NEAT EMBEDDING IS NOT SUFFICIENT FOR COMPLETE REPRESENTABILITY

Abstract

In [4] we characterize the class of countable completely representable relation and cylindric algebras via special neat embeddings. In this note we provide a counterexample showing that the condition of countability cannot be omitted.

1. Introduction

There are two kinds of algebras of relations mainly due to Tarski. Relation algebras (RA) and cylindric algebras of dimension $n$ (CA$_n$) for various $n$. RA are closely related to fields of binary relations while CA$_n$ are related to fields of $n$-ary relations.

For $n \geq 3$ the class of representable cylindric algebras of dimension $n$ cannot be defined by finitely many axioms [10] and similarly the class of representable relation algebras is not finitely axiomatisable [19]. However, the Henkin-Tarski Neat Embedding Theorem gives an algebraic characterisation of these representation classes. It states that a CA$_n$ is representable iff it is a subalgebra of a neat reduct of a CA$_{n+\omega}$ [18] and, correspondingly, a relation algebra is representable iff it is a subalgebra of the relation algebra reduct of a CA$_\omega$ [10]. (Relevant definitions will be recalled below.)

A complete representation of a relation algebra or of a cylindric algebra is a representation which preserves arbitrary suprema, whenever they exist in the algebra. Complete representations are harder to characterise — neither the class of completely representable relation algebras nor the class of completely representable cylindric algebras of dimension $n$, $n \geq 3$ can be
defined by any first order theory — these classes are not elementary [17].

We recall our main result in [4] which gives an algebraic characterization of completely representable algebras. To formulate it we need:

**Definition 1.** Let \( C \) be any \( n \) dimensional cylindric algebra, where \( n \geq 3 \). For \( i, j < \mu, i \neq j \), let \( s_i^j x = c_i(x.d_{ij}) \).

1. For \( \lambda < n \), the neat \( \lambda \)-reduct of \( C \) - in symbols \( \text{Nr}_\lambda C \) - is the \( CA_\lambda \) with domain \( \{ a \in C : c_k a = a \text{ for all } \lambda \leq k < n \} \) and with operations \( +, -, 0, 1, c_j, d_{jl}, j, l < \lambda \) induced from \( C \).

2. The relation algebra reduct of \( C \) - in symbols \( \text{RaC} \) - is the algebra \( \langle \text{dom}(\text{Nr}_2 C), +, -, 0, 1, 'r, r; s \rangle \),

where

- \( +, -, 0, 1 \) are as in \( C \)
- \( 'r = d_01 \)
- \( r; s = c_2(s_1 r.s_2 s) \) for all \( r, s \in \text{Nr}_2 C \).

For any boolean algebras with operators \( A, B \) we say that \( A \) is complete subalgebra of \( B \), in symbols \( A \subseteq_c B \), if whenever the supremum \( \sum^A X \) exists in \( A \), then the supremum exists in \( B \) and \( \sum^A X = \sum^B X \). We write \( S, K \) for \( \{ A : \exists B \in K, A \subseteq_c B \} \). Let \( \text{Nr}_3 K = \{ \text{Nr}_3 C : C \in K \} \) and \( \text{RaK} = \{ \text{RaC} : C \in K \} \). The following algebraic characterization (in analogy to the Neat Embedding Theorem) of completely representable algebras is proved in [4].

**Proposition 1.**

1. Let \( n < \omega \). Let \( A \in CA_n \) be countable. Then \( A \) is completely representable if and only if \( A \) is atomic and \( A \in S, \text{Nr}_n CA_\omega \).

2. Let \( A \in RA \) be countable. Then \( A \) is completely representable if and only if \( A \) is atomic and \( A \in S, \text{RaCA}_\omega \).

Our proof in [4] of Proposition 1, heavily relies on countability since it is a Baire category argument at heart. In this paper we show that, indeed, countability is essential.

The class \( \text{RaCA}_n \) is studied in e.g [11], [16] and [12]. On the other hand, we note that the notion of neat reducts is a venerable old notion in
cylindric algebras that is gaining some momentum. Indeed, the notion of neat reducts has been studied quite intensely lately [1], [2], [3], [9], [6], [13], [14]. [1], [2] and [3] deal with the notion of neat reducts on its own, while [9], and [6] deal with the related notion of neat embeddings in connection to interpolation and definability In [13] and [14] neat reducts are studied in connection to proof theory. Such results are surveyed in [7] and [8].

2. A Counterexample

We assume familiarity with constructing atomic relation algebras by specifying their atom structure via listing the set of forbidden triples [15] 3.3.7. We define an atomic relation algebra $A$ with uncountably many atoms. The atoms are $1'_{i}$, $a_i$ for $0 < i < \omega_1$ and $a_j$ for $1 \leq j < \omega$, all symmetric. The forbidden triples of atoms are all permutations of $(1', x, y)$ for $x \neq y$, $(a_{j}, a_{i}, a_{j})$ for $1 \leq j < \omega$ and $(a_{0}', a_{i}', a_{0}')$ for $i, i', i'' < \omega_1$. In other words, if you think of the subscript of a non-identity atom as its colour, then we forbid all the monochromatic triangles. Write $a_0$ for $\{a_i: i < \omega_1\}$ and $a_+ = \{a_j: 1 \leq j < \omega\}$. Call this atom structure $\alpha$. Let $A$ be the term algebra (the algebra generated by the atoms) on this atom structure. $A$ is a dense subalgebra of the complex algebra $m_\alpha$.

We recall that a relation algebra is representable if there is a structure $M$ in which each element $a \in A$ is interpreted as a binary relation $a^M$ over the domain $M$ faithfully (i.e. $a \neq b \rightarrow a^M \neq b^M$) so as to preserve all the relation algebra operations. In particular, i.e $0^M = \emptyset$, $(a + b)^M = a^M \cup b^M$, $(-a)^M = 1^M \setminus a^M$. $M$ is a complete representation if in addition $(\sum X)^M = \bigcup_{a \in X} a^M$ whenever $\sum X$ exists.

**Lemma 2.** $A$ has no complete representation.

**Proof.** Suppose, seeking a contradiction, that $A$ has a complete representation $M$. Let $x, y$ be points in the representation with $M \models a_1(x, y)$. For each $i < \omega_1$ there is a point $z_i \in M$ such that $M \models a_0'(x, z_i) \land a_1(z_i, y)$. Let $Z = \{z_i : i < \omega_1\}$. Within $Z$ there can be no edges labelled by $a_0$ so each edge is labelled by one of the countable number of atoms in $a_+$. Ramsey’s theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models a_j(z^1, z^2) \land a_j(z^2, z^3) \land a_j(z^3, z_1)$, for some single $j < \omega$. This contradicts the definition of composition in $A$. $\square$
We refer the reader to [15] p. 338 for the definition of an atomic network. For such networks \( N \) and \( M \) on a set of nodes \( n \), we write \( N \equiv_{x_1,...,x_n} M \) if \( N(\bar{a}) = M(\bar{a}) \) for all \( a \in 2^{(n - \{x_1,...,x_n\})} \). Let \( S \) be the set of all atomic A-networks \( N \) with nodes \( \omega \) such that \( \{a_i : 1 \leq i < \omega, a_i \text{ is the label of some edge in } N\} \) is finite.

**Lemma 3.** \( S \) is an amalgamation class, that is for all \( M, N \in S \) if \( M \equiv_{ij} N \) then there is \( L \in S \) with \( M \equiv_i L \equiv_j N \).

**Proof.** Straightforward. \( \square \)

Define an algebra \( \text{Ca}(S) \) of the signature of \( \omega \)-dimensional cylindric algebras. The boolean operations are defined as complement and union of sets. For \( i, j < \omega \) define the diagonal

\[
d_{ij} = \{ N \in S : N(i,j) \leq Id \},
\]

and for \( i < n \) define the cylindrifier \( c_i \) for \( T \subseteq S \), by

\[
c_iT = \{ N \in S : (\exists M \in T)(N \equiv_i M) \}.
\]

Hence from Lemma 3 we have

**Lemma 4.** The (complex) cylindric algebra \( \text{Ca}(S) \in \text{CA}_\omega \).

Now let \( X \) be the set of finite A-networks \( N \) with nodes \( \omega \) such that

1. each edge of \( N \) is either (a) an atom of \( A \) or (b) a cofinite subset of \( a_+ = \{a_j : 1 \leq j < \omega \} \) or (c) a cofinite subset of \( a_0 = \{a_i^0 : i < \omega_1 \} \) and

2. \( N \) is ‘triangle-closed’, i.e. for all \( l, m, n \in \text{nodes}(N) \) we have \( N(l,n) \leq N(l,m) \wedge N(m,n) \). That means if an edge \( (l,m) \) is labelled by \( 1' \) then \( N(l,n) = N(m,n) \) and if \( N(l,m), N(m,n) \leq a_0 \) then \( N(l,n).a_0 = 0 \) and if \( N(l,m) = N(m,n) = a_j \) (some \( 1 \leq j < \omega \)) then \( N(l,n)a_j = 0 \).

For \( N \in X \) let \( N' \in \text{Ca}(S) \) be defined by

\[
\{ L \in S : L(m,n) \leq N(m,n) \text{ for } m, n \in \text{nodes}(N) \}
\]
Lemma 5. If $N \in X$, $i < \omega$ then $c_i N' = (N \upharpoonright_{-i})'$. Here $N \upharpoonright_{-i}$ is the graph obtained from $N$ by deleting the node $i$.

Proof. The inclusion $c_i N' \subseteq (N \upharpoonright_{-i})'$ is clear. Conversely, let $L \in (N \upharpoonright_{-i})'$ We seek $M \equiv_i L$ with $M \in N'$. This will prove that $L \in c_i N'$ as required. Since $L \in S$ the set $X = \{a_i \notin L\}$ is infinite. Let $X$ be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the $\omega$-network $M$ we must define the labels of all edges involving the node $i$ (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labelling them one at a time. So let $j \neq i < \omega$. Suppose $j \in \text{nodes}(N)$. We must choose $M(i, j) \leq N(i, j)$. If $N(i, j)$ is an atom then of course $N(i, j) = N(i, j)$. Since $N$ is finite, this defines only finitely many labels of $N$. If $N(i, j)$ is a cofinite subset of $a_f$ then let $M(i, j)$ be an arbitrary atom in $N(i, j)$. And if $N(i, j)$ is a cofinite subset of $a_+$ then let $M(i, j)$ be an element of $N(i, j) \cap Y$ which has not been used as the label of any edge of $M$ which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \text{nodes}(N)$ then we can let $M(i, j) = a_k \in Y$ some $1 \leq k < \omega$ such that no edge of $M$ has already been labelled by $a_k$. It is not hard to check that each triangle of $M$ is consistent (we have avoided all monochromatic triangles) and clearly $M \in N'$ and $M \equiv_i L$. The labelling avoided all but finitely many elements of $Y'$, so $M \in S$. So $(N \upharpoonright_{-i})' \subseteq c_i N'$. □

Let $X' = \{N' : N \in X\} \subseteq \text{Ca}(S)$.

Lemma 6. The subalgebra of $\text{Ca}(S)$ generated by $X'$ is obtained from $X'$ by closing under finite unions.

Proof. Clearly all these finite unions are generated by $X'$. We must show that the set of finite unions of $X'$ is closed under all cylindric operations. Closure under unions is given. For $N' \in X$ we have $-N' = \bigcup_{m,n \in \text{nodes}(N)} N_{mn}'$ where $N_{mn}$ is a network with nodes $\{m, n\}$ and labelling $N_{mn}(m, n) = -N(m, n)$. $N_{mn}$ may not belong to $X$ but it is equivalent to a union of at most three members of $X$. The diagonal $d_{ij} \in \text{Ca}(S)$ is equal to $N'$ where $N$ is a network with nodes $\{i, j\}$ and labelling $N(i, j) = 1'$. Closure under cylindrification is given by Lemma 5. □
Let $C$ be the subalgebra of $Ca(S)$ generated by $X'$.

**Lemma 7.** $A = Ra(C)$.

**Proof.** Each element of $A$ is a union of a finite number of atoms and possibly a co-finite subset of $a_0$ and possibly a co-finite subset of $a_+$. Clearly $A \subseteq Ra(C)$. Conversely, each element $z \in Ra(C)$ is a finite union $\bigcup_{N \in F} N'$, for some finite subset $F$ of $X$, satisfying $c_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in $F$. Then by Lemma 5, $c_{i_0} \cdots c_{i_k} z = \bigcup_{N \in F} c_{i_0} \cdots c_{i_k} N' = \bigcup_{N \in F} (N \setminus \{0,1\})' \in A$. So $Ra(C) \subseteq A$. □

Hence,

**Theorem 8.** $A$ is relation algebra reduct of $C \in CA_\omega$ but has no complete representation.

Let $n > 2$. Let $B = Nr_n C$. Then

**Theorem 9.** $B \in Nr_n CA_\omega$ has no complete representation.

**Proof.** Else this complete representation would induce a complete representation of $A = RaB$. □

By noting that $Nr_n CA_\omega \subseteq S_n Nr_n CA_\omega$ and similarly for $Ra CA_\omega$, we are done. We conjecture the following: Let $n < \omega$. Let $A \in CA_n$. Then $A$ is completely representable if $A$ is atomic and $A \subseteq e Nr_n B$, for some atomic $B \in CA_\omega$. And similarly for $RA$. ($C$ above is not atomic. In fact it is atomless.)

**References**


Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt