7] that $B_{K+}$ is complete relative to the semantics defined above.

## 3. The logic $\mathrm{B}_{\mathbf{Kc4}_F}$

We add the falsity constant $F$ to the positive language together with the definition:

D$\neg$ : $\neg A \leftrightarrow (A \to F)$

Then, the logic $\mathrm{B}_{\mathrm{Kc4}_F}$ can be axiomatized by adding to $\mathrm{B}_{\mathrm{K}+}$ the following axioms:

A7. $F \to (A \to F)$
A8. $[A \wedge (A \to F)] \to F$

We note the following theorems of $\mathrm{B}_{\mathrm{Kc4}_F}$:

T1. $(A \wedge \neg A) \to \neg (A \to A)$ A7, A8
T2. $[A \to (B \wedge \neg B)] \to \neg A$ A8

T3. $\neg A \to [A \to (A \wedge \neg A)]$

Proof. By the theorem of $\mathrm{B}_{\mathrm{K}+}$:

$(A \to B) \to [A \to (A \wedge B)]$

we have

$(A \to F) \to [A \to (A \wedge F)]$

So, by A7,

$(A \to F) \to [A \to [A \wedge (A \to F)]]$ ■

T4. $[(A \to A) \to F] \to F$

Proof. By the theorem of $\mathrm{B}_{\mathrm{K}+}$ used in the previous proof,

1. $[(F \to F) \to F] \to [(F \to F) \wedge F)]$

So, by A8,

2. $[(F \to F) \to F] \to F$

By A1 and the K rule:

3. $(F \to F) \to (A \to A)$

Now, T4 is immediate from 2 and 3. ■

# 4. Semantics for $\mathbf{B_{Kc4_F}}$

A $B_{\mathrm{Kc4F}}$ *model* is a quadruple $\langle K, S, R, \models\rangle$ where $K$, $R$ and $\models$ are defined similarly as in a $\mathrm{B_{K+}}$ model and $S$ is a subset of $K$. The clauses:

(v). $(a \leq b \ \&\ a \models F) \Rightarrow b \models F$
(vi). $a \models F$ iff $a \notin S$

and the postulates:

P3. $(Rabc \ \&\ c \in S) \Rightarrow a \in S$
P4. $a \in S \Rightarrow (\exists x \in S)\ Raax$

are added to clauses (i)-(iv) and postulates P1-P2.

$A$ is $B_{\mathrm{Kc4}_F}$ *valid* ($\models_{\mathrm{B_{Kc1_F}}} A$) iff $a \models A$ for all $a \in K$ in all models.

We note that $F$ is not valid (in fact, it is insatisfiable): let $\mathcal{M}$ be any model and $a \in S$. Then, $a \not\models F$.

In order to prove soundness, we previously prove the following two lemmas:

Lemma 1. $(a \leq b \ \&\ a \models A) \Rightarrow a \models B.$

Proof. As in the standard semantics (see, e.g. [11]), by induction on the length of $A$. The conditional clause is proved with P2, and the $F$ case, with clause (v).

Lemma 2. $\models_{B_{Kc4_F}} A \to B$ *iff for all* $a \in K$ *in all models,* $a \models A \Rightarrow a \models B$.

Proof. By using lemma 1, P1 and d1 similarly as in the standard semantics. ■

Then, we shall prove soundness of $\mathrm{B_{Kc4_F}}$.

Theorem 1. *If* $\vdash_{B_{Kc4_F}} A$, *then* $\models_{B_{Kc4_F}} A$.

Proof. A1-A6 are proved valid as in $\mathrm{B_{K+}}$ (similarly, as in $\mathrm{B_+}$); the rules MP. Adj., Suf. and Pref. are shown to preserve validity as in $\mathrm{B_{K+}}$ (similarly, as in $\mathrm{B_+}$). That the K rule preserves validity is proved as follows: suppose $\models_{\mathrm{B_{Kc4_F}}} A$, $\not\models_{\mathrm{B_{Kc4_F}}} B \to A$ for some wff $A$, $B$. Then, $a \models B$, $a \not\models A$ for $a \in K$ in some model. But, as $A$ is $\mathrm{B_{Kc4_F}}$ valid, $a \models A$, which contradicts $a \not\models A$ above. Now, it remains to prove that A7 and A8 are valid.

*A7 is valid*: Suppose $a \models F$, $a \not\models A \rightarrow F$ for some wff $A$ and $a \in K$ in some model. Then, $Rabc$, $b \models A$, $c \not\models F$ for $b, c \in K$. So, $c \in S$ and by P3, $a \in S$, which is impossible.

*A8 is valid:* Suppose $a \models A \wedge (A \rightarrow F)$, $a \not\models F$ for some wff $A$ and $a \in K$ in some model. Then, $a \models A$, $a \models A \rightarrow F$ and $a \in S$. By P4, $Raax$ for some $x \in S$. By clause (iv), $F \in x$, which is impossible. ■

## 5. Completeness of $\mathrm{B}_{\mathbf{Kc4}_F}$

First, we state some definitions. A *theory* is a set of formulas closed under adjunction and provable entailment (that is $a$ is a theory if whenever $A$, $B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem and $A \in a$, then $B \in a$); a theory is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory is *regular* iff all theorems of $\mathrm{B}_{\mathrm{Kc4}_F}$ belong to $a$; a theory is *null* iff no wff belongs to $a$. Finally, $a$ is *inconsistent* iff $F \in a$.

Now, we define the canonical model. Let $K^T$ be the set of all theories and $R^T$ be defined on $K^T$ as follows: for all formulas $A$, $B$ and $a, b \in K^T$, $R^T abc$ iff if $A \rightarrow B \in a$ and $A \in b$, then $B \in c$. Further, let $K^C$ be set of all prime non-null theories, $S^C$ the set of all consistent theories and $R^C$ the restriction of $R^T$ to $K^C$. Finally, let $\models^C$ be defined as follows: for any wff $A$ and $a \in K^C$, $a \models^C A$ iff $A \in a$. Then, the $\mathrm{B}_{\mathrm{Kc4}_F}$ *canonical model* is the quadruple $\langle K^C, S^C, R^C, \models^C\rangle$.

In order to prove completeness, we shall need some previous lemmas.

LEMMA 3. *Let $a \in K^T$. Then, $a$ is non-null iff $a$ is regular.*

PROOF. (a) Let $A \in b$ and $B$ be a theorem. By the K rule, $A \rightarrow B$ is a theorem. So, $B \in b$. (b) If $a$ is regular, $a$ is obviously non-null. ■

LEMMA 4. *Let $a$, $b$ be non-null theories, the set $x = \{B \mid \exists A[A \rightarrow B \in a$ & $A \in b]\}$ is a non-null theory such that $R^T abx$.*

PROOF. It is easy to prove that $x$ is a theory such that $R^T abx$. We prove that $x$ is non-null. Let $A \in b$. By lemma 3, $A \rightarrow A \in a$. So, $A \in x$, by $R^T abx$. ■

The following four lemmas are an easy adaptation of the corresponding $B_+$ lemmas (see, e.g. [4]). They are restricted to the case of non-null theories (as it is known, theories are not necessarily non-null in the $B_+$ canonical model and, in fact, in the canonical model of any standard relevance logic).

LEMMA 5. *Let $A$ be a wff, $a$ a non-null element in $K^T$ and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

LEMMA 6. *Let $a$ be a non-null element in $K^T$, $b \in K^T$ and $c$ a prime member in $K^T$ such that $R^T abc$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

LEMMA 7. *Let $a \in K^T$, $b$ a non-null element in $K^T$ and $c$ a prime member in $K^T$ such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

LEMMA 8. *$a \leq^C b$ iff $a \subseteq b$.*

(Note that $b$ and $c$ in lemma 6 and $a$ and $c$ in lemma 7 need not be non-null).

We remark the following corollary of lemma 5:

COROLLARY 1. (Primeness lemma) *If $a$ is a non-null consistent theory, then there is a prime non-null consistent theory $x$ such that $a \subseteq x$.*

PROOF. Suppose $a$ is a non-null consistent theory. Then, $F \notin a$. So, by lemma 5, there is a prime non-null theory $x$ such that $a \subseteq x$ and $F \notin x$. Then, $x$ is consistent. ■

Now, in order to prove the completeness of $B_{Kc4_F}$, we have to prove:

1. $S^C$ is not empty.
2. Postulates P1-P4 hold canonically.
3. Clauses (i)-(vi) are canonically valid.

*1. $S^C$ is not empty:*

PROOF. Let $B_{Kc4_F}$ be the set of its theorems. As $\not\models_{B_{Kc4_F}} F$, $\nvdash_{B_{Kc4_F}} F$, by the soundness theorem, i.e. $F \notin B_{Kc4_F}$. Then, by lemma 5, there is a prime non-null theory $x$ such that $a \subseteq x$ and $F \notin x$. As $x$ is non-null, consistent and prime, $x \in S^C$. ■

*2. Postulates P1-P4 hold canonically:*

PROOF. P1 and P2 are immediate from lemma 8. So, we prove that P3 and P4 hold canonically. It follows immediately from the following lemma. ■

LEMMA 9.

1. *Let $a$, $b$ be non-null theories and $c$ a consistent non-null theory such that $R^T abc$. Then, $a$ is consistent as well.*
2. *Let $a$ be a non-null consistent element in $K^T$. Then, there is some non-null member in $K^T$ such that $R^T aax$.*

PROOF. *Case 1:* Assume the hypothesis of case 1. Suppose by reductio that $a$ is inconsistent, i.e. $F \in a$. By A7, $\vdash_{\mathrm{BKc4}_F} F \to [(F \to F) \to F]$. So, $(F \to F) \to F \in a$. Now, $F \to F \in b$ (cf. lemma 3). Therefore $F \in c$ contradicting the consistency of $c$.

*Case 2:* Let $a$ be a non-null consistent theory. Define the non-null theory $x$ such that $R^T aax$ (cf. lemma 4). Suppose $F \in x$. Then, $A \to F \in a$ for some $A \in a$. Then, $F \in a$ contradicting the consistency of $a$. ■

Next, we prove the canonical adequacy of P3 and P4. They read canonically as follows:

P3: Let $a$, $b \in K^C$, $c \in S^C$ and $R^C abc$. Then, $a \in S^C$.

PROOF. Immediate by lemma 9 (1). ■

P4: Let $a \in S^C$. Then, there is some $x \in S^C$ such that $R^C aax$.

PROOF. Let $a \in S^C$. By lemma 9 (2), there is a non-null consistent theory $y$ such that $R^T aay$. By using corollary 1, there is a prime non-null consistent theory $x$ such that $y \subseteq x$. Obviously, $R^T aax$. ■

*3. The clauses hold canonically:*

PROOF. Clauses (ii), (iii) and (vi) are trivial, and (i) and (v) are immediate by lemma 8. So, let us prove clause (iv):

(a) *If $a \models^C A \to B$, then $(R^C abc$ & $b \models^C A) \Rightarrow c \models^C B$:*

The proof is immediate by definitions.

(b) *If $a \not\models^C A \to B$, then there are $b$, $c \in K^C$ such that $R^C abc$, $b \models^C A$ and $c \not\models^C B$:*

Suppose $a \not\models^C A \to B$. The sets $x = \{B \mid \vdash_{\mathrm{B_{Kc4_F}}} A \to B\}$, $y = \{B \mid \exists C[C \to B \in a \ \&\ C \in x]\}$ are theories such that $R^T axy$. Now, $A \in x$ ($\vdash_{\mathrm{B_{Kc4_F}}} A \to A$, by A1) and $B \notin y$ (if $B \in y$, then $A \to B \in a$ contradicting the hypothesis). As $x$ is non-null, by lemma 4, $y$ is non-null as well. Thus, we have non-null theories $x$, $y$ such that $R^T axy$, $A \in x$, $B \notin y$. Now, by lemma 5, $y$ is extended to a prime theory $c$ such that $y \subseteq c$ and $B \notin c$. Obviously, $R^T axc$. Next, by lemma 7, $x$ is extended to a prime theory $b$ such that $x \subseteq b$ and $R^C abc$. Therefore, we have non-null prime theories $b$, $c$ such that $A \in b$, $B \notin c$ and $R^C abc$, as required. ■

Now, by 1, 2 and 3, we have:

THEOREM 2. (Completeness of $\mathrm{B_{Kc4_F}}$) *If* $\models_{B_{Kc4_F}} A$, *then* $\vdash_{B_{Kc4_F}} A$.

We end this section by proving a proposition on the meaning of $F$:

PROPOSITION 1. *Let* $a \in K^T$. *Then,* $a$ *is inconsistent* ($F \in a$) *iff* $a$ *contains a contradiction.*

PROOF. (a) Suppose $F \in a$. As $\neg F$ is a theorem (A1 and D¬), $\vdash_{\mathrm{B_{Kc4_F}}} F \to \neg F$ by the K rule. So, $\neg F \in a$, and $a$ contains the contradiction $F \wedge \neg F$. (b) Suppose for some wff $A$, $A \wedge \neg A \in a$, then $F \in a$ by A8. ■

Thus, as in $\mathrm{B_{Kc4}}$, $a$ is inconsistent iff $a$ contains a contradiction.

## 6. The logic $\mathbf{B_{Kc4}}$

We add the unary connective $\neg$ to the positive language of $\mathrm{B_{K+}}$. Next, $\mathrm{B_{Kc4}}$ is axiomatized by adding the following axioms to $\mathrm{B_{K+}}$ (see [6]):

A9. $\neg A \to [A \to (A \wedge \neg A)]$
A10. $[B \to (A \wedge \neg A)] \to \neg B$
A11. $(A \wedge \neg A) \to \neg(A \to A)$

We note the following theorems and rules of $\mathrm{B_{Kc4}}$:

| | | |
|---|---|---|
| t1. | $\vdash A \to B \Rightarrow \ \vdash \neg B \to \neg A$ | A9, A10 |
| t2. | $\neg A \to [A \to \neg(A \to A)]$ | A9, A11 |
| t3. | $\neg A \to [A \to \neg(B \to B)]$ | t2, A1, K, t1 |
| t4. | $(A \to \neg A) \to \neg A$ | |

PROOF. By the theorem of $B_{K+}$

$$(A \to B) \to [A \to (A \wedge B)]$$

we have

$$(A \to \neg A) \to [A \to (A \wedge \neg A)]$$

Then, t4 follows by A10. ■

| | | |
|---|---|---|
| t5. | $[B \to \neg(A \to A)] \to \neg B$ | A1, K, t1, t4 |
| t6. | $\neg(A \to A) \to [B \to \neg(A \to A)]$ | A1, K, t1, t3 |
| t7. | $(A \wedge \neg A) \to \neg(B \to B)$ | A1, K, t1, A10 |
| t8. | $[A \wedge [A \to \neg(B \to B)]] \to \neg(B \to B)$ | t5, t7 |

We end this section by introducing $F$ by definition in $B_{Kc4}$:

DEFINITION 2. (D$F$) Let $A$ be a wff. Then, $F \leftrightarrow \neg(A \to A)$.

That is to say, $F$ replaces any wff of the form $\neg(A \to A)$. Now, we recall that in $B_{Kc4}$, inconsistency is defined as follows (see [6]): a theory $a$ is inconsistent iff it contains a contradiction. Then we prove:

PROPOSITION 2. *Let $a \in K^T$. Then, $a$ is inconsistent iff for some wff $A$, $\neg(A \to A) \in a$.*

PROOF. (a) Suppose $a$ is inconsistent, i.e. suppose that $B \wedge \neg B \in a$ for some wff $B$. By t7, $\neg(A \to A) \in a$. (b)Now, suppose $\neg(A \to A) \in a$ for some wff $A$. By the K rule, $\vdash_{B_{Kc4}} \neg(A \to A) \to (A \to A)$. So, $A \to A \in a$. Therefore, $(A \to A) \wedge \neg(A \to A) \in a$. ■

In other words, $a$ is inconsistent iff $F \in a$, as one should expect.

## 7. The definitional equivalence between $B_{Kc4}$ and $B_{Kc4F}$

We shall understand the notion of definitional equivalence as "definitional equivalence via translations" (see, e.g. [5]). For the purposes of the present paper this notion can be explained as follows. Let L1 and L2 be two logics in different languages, t1 the set of terms of L1 absent in L2, and t2, the set of terms of L2 absent in L1. Then, L1 and L2 are definitionally equivalent iff there are definitions of t1 in terms of L2 (Dt1) and definitions of t2 in terms of L1 (Dt2) such that L1 $\cup$\{Dt2\} = L2 $\cup$\{Dt1\} ($x \cup y$ is the deductive closure of the union of $x$ and $y$, and definitions are expressed as a set of suitable biconditionals). It is important to note that it is not sufficient to prove L1 $\subseteq$ L2 $\cup$\{Dt1\} and L2 $\subseteq$ L1 $\cup$\{Dt2\}. It additionally has to be shown that Dt2 is provable in L2 $\cup$ \{Dt1\} and Dt1 is provable in L1 $\cup$ \{Dt2\} (cf. [1]).

Therefore, we have to prove in the present case:

1. $\mathrm{B}_{\mathrm{Kc4}_F} \subseteq \mathrm{B}_{\mathrm{Kc4}} \cup \{\mathrm{D}F\}$.
2. $\mathrm{B}_{\mathrm{Kc4}} \subseteq \mathrm{B}_{\mathrm{Kc1}_F} \cup \{\mathrm{D}\neg\}$.
3. $\mathrm{D}\neg$ is provable in $\mathrm{B}_{\mathrm{Kc4}} \cup \{\mathrm{D}F\}$.
4. $\mathrm{D}F$ is provable in $\mathrm{B}_{\mathrm{Kc4}_F} \cup \{\mathrm{D}\neg\}$.

Proposition 3. $\mathrm{B}_{\mathrm{Kc4}_F} \subseteq \mathrm{B}_{\mathrm{Kc4}} \cup \{\mathrm{D}F\}$.

Proof. Theorems t6 and t8 are A7 and A8, respectively, when defined. ■

Proposition 4. $\mathrm{B}_{\mathrm{Kc4}} \subseteq \mathrm{B}_{\mathrm{Kc4}_F} \cup \{\mathrm{D}\neg\}$.

Proof. T3 and T2 and T1 are A9, A10 and A11, respectively. ■

Proposition 5. *$\mathrm{D}\neg$ is provable in* $\mathrm{B}_{\mathrm{Kc4}} \cup \{\mathrm{D}F\}$.

Proof. By t2 and D$F$, $\neg A \to (A \to F)$. By t5 and D$F$, $(A \to F) \to \neg A$. So, $\neg A \leftrightarrow (A \to F)$ by Adj. and definition of the biconditional. ■

Proposition 6. *$\mathrm{D}F$ is provable in* $\mathrm{B}_{\mathrm{Kc4}_F} \cup \{\mathrm{D}\neg\}$.

Proof. By A7 and D$\neg$, $F \to \neg(A \to A)$. By T4 and D$\neg$, $\neg(A \to A) \to F$. So, $F \leftrightarrow \neg(A \to A)$ by Adj. and definition of the biconditional. ■

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Universidad de Salamanca
Campus Unamuno, Edificio FES, Salamanca, E-37007, Spain
http://www.usal.es/glf
e-mail: sefus@usal.es; gemmarobles@gmail.com

Department of Psychology, Sociology and Philosophy
Universidad de León
Campus Vegazana, 24071, León, Spain
e-mail: dfcfsa@unileon.es