Anetta Górnicka

AXIOMATIZATION OF THE SENTENTIAL LOGIC DUAL WITH RESPECT TO LUKASIEWICZ’S THREE–VALUED LOGIC

Abstract

Dual logics with respect to Lukasiewicz’s logics were investigated by G. Malinowski, M. Spasowski and R. Wójcicki in [3], [5]. On the basis of their work no axiomatics for these logics were given. They were only referred according to the set of tautologies.

Our aim is to build such an axiomatic system, for which the matrix dual to Lukasiewicz’s three–valued matrix is strongly adequate.

1. Introduction

The definition of a dual consequence applied here was given by R. Wójcicki [5].

Let $Cn$ be an arbitrary consequence operation in $S$, where $S$ is the set of all formulas in some language $L$.

The consequence $dCn$ dual to the consequence $Cn$ is defined as follows:

$$\alpha \in dCn(X) \iff \exists Y [Y \subseteq X \land \text{card}(Y) < \aleph_0 \land \bigcap_{\beta \in Y} Cn(\{\beta\}) \subseteq Cn(\{\alpha\})]$$

for all $\alpha, \beta \in S$ and every $X \subseteq S$.

By the logic dual to a logic $(S, Cn)$ we will mean the logic $(S, dCn)$.

Let $J = (S, \{\Rightarrow, \neg, \land\})$ be the language of Lukasiewicz’s three–valued sentential calculus, where connectives $\Rightarrow, \neg, \land$ stand for implication, negation and conjunction, respectively.
The Łukasiewicz’s three-valued matrix has a form:

\[ M_3 = (\{0, \frac{1}{2}, 1\}, \{f_\Rightarrow, f_\neg, f_\wedge\}), \]

where functions \( f_\Rightarrow, f_\neg, f_\wedge \) are defined as follows:
\[
\begin{align*}
  f_\Rightarrow(x, y) &= \min\{1, 1 - x + y\}, \\
  f_\neg(x) &= 1 - x, \\
  f_\wedge(x, y) &= \min\{x, y\}.
\end{align*}
\]

The matrix dual to Łukasiewicz’s three-valued matrix has a form:

\[ M_3^d = (\{0, \frac{1}{2}, 1\}, \{0, \frac{1}{2}\}, \{f_\Rightarrow, f_\neg, f_\wedge\}), \]

where functions \( f_\Rightarrow, f_\neg, f_\wedge \) are defined in the same way as in the matrix \( M_3 \).

The disjunction functor \( \lor \) can be defined as follows:
\[
\alpha \lor \beta \overset{df}{=} \neg(\neg \alpha \land \neg \beta).
\]

Additionally, let us define the functor \( \rightarrow^d \) as follows:
\[
\begin{align*}
  \alpha \rightarrow^d \beta &\overset{df}{=} \neg_\beta(\alpha \rightarrow \beta), \\
  \neg_\alpha \overset{df}{=} \alpha \Rightarrow (\alpha \land \neg \alpha), \\
  \alpha \rightarrow \beta &\overset{df}{=} \alpha \Rightarrow (\alpha \Rightarrow \beta).
\end{align*}
\]

It can be easily seen from the definition of the functor \( \rightarrow^d \) that for any homomorphism \( h \) from the language \((S, \{\Rightarrow, \neg, \land\})\) into the algebra \((\{0, \frac{1}{2}, 1\}, \{f_\Rightarrow, f_\neg, f_\wedge\})\)

\[
1 \quad \text{for} \quad h(\alpha) = 1 \quad \text{and} \quad h(\beta) \in \{0, \frac{1}{2}\}, \\
0 \quad \text{otherwise}.
\]

We also consider two inference rules:

\[
\begin{align*}
  r_{\text{mp}} : \frac{\alpha \Rightarrow \beta, \alpha}{\beta}, \\
  r_{\text{mp}}^d : \frac{\alpha \rightarrow^d \beta, \beta}{\alpha}.
\end{align*}
\]

We will use the following traditional notation for the content of a matrix, the matrix consequence and the consequence based on inference rules.

I. \[ E(M_3) = \{\alpha \in S : \forall h : J \rightarrow M_3 h(\alpha) = 1\}. \]

II. \[ E(M_3^d) = \{\alpha \in S : \forall h : J \rightarrow M_3^d h(\alpha) \in \{0, \frac{1}{2}\}\}. \]
For any $X \subseteq S$

III. $C_{M_3}(X) = \{ \alpha \in S: \forall h, J \rightarrow M_3(h(X) \subseteq \{1\} \Rightarrow h(\alpha) = 1) \}$.

IV. $C_{M_3^d}(X) = \{ \alpha \in S: \forall h, J \rightarrow M_3^d(h(X) \subseteq \{0, \frac{1}{2}\} \Rightarrow h(\alpha) \in \{0, \frac{1}{2}\}) \}$.

V. $C_R(X)$ is the least set $Y$ which is closed under the rule $r_{mp}$ and which satisfies $E(M_3) \cup X \subseteq Y$.

VI. $C_{R^d}(X)$ is the least set $Y$ which is closed under the rule $r_{mp}^d$ and which satisfies $E(M_3^d) \cup X \subseteq Y$.

In [3] it was shown that

**Theorem 1.** $C_{R^d} = C_{M_3^d} = dC_{M_3} = dC_R$.

Thus, the sentential logic $(S, C_{R^d})$ will be called the dual logic with respect to Łukasiewicz’s three – valued logic and will be denoted by $L_3^d$.

(The dual sentential calculus with respect to Łukasiewicz’s three – valued calculus will be denoted by $L_3^d$, as well).

2. **Axiomatization**

The primitive functors of the system $L_3^d$ are the same as primitive functors, which appear in Łukasiewicz’s three – valued calculus.

Let us consider in $L_3^d$ the functor $d$, which according to (1), corresponds in the matrix $M_3^d$ to a function $f_{\downarrow}$ with the following property:

$$f_{\downarrow}(x, y) = \begin{cases} 1 & \text{for } x = 1 \text{ and } (y = 0 \text{ or } y = \frac{1}{2}), \\ 0 & \text{otherwise.} \end{cases}$$

We will show that the following system of formulas is an axiomatization of $L_3^d$:

A1. $(\alpha \rightarrow \beta) \rightarrow \alpha$.

A2. $[(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)] \rightarrow (\gamma \rightarrow \beta)$.

A3. $\alpha \rightarrow [\alpha \rightarrow (\beta \rightarrow \alpha)]$.

A4. $(\beta \rightarrow \neg \alpha) \rightarrow (\beta \rightarrow \alpha)$.

A5. $(\beta \rightarrow \alpha) \rightarrow (\beta \rightarrow \neg \alpha)$.

A6. $[\beta \rightarrow (\beta \rightarrow \alpha)] \rightarrow (\beta \rightarrow \neg \alpha)$.

A7. $[\neg (\alpha \land \beta) \rightarrow \neg \alpha] \rightarrow \neg \beta$.
\[ A8 \quad \neg \alpha \overset{d}{\rightarrow} \neg (\alpha \land \beta), \]
\[ A9 \quad \neg \beta \overset{d}{\rightarrow} \neg (\alpha \land \beta), \]
\[ A10 \quad (\gamma \overset{d}{\rightarrow} \alpha) \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \alpha \land \beta), \]
\[ A11 \quad (\gamma \overset{d}{\rightarrow} \beta) \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \alpha \land \beta), \]
\[ A12 \quad [(\gamma \overset{d}{\rightarrow} \alpha \land \beta) \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \alpha)] \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \beta), \]
\[ A13 \quad (\gamma \overset{d}{\rightarrow} \alpha) \overset{d}{\rightarrow} [\gamma \overset{d}{\rightarrow} \neg (\alpha \Rightarrow \beta)], \]
\[ A14 \quad (\gamma \overset{d}{\rightarrow} \neg \beta) \overset{d}{\rightarrow} [\gamma \overset{d}{\rightarrow} \neg (\alpha \Rightarrow \beta)], \]
\[ A15 \quad [\gamma \overset{d}{\rightarrow} (\alpha \Rightarrow \beta)] \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \neg \alpha), \]
\[ A16 \quad [\gamma \overset{d}{\rightarrow} (\alpha \Rightarrow \beta)] \overset{d}{\rightarrow} \alpha \overset{d}{\rightarrow} \neg \beta, \]
\[ A17 \quad [\gamma \overset{d}{\rightarrow} (\alpha \Rightarrow \beta)] \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \alpha) \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \neg \beta), \]
\[ A18 \quad [\gamma \overset{d}{\rightarrow} \neg \alpha] \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \neg \beta) \overset{d}{\rightarrow} \neg [\gamma \overset{d}{\rightarrow} (\alpha \Rightarrow \beta)], \]
\[ A19 \quad [\gamma \overset{d}{\rightarrow} \beta] \overset{d}{\rightarrow} (\gamma \overset{d}{\rightarrow} \alpha) \overset{d}{\rightarrow} [\gamma \overset{d}{\rightarrow} (\alpha \Rightarrow \beta)], \]
\[ A20 \quad \beta \overset{d}{\rightarrow} (\alpha \Rightarrow \beta). \]

Let’s denote \( A = \{A1, \ldots, A20\}. \)

The only inference rule is the rule

\[ r_{mp}^d: \frac{\alpha \overset{d}{\rightarrow} \beta, \beta}{\alpha} \]

The consequence function based on the rule \( r_{mp}^d \) is defined as follows:

\( C_{R^d}(X) \) is the least set \( Y \) which is closed under the rule \( r_{mp}^d \) and which satisfies \( A \cup X \subseteq Y \).

\( T^d \) stands for the set of all theorems of the logic \( \mathcal{L}^d_3 \).

It is easy to see that \( T^d = C_{R^d}(\emptyset) \).

One can prove, using axioms \( A1, A2 \) and \( A3 \), that the deduction theorem holds for the consequence \( C_{R^d} \).

**Theorem 2.** \( \alpha \in C_{R^d}(X \cup \{\beta\}) \iff \alpha \overset{d}{\rightarrow} \beta \in C_{R^d}(X) \), for every \( X \subseteq S \) and all \( \alpha, \beta \in S \).

### 3. Completeness

Now, we are going to prove the completeness theorem for the logic \( \mathcal{L}^d_3 \) dual to Łukasiewicz’s three-valued logic.
First, we prove that the set of theorems of $L_3^d$ is the same as the content $E(M_3^d)$ of the matrix $M_3^d$.

**Theorem 3.** $T^d = E(M_3^d)$.

**Proof.** Inclusion $T^d \subseteq E(M_3^d)$ is obvious.

To prove that the inclusion $E(M_3^d) \subseteq T^d$ holds we will apply Łos’s method (see [2]).

Let’s assume that $\gamma \notin T^d$. Our aim is to show that $\gamma \notin E(M_3^d)$.

Since $C_{R_3}$ is a finitary consequence operator, according to the relative Lindenbaum’s theorem [4], there is a set $Y \subseteq S$, such that:

(i) $\gamma \notin Y$,
(ii) $C_{R_3}(Y) = Y$,
(iii) $\forall \alpha \in Y$ $\gamma \in C_{R_3}(Y \cup \{\alpha\})$.

Additionally, it can be shown that the set $Y$ has the following property:

(iv) $\forall \alpha \in S (\alpha \in Y$ or $\neg \alpha \in Y$).

To prove this fact we will use Theorem 2 and the axiom $A_6$.

Let’s define three sets of formulas: $Y_0, Y_2, Y_1$ in the following way:

$Y_1 = S \setminus Y$,

$Y_0 = \{ \alpha \in S : \neg \alpha \in Y_1 \}$,

$Y_2 = Y \setminus Y_0$.

Let us observe that:

$Y_0 \cup Y_2 = Y$, $Y \cup Y_1 = S$ and sets $Y_0, Y_2, Y_1$ are pairwise disjoint.

Moreover, using axioms and the deduction theorem one can prove that for all formulas $\alpha, \beta \in S$ the following conditions hold:

1. $\neg \alpha \in Y_1 \iff \alpha \in Y_0$,
2. $\neg \alpha \in Y_0 \iff \alpha \in Y_1$,
3. $\neg \alpha \in Y_2 \iff \alpha \in Y_1$,
4. $\alpha \in Y_2 \iff (\alpha \in Y \land \neg \alpha \in Y_1)$,
5. $\alpha \land \beta \in Y_1 \iff (\alpha \in Y_1 \land \beta \in Y_1)$,
6. $\alpha \land \beta \in Y_0 \iff (\alpha \in Y_0 \lor \beta \in Y_0)$,
7. $\alpha \land \beta \in Y_2 \iff [(\alpha \in Y_2 \land \beta \notin Y_0) \lor (\alpha \notin Y_0 \land \beta \in Y_2)]$,
8. $\alpha \Rightarrow \beta \in Y_1 \iff [\alpha \in Y_0 \lor (\alpha \in Y_2 \land \beta \notin Y_0) \lor \beta \in Y_1]$. 

9. $\alpha \Rightarrow \beta \in Y_0 \iff (\alpha \in Y_1 \land \beta \in Y_0)$,

10. $\alpha \Rightarrow \beta \in Y_\frac{1}{2} \iff [(\alpha \in Y_\frac{1}{2} \land \beta \in Y_1) \lor (\alpha \in Y_1 \land \beta \in Y_\frac{1}{2})]$

The condition 1. is obtained by the definition of the set $Y_0$.

To prove the condition 2. we apply definitions of sets $Y_0$, $Y_1$, axioms $A5$, $A4$ and Theorem 2. We conclude that $\neg \alpha \in Y_0 \iff \neg \alpha \in Y_1 \iff \neg \alpha \notin Y \iff \gamma \in C_{R^4}(Y \cup \{\neg \alpha\}) \Rightarrow \gamma \frac{d}{d} \neg \alpha \in C_{R^4}(Y)$.

The condition 3. is obtained just by conditions 1., 2. and the definition of $Y_\frac{1}{2}$.

4. If $\alpha \in Y_\frac{1}{2}$, then from 3. $\neg \alpha \in Y_\frac{1}{2}$. Thus, by the definition of $Y_\frac{1}{2}$, we conclude that $\alpha \in Y$ and $\neg \alpha \in Y$.

Let $\alpha \in Y$ and suppose that $\alpha \notin Y_\frac{1}{2}$. Then $\alpha \in Y_0$ or $\alpha \in Y_1$.

In the first case, by 1., we have $\neg \alpha \in Y_1$, so $\neg \alpha \notin Y$.

In the second case, from the definition of $Y_1$, we have $\alpha \notin Y$.

So, in both cases we have contradiction.

5. "⇒" Let $\alpha \land \beta \in Y_1$. By the definition of $Y_1$, (iii) and Theorem 2 we conclude that $\alpha \land \beta \notin Y, \gamma \in C_{R^4}(Y \cup \{\alpha \land \beta\}), \gamma \frac{d}{d} \alpha \land \beta \in C_{R^4}(Y)$.

Using axioms $A10$ and $A11$ we have $\gamma \frac{d}{d} \alpha, \gamma \frac{d}{d} \beta \in C_{R^4}(Y)$, thus by Theorem 2 $\gamma \in C_{R^4}(Y \cup \{\alpha\})$ and $\gamma \in C_{R^4}(Y \cup \{\beta\})$. If $\alpha \in Y$ or $\beta \in Y$, then $\gamma \in C_{R^4}(Y) = Y$. Since from (i) $\gamma \notin Y$, thus $\alpha \notin Y$ and $\beta \notin Y$, that is $\alpha \in Y_1$ and $\beta \in Y_1$.

"⇒" If $\alpha, \beta \in Y_1$, then $\alpha \notin Y$ and $\beta \notin Y$. Thus, by (iii) $\gamma \in C_{R^4}(Y \cup \{\alpha\})$ and $\gamma \in C_{R^4}(Y \cup \{\beta\})$.

Therefore, from Theorem 2, we have $\gamma \frac{d}{d} \alpha, \gamma \frac{d}{d} \beta \in C_{R^4}(Y)$.

Applying the axiom $A12$, we obtain that $\gamma \frac{d}{d} \alpha \land \beta \in C_{R^4}(Y)$, thus $\gamma \in C_{R^4}(Y \cup \{\alpha \land \beta\})$. If $\alpha \land \beta \in Y$, then $\gamma \in C_{R^4}(Y) = Y$. Since, by (i), $\gamma \notin Y$, then $\alpha \land \beta \notin Y$ and $\alpha \land \beta \notin Y_1$.

The proof of the other properties is analogous.

Now, we define a valuation $v$ of propositional variables by:

$$v(p_i) = \begin{cases} 1 & \text{for } p_i \in Y_1, \\ \frac{1}{2} & \text{for } p_i \in Y_\frac{1}{2}, \\ 0 & \text{for } p_i \in Y_0, \end{cases}$$

for $i = 1, 2, \ldots$. 

The homomorphic extension \( v' \) of \( v \) onto the set all formulas is given by:

\[
\begin{align*}
  & v'(p_i) = v(p_i), i = 1, 2, \ldots, \\
  & v'(-\alpha) = f_- v'(\alpha), \\
  & v'(\alpha \land \beta) = f_\land (v'(\alpha), v'(\beta)), \\
  & v'(\alpha \Rightarrow \beta) = f_\Rightarrow (v'(\alpha), v'(\beta)).
\end{align*}
\]

It can be proved, by induction on the length of formulas, that for any \( \delta \in S \)

\[
(*) \quad v'(\delta) = \begin{cases} 
  1 & \text{for } \delta \in Y_1, \\
  \frac{1}{2} & \text{for } \delta \in Y_2, \\
  0 & \text{for } \delta \in Y_0.
\end{cases}
\]

Let \( \delta = p_i \). We get

\[
v'(\delta) = v'(p_i) = v(p_i) = \begin{cases} 
  1 & \text{for } p_i \in Y_1, \\
  \frac{1}{2} & \text{for } p_i \in Y_2, \\
  0 & \text{for } p_i \in Y_0,
\end{cases}
\]

for \( i = 1, 2, \ldots \)

Let \( \delta = \neg \alpha \).

Assume that the (*) holds for the formula \( \alpha \).

Then,

\[
v'(\delta) = v'(\neg \alpha) = f_- v'(\alpha).
\]

- If \( \delta \in Y_1 \), then \( \neg \alpha \in Y_1 \), thus from the condition 1. \( \alpha \in Y_0 \), therefore,

  \[
  f_- v'(\alpha) = f_- (0) = 1.
  \]

- If \( \delta \in Y_2 \), then \( \neg \alpha \in Y_2 \), thus from the condition 3. \( \alpha \in Y_2 \), therefore,

  \[
  f_- v'(\alpha) = f_- (\frac{1}{2}) = \frac{1}{2}.
  \]

- If \( \delta \in Y_0 \), then \( \neg \alpha \in Y_0 \), thus from the condition 2. \( \alpha \in Y_1 \), therefore,

  \[
  f_- v'(\alpha) = f_- (1) = 0.
  \]

Let \( \delta = \alpha \land \beta \).

Assume that (*) holds for formulas \( \alpha, \beta \).
Then,

\[ v'(\delta) = v'(\alpha \land \beta) = f_\land(v'(\alpha), v'(\beta)). \]

- If \( \delta \in Y_1 \), then \( \alpha \land \beta \in Y_1 \), thus from the condition 5., \( \alpha \in Y_1 \) and \( \beta \in Y_1 \), therefore,

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(1, 1) = 1. \]

- If \( \delta \in Y_2 \), then \( \alpha \land \beta \in Y_1 \), thus from the condition 7. \( (\alpha \in Y_2 \) and \( \beta \in Y_2 \)) or \( (\alpha \in Y_2 \) and \( \beta \in Y_1 \)) or \( (\alpha \in Y_1 \) and \( \beta \in Y_2 \)).

If \( (\alpha \in Y_2 \) and \( \beta \in Y_2 \)), then

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}. \]

If \( (\alpha \in Y_1 \) and \( \beta \in Y_1 \)), then

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(\frac{1}{2}, 1) = \frac{1}{2}. \]

If \( (\alpha \in Y_1 \) and \( \beta \in Y_2 \)), then

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(1, \frac{1}{2}) = \frac{1}{2}. \]

- If \( \delta \in Y_0 \), then \( \alpha \land \beta \in Y_0 \), thus from the condition 6. \( \alpha \in Y_0 \) or \( \beta \in Y_0 \).

If \( \alpha \in Y_0 \), then

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(0, v'(\beta)) = 0. \]

If \( \beta \in Y_0 \), then

\[ f_\land(v'(\alpha), v'(\beta)) = f_\land(v'(\alpha), 0) = 0. \]

The proof for the other cases, based on conditions 1. - 10., is analogous.

The mapping \( v': S \to \{0, \frac{1}{2}, 1\} \) is a homomorphism from the language \( (S, \{\Rightarrow, \neg, \land\}) \) into the algebra \( (\{0, \frac{1}{2}, 1\}, \{f_\Rightarrow, f_\neg, f_\land\}) \). Since \( \gamma \not\in Y \), therefore \( \gamma \in Y_1 \) and \( v'(\gamma) = 1 \). We have thus shown that \( v' \) falsifies the formula
\( \gamma \) in the matrix \( M^d_3 \). Finally, we have \( \gamma \notin E(M^d_3) \), which completes the proof.

Therefore, the matrix \( M^d_3 \) is weakly adequate with respect to \( L^d_3 \). \( \square \)

Now, we are going to strengthen the result and prove that the dual matrix \( M^d_3 \) is strongly adequate with respect to \( L^d_3 \).

First, let us observe that, from the definition IV. and the property (1), the following lemma holds:

**Lemma 1.** For every \( Z \subseteq S \)

\[ \alpha \overset{d}{\rightarrow} \beta \in C_{M^d_3}(Z) \iff \alpha \in C_{M^d_3}(Z \cup \{ \beta \}) \]

Now, we are ready to prove:

**Theorem 4.**

\( \forall X \subseteq S \left[ C_{R^d}(X) = C_{M^d_3}(X) \right] \)

**Proof.** Let \( \alpha \in C_{R^d}(X) \). Since the operator \( C_{R^d} \) is finitary, there is a set \( Z \), such that \( Z \subseteq X, \text{card}(Z) < \aleph_0 \) and \( \alpha \in C_{R^d}(Z) \).

We consider the following two cases:

a) \( Z = \emptyset \),

b) \( Z = \{ \alpha_1, \ldots, \alpha_n \} \).

In the first case, we have

\[ \alpha \in C_{R^d}(\emptyset) = T^d = E(M^d_3) = C_{M^d_3}(\emptyset) \]

Therefore, \( \alpha \in C_{M^d_3} \).

In the second case, by using the deduction theorem to the formula \( \alpha \in C_{R^d}(\{ \alpha_1, \ldots, \alpha_n \}) \), we obtain:

\[ (\ldots((\alpha \overset{d}{\rightarrow} \alpha_1) \overset{d}{\rightarrow} \alpha_2) \ldots \overset{d}{\rightarrow} \alpha_n) \in C_{R^d}(\emptyset) \]

From the fact that the matrix \( M^d_3 \) is weakly adequate, we have

\[ (\ldots((\alpha \overset{d}{\rightarrow} \alpha_1) \overset{d}{\rightarrow} \alpha_2) \ldots \overset{d}{\rightarrow} \alpha_n) \in C_{M^d_3}(\emptyset) \]
Using Lemma 1, we get
\[ \alpha \in C_{M_3^d}(\{\alpha_1, \ldots, \alpha_n\}). \]
Thus, \( \alpha \in C_{M_3^d}(Z) \). Therefore, \( C_{R^d}(X) \subseteq C_{M_3^d}(X) \). The proof of the inclusion \( C_{M_3^d}(X) \subseteq C_{R^d}(X) \) is analogous.

We have thus shown that matrix \( M_3^d \) is strongly adequate. Therefore, the completeness theorem holds for \( L_3^d \).

References