AN EQUIVALENCE BETWEEN SEQUENT CALCULI FOR LINEAR-TIME TEMPORAL LOGIC

Abstract

The equivalence between Kawai’s sequent calculus $LT_\omega$ and Baratella-Masini’s 2-sequent calculus $2S_\omega$ is shown for the until-free linear-time temporal logic. By using this equivalence, an alternative proof of the cut-elimination theorems for $LT_\omega$ and $2S_\omega$ is obtained.

1. Introduction

Linear-time (propositional) temporal logic (LTL) has widely been studied by many researchers. For a comprehensive review of LTL, see e.g. [3]. Several sequent calculi for LTL and its neighbors including first-order predicate extensions have been introduced by some researchers e.g. [2,1]. In [2], two sequent calculi $LT_\omega$ and $LT^*_\omega$ for first-order infinitary temporal logics were introduced, and the cut-elimination and completeness theorems for these calculi were proved using Schütte’s method. The equivalence between these calculi and Kröger’s temporal logics with or without Barcan formulas were also shown. In [1], the 2-sequent calculi $2S_\omega$ and $2SP^\Phi_\omega$, which are natural extensions of the usual sequent calculus, were introduced for the propositional and first-order predicate LTLs without the until operator. The cut-elimination and completeness theorems for these calculi were also proved, presenting an analogy between LTL and Peano arithmetic with $\omega$-rule.
In this paper, the equivalence between Kawai’s $\text{LT}_\omega$ (propositional version) and Baratella-Masini’s $\text{2S}_\omega$ is shown. By using this equivalence, an alternative proof of the cut-elimination theorems for $\text{LT}_\omega$ and $\text{2S}_\omega$ is obtained.

2. $\text{L}_\omega$ and $\text{LT}_\omega$

Formulas for the until-free propositional LTL is constructed from propositional variables, $\rightarrow$ (implication), $\land$ (conjunction), $\lor$ (disjunction), $\neg$ (negation), $G$ (globally), $F$ (eventually) and $X$ (next-time). Lower-case letters $p, q, ...$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sequences of formulas. For any $\sharp \in \{G, F, X\}$, an expression $\sharp \Gamma$ is used to denote the sequence $\langle \sharp \gamma \mid \gamma \in \Gamma \rangle$. The symbol $\equiv$ is used to denote equality as sequences of symbols. The symbol $\omega$ is used to represent the set of natural numbers. An expression $X^i\alpha$ for any $i \in \omega$ is used to denote $X^i \alpha$, e.g. $(X^0 \alpha \equiv \alpha)$, $(X^1 \alpha \equiv X \alpha)$ and $(X^{n+1} \alpha \equiv X^n X \alpha)$. Lower-case letters $i, j$ and $k$ are used to denote any natural numbers.

An expression of the form $\Gamma \Rightarrow \Delta$ is called a sequent. An expression $L \vdash S$ is used to denote the fact that a sequent $S$ is provable in a sequent calculus $L$.

First, an auxiliary calculus $\text{L}_\omega$, which is equivalent to $\text{LT}_\omega$, is introduced below.

**Definition 1.** ($\text{L}_\omega$) The initial sequents of $\text{L}_\omega$ are of the form:

$$X^i\alpha \Rightarrow X^i\alpha.$$ 

The structural rules of $\text{L}_\omega$ are of the form:

$$\begin{align*}
\frac{\Gamma \Rightarrow \Delta, X^i\alpha, X'^i\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} & \quad \text{(cut)} \\
\frac{\Gamma \Rightarrow \Delta}{X^i\alpha, \Gamma \Rightarrow \Delta} & \quad \text{(we-left)} \\
\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, X^i\alpha} & \quad \text{(we-right)}
\end{align*}$$
An Equivalence Between Sequent Calculi for Linear-Time Temporal Logic

\[ \frac{X^i_\alpha, X^i_\beta, \Delta \Rightarrow \Sigma}{X^i_\alpha, \Delta \Rightarrow \Sigma} \text{ (co-left)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha, X^i_\beta}{\Gamma \Rightarrow \Delta, X^i_\alpha} \text{ (co-right)} \]

\[ \frac{\Gamma, X^i_\alpha, X^k_\beta, \Delta \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, X^i_\alpha, X^k_\beta, \Delta \Rightarrow \Sigma} \text{ (ex-left)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha, X^k_\beta, \Sigma}{\Gamma \Rightarrow \Delta, X^i_\alpha, X^k_\beta, \Sigma} \text{ (ex-right)} \]

The logical inference rules of \( L_\omega \) are of the form:

\[ \frac{\Gamma \Rightarrow \Sigma, X^i_\alpha, X^j_\beta, \Delta \Rightarrow \Pi}{X^i(\alpha \rightarrow \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi} \text{ (→left)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha, X^j_\beta}{\Gamma \Rightarrow \Delta, X^i_\alpha (\alpha \rightarrow \beta)} \text{ (→right)} \]

\[ \frac{X^i_\alpha, \Gamma \Rightarrow \Delta}{X^i(\alpha \land \beta), \Gamma \Rightarrow \Delta} \text{ (\&left1)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha}{X^i(\alpha \lor \beta), \Gamma \Rightarrow \Delta} \text{ (\&left2)} \]

\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha}{\Gamma \Rightarrow \Delta, X^i(\alpha \land \beta)} \text{ (\&right)} \]
\[ \frac{X^i_\alpha, \Gamma \Rightarrow \Delta}{X^i(\alpha \lor \beta), \Gamma \Rightarrow \Delta} \text{ (\&right1)} \]
\[ \frac{X^i_\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, X^i_\alpha (\alpha \lor \beta)} \text{ (\&right2)} \]

\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha}{\Gamma \Rightarrow \Delta, X^i\neg_\alpha} \text{ (¬left)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^i_\alpha}{\Gamma \Rightarrow \Delta, X^i\neg_\alpha} \text{ (¬right)} \]

\[ \frac{X^{i+j}_\alpha, \Gamma \Rightarrow \Delta}{X^i\neg\alpha, \Gamma \Rightarrow \Delta} \text{ (Gleft)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^{i+j}_\alpha}{\Gamma \Rightarrow \Delta} \text{ \{ \( j \in \omega \) \}} \text{ (Grigh)} \]

\[ \frac{\{ X^{i+j}_\alpha, \Gamma \Rightarrow \Delta \}}{X^i\neg\alpha, \Gamma \Rightarrow \Delta} \text{ (Fleft)} \]
\[ \frac{\Gamma \Rightarrow \Delta, X^{i+k}_\alpha}{\Gamma \Rightarrow \Delta} \text{ (Frigh)} \]

It is remarked that (Grigh) and (Fleft) have infinite premises.

**Definition 2.** (LT\(_\omega\)) Kawai’s sequent calculus LT\(_\omega\) [2] is obtained from \( L_\omega \) by replacing the initial sequents and structural rules by the initial sequents and structural rules only with \( i = k = 0 \), i.e. every \( X^i \) is deleted from the forms of the initial sequents and structural rules. \(^1\) The cut rule with \( i = 0 \) is denoted as (cut0).

\(^1\) Obviously, this restriction does not change the the provability of sequents. In other words, it is not a restriction.
It is remarked that the original system LT_ω is a first-order predicate sequent calculus with Barcan formula, i.e. the system in Definition 2 is the propositional-fragment of the original one. It is also remarked that in [2], the next-time operator is not used as a modal operator but used as a special symbol.

The following theorem is known [2].

**Theorem 3.** (Cut-elimination for LT_ω) The rule (cut0) is admissible in cut-free LT_ω.

**Proposition 4.** (Equivalence between L_ω and LT_ω) (1) for any sequent S, L_ω ⊢ S iff LT_ω ⊢ S. (2) for any sequent S, L_ω−(cut) ⊢ S iff LT_ω−(cut0) ⊢ S.

Using Proposition 4 and Theorem 3, we can derive the following theorem.

**Theorem 5.** (Cut-elimination for L_ω) The rule (cut) is admissible in cut-free L_ω.

### 3. 2S_ω

Baratella and Masini’s 2-sequent calculus 2S_ω [1] is presented below. The language of 2S_ω and the notations used are almost the same as those of L_ω.

**Definition 6.** An expression α^i (α is a formula and i ∈ ω) is called an indexed formula. An expression Γ ⇒^2 Δ, where Γ and Δ are finite (possibly empty) sequences of indexed formulas is called a 2-sequent.

**Definition 7.** (2S_ω) The initial sequents of 2S_ω are of the form:

\[ α^i ⇒^2 α^i. \]

The structural rules of 2S_ω are of the form:

\[
\frac{Γ ⇒^2 Δ, α^i, α^i, Σ ⇒^2 Π}{Γ, Σ ⇒^2 Δ, Π} \quad \text{(cut2)}
\]
An Equivalence Between Sequent Calculi for Linear-Time Temporal Logic

The logical inference rules of \(2S_\omega\) are of the form:

\[
\frac{\Gamma \implies \Sigma, \alpha^i, \beta^i, \Delta \implies \Pi}{(\alpha \implies \beta)^i, \Gamma, \Delta \implies \Sigma, \Pi} \quad (\leftarrow \text{left2})
\]

\[
\frac{\alpha^i, \Gamma \implies \Delta, \beta^i, \Delta \implies \Sigma}{\alpha^i, \beta^i, \Delta \implies \Sigma} \quad (\wedge \text{left12})
\]

\[
\frac{\Gamma \implies \Delta, \alpha^i}{\Gamma \implies \Delta, (\alpha \wedge \beta)^i} \quad (\wedge \text{right2})
\]

\[
\frac{\Gamma \implies \Delta, \alpha^i}{\Gamma \implies \Delta, (\alpha \lor \beta)^i} \quad (\lor \text{left12})
\]

\[
\frac{\Gamma \implies \Delta, \alpha^i}{\Gamma \implies \Delta, (\neg \alpha)^i} \quad (\neg \text{left2})
\]

\[
\frac{\alpha^{i+1}, \Gamma \implies \Delta}{(X \alpha)^i, \Gamma \implies \Delta} \quad (X \text{left})
\]

\[
\frac{\alpha^{i+k}, \Gamma \implies \Delta}{(G \alpha)^i, \Gamma \implies \Delta} \quad (G \text{left2})
\]

\[
\frac{\{ \alpha^{i+j}, \Gamma \implies \Delta \}_{j \in \omega}}{(F \alpha)^i, \Gamma \implies \Delta} \quad (F \text{left2})
\]

\[
\frac{\Gamma \implies \Delta, \alpha^i}{\Gamma \implies \Delta, (\alpha \implies \beta)^i} \quad (\rightarrow \text{right2})
\]

\[
\frac{\beta^i, \Gamma \implies \Delta}{(\alpha \wedge \beta)^i, \Gamma \implies \Delta} \quad (\wedge \text{right2})
\]

\[
\frac{\alpha^i, \Gamma \implies \Delta}{(\alpha \lor \beta)^i, \Gamma \implies \Delta} \quad (\lor \text{right2})
\]

\[
\frac{\Gamma \implies \Delta, \beta^i}{\Gamma \implies \Delta, (\neg \alpha)^i} \quad (\neg \text{right2})
\]

\[
\frac{\Gamma \implies \Delta, \alpha^i}{\Gamma \implies \Delta, (\alpha \implies \beta)^i} \quad (\rightarrow \text{left2})
\]

An expression \(L \vdash \Gamma \implies \Delta\) is used to denote the fact that \(\Gamma \implies \Delta\) is provable in a 2-sequent calculus \(L\). The provability "\(\vdash\)" of a 2-sequent for \(2S_\omega\) or \(2S_\omega)-(\text{cut2})\) is defined by the existence of an \(\omega\)-proof for the 2-sequent. For the precise definition of the notion of \(\omega\)-proof, see [1].

**Definition 8.** Let \(L_1\) be the set of formulas of \(L_\omega\), and \(L_2\) be the set of indexed formulas of \(2S_\omega\).
A mapping \( f \) from \( \mathcal{L}_1 \) to \( \mathcal{L}_2 \) is defined by \( f(X^i \alpha) := \alpha^i \) for any formula \( \alpha \).

A mapping \( g \) from \( \mathcal{L}_2 \) to \( \mathcal{L}_1 \) is defined by \( g(\alpha^i) := X^i \alpha \) for any formula \( \alpha \).

It is remark that \( fg(\alpha^i) = \alpha^i \) and \( gf(X^i \alpha) = X^i \alpha \) hold for any formula \( \alpha \).

**Theorem 9.** (Equivalence between \( L_\omega \) and \( 2S_\omega \))

(1) For any 2-sequent \( \Gamma \Rightarrow^2 \Delta \), if \( 2S_\omega \vdash \Gamma \Rightarrow^2 \Delta \), then \( L_\omega \vdash g(\Gamma) \Rightarrow g(\Delta) \).

(2) For any sequent \( \Gamma \Rightarrow \Delta \), if \( L_\omega - (cut) \vdash \Gamma \Rightarrow \Delta \), then \( 2S_\omega - (cut2) \vdash f(\Gamma) \Rightarrow^2 f(\Delta) \).

**Proof.** We show only (1) by induction on a cut-free \( \omega \)-proof \( P \) of \( \Gamma \Rightarrow^2 \Delta \) in \( 2S_\omega \). We show only the following case.

Case (Xleft): The last inference of \( P \) is of the form:

\[
\frac{\alpha^{i+1}, \Sigma \Rightarrow \Pi}{(X\alpha)^i, \Sigma \Rightarrow \Pi} \ (\text{Xleft}).
\]

By the hypothesis of induction, we obtain \( L_\omega \vdash g(\alpha^{i+1}), g(\Sigma) \Rightarrow g(\Pi) \), and hence obtain \( L_\omega \vdash g((X\alpha)^i), g(\Sigma) \Rightarrow g(\Pi) \) by \( g(\alpha^{i+1}) = X^{i+1} \alpha = X^i(X\alpha) = g((X\alpha)^i) \). Q.E.D.

By Theorems 5 and 9, an alternative proof of the following theorem [1] is obtained.

**Theorem 10.** (Cut-elimination for \( 2S_\omega \)) The rule \( \text{(cut2)} \) is admissible in cut-free \( 2S_\omega \).

**Proof.** Suppose \( 2S_\omega \vdash \Gamma \Rightarrow^2 \Delta \) for an arbitrary 2-sequent \( \Gamma \Rightarrow^2 \Delta \). Then we have \( L_\omega \vdash g(\Gamma) \Rightarrow g(\Delta) \) by Theorem 10 (1). By Theorem 5, we obtain \( L_\omega - (cut) \vdash g(\Gamma) \Rightarrow g(\Delta) \). We thus obtain \( 2S_\omega - (cut2) \vdash f(\Gamma) \Rightarrow^2 f(\Delta) \) by Theorem 9 (2). Therefore \( 2S_\omega - (cut2) \vdash \Gamma \Rightarrow^2 \Delta \). Q.E.D.

Conversely, by Theorem 10 and an appropriate modification of Theorem 9, an alternative proof of Theorem 3 or 5 is also derived.
References


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