Abstract

Several authors have pointed out that the class of Nemitz’s implicative semilattices are not the same as the class of Hilbert algebras with the property that for each pair of elements there exists its infimum, which we called Hilbert algebras with infimum. To the best of our knowledge, the first who realized this fact was Marsden in 1972.

In a previous paper we have shown that the class of Hilbert algebras with infimum is equational and that the class of implicative semilattices is strictly contained in this variety.

In this article, bearing in mind the relationship between the implicative semilattices and the \(\{\rightarrow, \wedge\}\)-fragment of the intuitionistic propositional calculus, we describe a Hilbert style \(\{\rightarrow, \wedge\}\)-propositional calculus weaker than the intuitionistic fragment and we show that the algebraic models of this new calculus are Hilbert algebras with infimum.

1. Introduction and preliminaries

It is well-known that the \(\{\rightarrow\}\)-fragment of the intuitionistic propositional calculus (or iIPC) can be described by means of the axiom schemata:

\[(I1) \; p \rightarrow (q \rightarrow p),\]
\[(I2) \; (p \rightarrow (q \rightarrow t)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow t)),\]

and only modus ponens as a rule of inference

\[(MP) \; \frac{p, p \rightarrow q}{q}.\]

This work was partially supported by the Universidad Nacional del Sur, Bahía Blanca, Argentina.
Besides, A. Diego ([1]) called Hilbert algebras (or $H$-algebras, for short) the algebraic models of $\text{iIPC}$. By an $H$-algebra we mean an algebra $\langle A, \to, 1 \rangle$ of the type $(2, 0)$ which satisfies the following conditions:

(H1) $x \to (y \to x) = 1$,
(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(H3) $x \to y = 1, y \to x = 1$ imply $x = y$.

In [2], we proved that the class of Hilbert algebras with infimum (or $iH$-algebras) is the variety of algebras $\langle A, \to, \wedge, 1 \rangle$ of type $(2, 2, 0)$ which satisfy the following conditions:

(i) the reduct $\langle A, \to, 1 \rangle$ is an $H$-algebra,
(ii) these identities are verified

(H4) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(H5) $x \wedge x = x$,
(H6) $x \wedge (x \to y) = x \wedge y$,
(H7) $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$.

In what follows we shall denote by $\mathcal{V}_{iH}$ the variety of these algebras.

On the other hand, it is easy to check that the variety of implicative semilattices is strictly contained in $\mathcal{V}_{iH}$. It is worth mentioning that implicative semilattices, introduced by Nemitz in [5] (see also [4]), are the algebraic models of the $\{\to, \wedge\}$-fragment of the intuitionistic propositional calculus.

In this paper, we describe a propositional calculus which we called $iH$-calculus, weaker than the $\{\to, \wedge\}$-fragment of the intuitionistic propositional calculus and we show that $iH$-algebras are the algebraic counterpart of this calculus.

2. $iH$-calculus

Next, we shall present a Hilbert style description of the $iH$-calculus.

Let $\mathcal{L} = \{\to, \wedge\}$ be a language of type $(2, 2)$ where $\to$ and $\wedge$ are called implication and conjunction respectively. Let $X$ be a set of propositional variables, $\text{For}[X]$ the set of all $\mathcal{L}$-formulas over $X$ and $\mathcal{F}[X] = \langle \text{For}[X], \to, \wedge \rangle$ the absolutely free algebra of $\mathcal{L}$-formulas.
Then, the \( iH \)-calculus (or \( iH \)) is the propositional calculus determined over \( X \) by means of the axiom schemas:

\begin{align*}
(T1) & \quad (I1), \\
(T2) & \quad (I2), \\
(T3) & \quad (p \land q) \to q, \\
(T4) & \quad (p \land (p \to q)) \to (p \land q), \\
(T5) & \quad (p \land q) \to (q \land p), \\
(T6) & \quad ((p \land q) \land t) \to ((p \land t) \land q),
\end{align*}

and the rules of inference

\begin{align*}
(R1) & \quad (MP), \\
(R2) & \quad \frac{p \to q}{p \to (p \land q)} \quad \text{(Restricted absorption)}.
\end{align*}

We shall denote by \( T_{iH} \) the set of all formulas which are derivable in \( iH \). As usual, if \( \alpha \in T_{iH} \) we shall say that \( \alpha \) is a theorem of \( iH \). Taking into account that the \( \{\to\} \)-fragment of \( iH \) where the rule of inference \( R2 \) is also omitted coincides with the \( \text{iIPC} \), we can assert that in \( iH \) the statements of Lemma 1 hold true.

**Lemma 1.** In \( iH \) the following rules and theorems are verified:

\begin{align*}
(R3) & \quad \frac{q}{p \to q} \\
(T7) & \quad p \to p, \\
(DT) & \quad \text{the following conditions are equivalent:} \\
& \quad \begin{align*}
(i) & \quad p \to q, \\
(ii) & \quad \frac{p}{q}.
\end{align*}
\end{align*}

\begin{align*}
(R4) & \quad \frac{p \to q}{r \to p} \to (r \to q) \\
(R5) & \quad \frac{p \to q}{(q \to r)} \to (p \to r).
\end{align*}
Taking into account (T1), (T2) and (R1) we prove that the relation

\[ \equiv = \{ (p, q) \in \text{For}[X] \times \text{For}[X] : p \to q, q \to p \in T_{\text{tr}} \} \]

is an equivalence relation on \( \text{For}[X] \) which is a congruence with respect to \( \to \). Then, \( (\text{For}[X]/\equiv, \to, 1) \) is an \( H \)-algebra where \( 1 = T_{\text{tr}} \). In addition, Lemma 2 allows us to conclude that \( \equiv \) is a congruence on \( \mathcal{F}[X] \).

**Lemma 2.** In \( \mathbf{iH} \) the following rules and theorems are verified:

1. **(T8)** \( p \land q \equiv q \land p, \)

2. **(R6)** \( p \to q \quad (s \land p) \to (s \land q) \)

3. **(R7)** \( p \equiv q \) implies \( s \land p \equiv s \land q, \)

4. **(R8)** \( p \equiv q \) implies \( p \land s \equiv q \land s, \)

5. **(R9)** \( p \to q \quad (p \land s) \to (q \land s) \)

6. **(R10)** \( p \to q, s \to t \quad (p \land s) \to (q \land t) \)

7. **(T9)** \( ((p \land q) \land t) \to (p \land (q \land t)) \),

8. **(T10)** \( p \to (p \land p), \)

9. **(R11)** \( p \to q, p \to t \quad p \to (q \land t) \)

10. **(T11)** \( (p \to (q \land t)) \to ((p \to t) \land (p \to q)) \).

**Proof.**

(T8) is a direct consequence of T5.

**Proof.**

1. **(R6)** \( 1. \quad p \to q, \quad \text{[premise]} \)

2. **(T2)** \( (s \land p) \land ((s \land p) \to q) \to ((s \land p) \land q), \quad [T4] \)

3. **(T3)** \( (s \land p) \to p \to ((p \to q) \to ((s \land q) \to q)), \quad [R5,\text{DT}] \)

4. **(T4)** \( (s \land p) \to p, \quad [T3] \)

5. **(T5)** \( (s \land p) \to q, \quad [(3),(4),(1),R1] \)

6. **(T6)** \( (s \land p) \to ((s \land p) \to q), \quad [(5),\text{R3}] \)
(7) \((s \land p) \rightarrow ((s \land p) \land ((s \land p) \rightarrow (s \land q))), \quad [2], R3\)
(8) \(((s \land p) \rightarrow ((s \land p) \land ((s \land p) \rightarrow q))) \rightarrow ((s \land p) \rightarrow (s \land q))), \quad [7], T2, R1\)
(9) \((s \land p) \rightarrow ((s \land p) \land ((s \land p) \rightarrow q)),\quad [6], R2\)
(10) \((s \land p) \rightarrow ((s \land p) \land q),\quad [5], R2\)
(11) \(((s \land q) \land p) \rightarrow (s \land q),\quad [T5, T3, R5, R1]\)
(12) \((s \land p) \rightarrow (((s \land q) \land p) \rightarrow (s \land q)),\quad [11], R3\)
(13) \(((s \land p) \rightarrow ((s \land q) \land p)) \rightarrow ((s \land p) \rightarrow (s \land q)),\quad [12], T2, R1\)
(14) \(((s \land p) \land q) \rightarrow ((s \land q) \land p),\quad [T6]\)
(15) \((s \land p) \rightarrow (((s \land q) \land q) \rightarrow (s \land q) \land p)),\quad [14], R3\)
(16) \(((s \land p) \rightarrow ((s \land p) \land q)) \rightarrow (((s \land p) \rightarrow (s \land q) \land p)),\quad [15], T2, R1\)
(17) \((s \land p) \rightarrow ((s \land q) \land p),\quad [16], (10), R1\)
(18) \((s \land p) \rightarrow (s \land q),\quad [13], (17), R1\)

(R7) follows from the premise and (R6).
(R8) follows from the premise, (T8) and (R7).
(R9) is a consequence of (R8).
(R10) follows from the premises, (R9), (R6), (R5) and (R1).

(T9) (1) \((p \land q) \land t,\quad \text{[premise]}\)
(2) \(p \land q,\quad [1], T3, T8\)
(3) \(t,\quad [1], T3\)
(4) \(q \rightarrow t,\quad [3], T3\)
(5) \(q \rightarrow (q \land t),\quad [4], R2\)
(6) \((p \land q) \rightarrow (p \land (q \land t)),\quad [5], R6\)
(7) \(p \land (q \land t),\quad [2], (6), R1\)
(8) \(((p \land q) \land t) \rightarrow (p \land (q \land t)).\quad [1], (7), DT\)

(T10) follows from (T7) and (R2).
(R11) follows from the premises, (R10), (R4) and (T10).

(T11) (1) \((q \land t) \rightarrow q,\quad [T5, T3, R1]\)
(2) \((p \rightarrow (q \land t)) \rightarrow (p \rightarrow q),\quad [1], R4\)
(3) \((p \rightarrow (q \land t)) \rightarrow (p \rightarrow t),\quad [T3, R4]\)
(4) \(((p \rightarrow (q \land t)) \rightarrow ((p \rightarrow t) \land (p \rightarrow q)).\quad [(3), (2), R11] □\)

Let \(\mathcal{L}_{iH} = \langle \text{For}[X] / \equiv, \rightarrow, \land, 1 \rangle\) be the Lindenbaum algebra of \(iH\). Then, from the above results and Lemma 2 we conclude Theorem 1.

Theorem 1. \(\mathcal{L}_{iH}\) is an \(iH\)-algebra.
Furthermore, it is well-known that the Lindenbaum algebra $L_{iH}$ is a natural algebraic model for $iH$. On the other hand, bearing in mind the results established in [6], it is simple to verify that the variety $V_{iH}$ coincides with the class $S$ of algebras which are the algebraic models for $iH$. Therefore, the following theorem holds true.

**Theorem 2.** Let $\alpha$ be a formula in $iH$. Then the following conditions are equivalent:

- (i) $\alpha$ is a propositional tautology,
- (ii) $\alpha$ is derivable in $iH$.

**References**


