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SYNTACTICAL AND SEMANTICAL CHARACTERIZATION OF A CLASS OF PARACONSISTENT LOGICS

Abstract

The paper presents a modal formulation of some propositional logics. The idea was used in [3] where the logic Z is formulated with the help of the logic **S5**. The formulation of the logic Z uses a transformation from the set of classical propositional formulae to the set of modal propositional formulae. This formulation is in a sense similar to Jaśkowki's formulation of logic $\mathbf{D_2}$. The idea also refers to works by Segerberg [13], Rasiowa [12] and Vakarelov [14]. The main results of the paper are Lemma 16 and Theorem 1 presenting a class of propositional logics obtained with the help of Béziau's transformation which is applied to other modal logics. As Remark 2 and Theorem 2 prove, a lot of presented propositional logics are paraconsistent.

The paper is a continuation of a work presented as [11]. It is a written version of results presented in [8] and [9].

Introduction

In [3] J.-Y. Béziau presented a formulation of the logic **Z**. Its definition in a sense refers to Jaśkowski's idea of a modalization of connectives. In the present paper we are using Béziau's transformation but in the case of all

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normal modal logics expressible by means of Kripke semantics. Since, as Lemma 16 shows, Béziau's method can be applied to any normal modal logic, in this way we obtain syntactical and semantical characterization of a broad class of propositional logics. A lot of these logics appear to be paraconsistent.

We use standard notions and results from the field of modal logic[1].

1. Class $\mathcal{K}$

Definition 1. Let For be a set of all propositional formulae in the language with connectives $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and the set of propositional variables Var.

Definition 2. Let $\mathcal{K}$ be a class of all logics being any set strictly contained in For, containing the full positive classical logic in the language $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$, including the following version of de Morgan's law:

$$\sim(p \wedge q) \rightarrow (\sim p \vee \sim q) \qquad (\mathrm{dM1}_{\rightarrow})$$

the following version of *Ex falso quodlibet*:

$$\sim(p \rightarrow p) \rightarrow p \qquad (\mathrm{EFQ})$$

and closed under Modus Ponens, any substitution, and the contraposition rule:

$$\frac{\vdash A \rightarrow B}{\vdash \sim B \rightarrow \sim A} \qquad (\mathrm{CONTR})$$

Further, we will prove that $\mathcal{K} \neq \emptyset$.

Obviously, for any $\mathbf{L} \in \mathcal{K}$, $\mathbf{L}$ is closed on the following rule:

$$\frac{\vdash A \leftrightarrow B}{\vdash \sim A \leftrightarrow \sim B} \qquad (\mathrm{EXT}_{\sim})$$

[1] See for example [5].

and contains the following versions of de Morgan's lows:

$$(\sim p \vee \sim q) \rightarrow \sim(p \wedge q) \tag{dM1$_\leftarrow$}$$

$$\sim(p \vee q) \rightarrow (\sim p \wedge \sim q) \tag{dM2$_\rightarrow$}$$

Theses (dM1$_\rightarrow$) and (dM2$_\rightarrow$) can be proved by Modus Ponens, substitution, (CONTR), and the following theses of the positive logic: $(p \wedge q) \rightarrow p, (p \wedge q) \rightarrow q, ((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r), p \rightarrow (p \vee q), q \rightarrow (p \vee q)$, and $((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$.

LEMMA 1. *For any* $\mathbf{L} \in \mathcal{K}$, $\mathbf{L}$ *contains:*

$$\sim(p \rightarrow p) \rightarrow q \tag{EFQ'}$$

$$q \leftrightarrow (q \vee \sim(p \rightarrow p)) \tag{‡}$$

PROOF. Let $\mathbf{L} \in \mathcal{K}$. $(q \rightarrow q) \rightarrow (p \rightarrow p) \in \mathbf{L}$ and also $\sim(p \rightarrow p) \rightarrow \sim(q \rightarrow q) \in \mathbf{L}$, by (CONTR). Since $\sim(q \rightarrow q) \rightarrow q \in \mathbf{L}$, thus also $\sim(p \rightarrow p) \rightarrow q \in \mathbf{L}$.

For the proof of (‡) we use (EFQ') and the positive logic. □

2. Semantics for $\mathcal{K}$. The class $\mathcal{P}$ and logic $\mathbf{P_K}$

DEFINITION 3. A *relational frame* (or briefly a *frame*) is a pair $\langle W, R\rangle$ consisting of a nonempty set W, and a binary relation R on W. Elements of the set W we call the *worlds*, while R is an *accessibility relation*.

DEFINITION 4. A *valuation* is any function $v : \mathrm{Var} \longrightarrow 2^W$.

DEFINITION 5. A *model* is a triple $\langle W, R, v\rangle$, where $\langle W, R\rangle$ is a frame and v is a valuation. We say that $\langle W, R, v\rangle$ is based on the frame $\langle W, R\rangle$.

Below we recall a definition used by J.-Y. Béziau.

DEFINITION 6. A formula A is *true* in a world $w \in W$ under a valuation v (notation: $w \Vdash_v A$) iff

1. if A a propositional variable, then
$w \Vdash_v A \iff w \in v(A)$.
2. if A has a form $\sim B$, for some formula B, then
$w \Vdash_v \sim B \iff$ there is a world w' such that wRw' and it is not the case that $w' \Vdash_v B$ (the notation $w' \nVdash_v B$).
3. if A is of the form $B \wedge C$, for some formulae B and C, then
$w \Vdash_v B \wedge C \iff w \Vdash_v B$ and $w \Vdash_v C$.
4. if A is of the form $B \vee C$, for some formulae B and C, then
$w \Vdash_v B \vee C \iff w \Vdash_v B$ or $w \Vdash_v C$.
5. if A is of the form $B \to C$, for some formulae B and C, then
$w \Vdash_v B \to C \iff w \nVdash_v B$ or $w \Vdash_v C$.
6. if A is of the form $B \leftrightarrow C$, for some formulae B and C, then
$w \Vdash_v B \leftrightarrow C \iff (w \Vdash_v B$ and $w \Vdash_v C)$ or $(w \nVdash_v B$ and $w \nVdash_v C)$.

Definition 7. A formula A is *true* in a model $M = \langle W, R, v\rangle$ (notation $M \Vdash A$) iff $w \Vdash_v A$ for each $w \in W$.

Definition 8. A formula A is *valid* in a frame $\langle W, R\rangle$ iff it is true in all models based on $\langle W, R\rangle$.

Lemma 2. *For any model $\langle W, R, v\rangle$, for any $w \in W$: $w \nVdash_v \sim(p \to p)$.*

Lemma 3.

a) ($dM1_{\to}$) *and* (EFQ) *are true in every frame.*
b) *All positive classical theses are true in every frame.*
c) *The set of all formulae true in a given world of a model based on a frame is closed under* MP.
d) *The set of all formulae true in a given model based on a frame is closed under contraposition rule.*
e) *The set of all formulae true in a given frame is closed under substitution.*

Proof. a) Let us consider any model $\langle W, R, v\rangle$ and a world $w \in W$. We assume that $w \Vdash_v \sim(A \wedge B)$. By the case 2 of Definition 6 it means that there is $w' \in W$, such that wRw' and $w' \nVdash_v (A \wedge B)$. Also by Definition 6 it means that either $w' \nVdash_v A$ or $w' \nVdash_v B$, therefore $w \Vdash_v \sim A \vee \sim B$.

The second point follows from Lemma 2.

b) and c) Standard.

d) Let us assume that a formula $A \to B$ is true in some model $\langle W, R, v\rangle$. Let $w \in W$ be any possible world in which $w \Vdash_v {\sim} B$. It means that there is $w' \in W$ such that wRw' and $w' \nVdash_v B$. But since $A \to B$ is true in the model, so is true in each world of this model, especially in w'. However, $w' \nVdash_v B$ therefore $w' \nVdash_v A$. It means that $w \Vdash_v {\sim} A$.

e) Standard. □

By Lemmas 2 and 3 we have:

Corollary 1. *Let $\langle W, R\rangle$ be any frame.*

(a) *The set of all formulae valid in $\langle W, R\rangle$ does not equal For.*

(b) *The set of all formulae valid in $\langle W, R\rangle$ belongs to $\mathcal{K}$.*

Thus, the class $\mathcal{K}$ is non-empty. So, we can consider the smallest logic from $\mathcal{K}$. Let us denote it by $\mathbf{P_K}$.

We will define the following class of paraconsistent logics being a subclass of $\mathcal{K}$:

Definition 9. For any logic $\mathbf{L} \in \mathcal{K}$, $\mathbf{L}$ is *paraconsistent* iff the formula

$$p \to ({\sim} p \to q) \tag{†}$$

does not belong to $\mathbf{L}$.

Let $\mathcal{P}$ be a class of all paraconsistent logics from $\mathcal{K}$. Directly from the definition of $\mathbf{P_K}$, by Lemma 3 we have:

Corollary 2. (Soundness) *For any $A \in \mathbf{P_K}$, A is true in every frame.*

Corollary 3. $\mathbf{P_K} \in \mathcal{P}$.

Proof. Let $W = \{w_1, w_2\}$, $R = \{\langle w_1, w_2\rangle\}$ and $v(p) = \{w_1\}$, $v(q) = \emptyset$. It is easy to see that the formula (†) is not true in the world w_1 of the model $\langle W, R, v\rangle$, thus, (†) is not valid in $\langle W, R\rangle$, i.e. (†) $\notin \mathbf{P_K}$. □

3. Canonical models for $\mathcal{K}$

Definition 10. Let $\mathbf{L} \in \mathcal{K}$. A set X of formulae is *inconsistent with respect to the logic* $\mathbf{L}$ (or briefly $\mathbf{L}$-inconsistent) iff there are $n \geq 1$ and formulae $A_1, \ldots, A_n \in X$, such that for any formula B we have $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow B$. The set X is *c*onsistent with respect to the logic $\mathbf{L}$ (or shortly $\mathbf{L}$-consistent) iff it not $\mathbf{L}$-inconsistent.

We have obvious:

LEMMA 4. *Let $\mathbf{L} \in \mathcal{K}$. If $X \subseteq Y \subseteq$ For and X is $\mathbf{L}$-inconsistent, then Y is $\mathbf{L}$-inconsistent.*

Besides, we have:

LEMMA 5. *If $\sim(p \rightarrow p) \in X$, then X is $\mathbf{L}$-inconsistent, for any logic $\mathbf{L} \in \mathcal{K}$.*

PROOF. Let $\mathbf{L} \in \mathcal{K}$ and $\sim(p \rightarrow p) \in X$. By (EFQ') and the substitution, for any $A \in$ For we have that $\sim(p \rightarrow p) \rightarrow A \in \mathbf{L}$. □

DEFINITION 11. Let $\mathbf{L} \in \mathcal{K}$. A set X is *maximally* consistent with respect to the logic $\mathbf{L}$ (or briefly a maximally $\mathbf{L}$-consistent set) iff $\mathbf{L}$-consistent and for any formula $A \notin X$ the set $X \cup \{A\}$ is $\mathbf{L}$-inconsistent.

The following lemma holds:

LEMMA 6. *For $\mathbf{L} \in \mathcal{K}$, every maximally $\mathbf{L}$-consistent set contains $\mathbf{L}$ and is closed under* MP.

PROOF. Let X be a maximally $\mathbf{L}$-consistent set. Assume that $A \in \mathbf{L}$ and $A \notin X$. By Definitions 10 and 11 it means that there are $n \geq 0$, formulae $A_1, \ldots A_n \in X$ such that for any formula E: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge A) \rightarrow E$. By the positive logic we see that $\vdash_{\mathbf{L}} A \rightarrow ((A_1 \wedge \ldots \wedge A_n) \rightarrow E)$. Thus, by MP we have that for any formula E: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow E$. So, X is $\mathbf{L}$-inconsistent. A contradiction.

Assume, for a contradiction, that $A \rightarrow B \in X$, $A \in X$, and $B \notin X$. By Definitions 10 and 11 there are $n \geq 0$ and formulae $A_1, \ldots A_n \in X$ such that for any formula E: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge B) \rightarrow E$. By the positive logic it means that for any formula E we obtain $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge (A \rightarrow B) \wedge A) \rightarrow E$. A contradiction. □

We have also:

LEMMA 7. *For any $\mathbf{L} \in \mathcal{K}$, a maximally $\mathbf{L}$-consistent set X, and any formulae A, B we have:*

1. $A \wedge B \in X$ *iff* $A \in X$ *and* $B \in X$,
2. $A \rightarrow B \in X$ *iff* $A \notin X$ *or* $B \in X$,
3. $A \vee B \in X$ *iff* $A \in X$ *or* $B \in X$,
4. $A \leftrightarrow B \in X$ *iff* $A, B \in X$ *or* $A, B \notin X$.

Proof. Ad 1. By the positive logic and Lemma 6.

Ad 2. Left-to-right implication is just the second point of Lemma 6.

Right-to-left implication. Assume that $A \notin X$. By Definition 10 there are $n \geq 0$ and formulae $A_1, \ldots A_n \in X$ such that for any formula E: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge A) \rightarrow E$ and for E equal B we get by the positive logic that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow (A \rightarrow B)$. By case 1 we have $(A_1 \wedge \ldots \wedge A_n) \in X$ and by Lemma 6 we see that $A \rightarrow B \in X$. If $B \in X$ we also obtain that $A \rightarrow B \in X$ by the positive logic and the second point of Lemma 6.

Cases 3 and 4 follows by the positive logic, cases 1-2 and Lemma 6. □

We have the following version of Lindenbaum Lemma.

Lemma 8. (Lindenbaum Lemma) *If* $A \in For$, $X \subseteq For$, *and there is no* $n \geq 0$ *and a set* $\{A_1, \ldots, A_n\} \subseteq X$ *such that* $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$[2] *then there is a maximally* **L**-*consistent set* Y *containing* X *such that* $A \notin Y$.

Proof. Assume that $A \in$ For, $X \subseteq$ For, and there is no $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq X$ such that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$. Let $C_1, \ldots, C_i, \ldots$ be an infinite sequence of all formulae from For. By induction we define a sequence of sets $(X_i)_{i \in \mathsf{N}}$.[3] We put: $X_0 = X$ and for any $i \geq 0$:

$$X_{i+1} = \begin{cases} X_i \cup \{C_{i+1}\} & \text{if there is no } n \geq 0 \text{ and a set } \{A_1, \ldots, A_n\} \subseteq X_i, \text{ such that } \vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge C_{i+1}) \rightarrow A. \\ X_i & \textit{otherwise.} \end{cases}$$

Of course, we have $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$. By induction, one can easily see that for any $i \in \mathsf{N}$ there is no $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq X_i$ such that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$.

We prove by induction that for any $i \geq 0$, X_i is **L**-consistent. By the assumption, the set X_0 is **L**-consistent. Indeed, if there are $n \geq 1$ and formulae $A_1, \ldots, A_n \in X$, such that for any formula B we have $\vdash_{\mathbf{L}}$

[2] The case $n = 0$ we treat as equivalent to the fact that $\nvdash_{\mathbf{L}} A$.

[3] Let N denotes the set of all natural numbers.

$(A_1 \wedge \ldots \wedge A_n) \rightarrow B$, then for B equal A we would receive a contradiction with the assumption of the lemma.

Let $i \geq 0$ and assume that X_i is **L**-consistent. Assume, for a contradiction, that there are $n \geq 1$ and formulae $A_1, \ldots, A_n \in X_{i+1}$, such that for any formula B we have $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow B$. Thus, X_{i+1} does not equal X_i and $C_{i+1} \in \{A_1, \ldots, A_n\}$. But again, for B equal A we obtain a contradiction. In the standard way we deduce, that the set $Y = \bigcup_{i \in \mathsf{N}} X_i$ is **L**-consistent. Obviously, for any $i \geq 0$, $A \notin X_i$ and so $A \notin Y$.

Now, let us observe that for any formula C, we have that $A \rightarrow C \in Y$. If for some formula C we have that $A \rightarrow C \notin Y$, then there are $i_0 \geq 1$, $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq X_{i_0 - 1}$ such that $C_{i_0} = A \rightarrow C$ and $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge (A \rightarrow C)) \rightarrow A$. However, by the positive logic we would have that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow ((A \rightarrow C) \rightarrow A)$ and by the transitivity of implication and Peirce's law we have $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$. A contradiction with the definition of the set $X_{i_0 - 1}$.[4]

We prove that Y is maximally **L**-consistent set. Let $B \notin Y$. By the definition of Y it means that for every $i \in \mathsf{N}$, $B \notin X_i$. Let us assume that B is appearing in the sequence $(C_i)_{i \geq 1}$ with a number k. By the definition of the sequence $(X_i)_{i \in \mathsf{N}}$, there is $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq X_{k-1}$ such that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge C_k) \rightarrow A$. Thus, by the observation from the previous paragraph we see that $A \rightarrow {\sim}(p \rightarrow p) \in Y$ and since by (EFQ) and substitution $\vdash_{\mathbf{L}} {\sim}(p \rightarrow p) \rightarrow E$, for any $E \in \mathrm{For}$, therefore we have $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n \wedge C_k \wedge (A \rightarrow {\sim}(p \rightarrow p))) \rightarrow E$, for any $E \in \mathrm{For}$. □

DEFINITION 12. Let $\mathbf{L} \in \mathcal{K}$.

1. Let W be the set of all maximally consistent sets with respect to **L**, and R be a binary relation on W defined as follows: $wRw' \iff \forall_A (A \notin w' \Rightarrow {\sim} A \in w)$.
   The pair $\langle W, R \rangle$ is called *the canonical frame of the logic* **L**.
2. The *canonical model* of the logic **L** is a model $\langle W, R, v \rangle$, where $\langle W, R \rangle$ is the canonical frame of the logic **L** and the following condition is satisfied for any variable A:

$$v(A) = \{w \in W : A \in w\}.$$

   i.e. $w \Vdash_v A \iff A \in w$.

[4] The idea of this part of the proof is borrowed from D. Batens [1].

We have:

LEMMA 9. *Let* $\mathbf{L} \in \mathcal{K}$ *and* $\langle W, R, v\rangle$ *be the canonical model of* $\mathbf{L}$.

a) *For each formula* A *and* $w \in W$ *the following holds:* $w \Vdash_v A \Longleftrightarrow A \in w$.

b) *A formula is true in* $\langle W, R, v\rangle$ *iff it is a theorem of* $\mathbf{L}$.

PROOF. a) Goes by induction.

1. if A is a propositional variable, then by Definition 12 we have: $w \Vdash_v A \Longleftrightarrow A \in w$.
2. if A has a form $\sim B$, for some formula B, then If $w \Vdash_v \sim B$ then there is a world w' such that wRw' and $w' \nVdash_v B$, therefore by the inductive hypothesis $B \notin w'$ and by the definition of the relation R: $\sim B \in w$.
   Let us assume that $\sim B \in w$. We consider the set $\mathcal{W} := \{A : \sim A \notin w\}$. By the definition of $\mathcal{W}$ we have $B \notin \mathcal{W}$.
   We prove that the assumptions of Lemma 8 are fulfilled for the formula B and the set $\mathcal{W}$. Let us assume, for a contradiction, that either $\vdash_{\mathbf{L}} B$ or there are $n \geq 1$ and formulae $A_1, \ldots, A_n$, such that for each $1 \leq i \leq n : \sim A_i \notin w$ and $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \to B$. The first case can not happen since we would have $\vdash_{\mathbf{L}} B$, then also $\vdash_{\mathbf{L}} (p \to p) \to B$, thus, by (CONTR) we would have $\vdash_{\mathbf{L}} \sim B \to \sim(p \to p)$ and using the initial assumption $\sim B \in w$, by Lemma 6 we would have that $\sim(p \to p) \in w$, thus, by Lemma 5 we would have a contradiction.
   We consider the second case. By the contraposition rule we get $\vdash_{\mathbf{L}} \sim B \to \sim(A_1 \wedge \ldots \wedge A_n)$. Thus, by (dM1$_{\to}$) and obvious induction we receive: $\vdash_{\mathbf{L}} \sim B \to (\sim A_1 \vee \ldots \vee \sim A_n)$. However, $\sim B \in w$, every maximally $\mathbf{L}$-consistent set contains every theorem of $\mathbf{L}$, and w is closed on MP, so $\sim A_1 \vee \ldots \vee \sim A_n \in w$, and by Lemma 7 case 3 we have that either $\sim A_1 \in w$, or $\sim A_2 \in w \ldots$ or $\sim A_n \in w$ contrary to the assumptions.
   The above arguments show that there is no $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq \mathcal{W}$ such that $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \to B$. Thus, by Lemma 8 the set $\mathcal{W}$ is contained is some maximally $\mathbf{L}$-consistent set w' such that $B \notin w'$. By the definition of $\mathcal{W}$ we see that wRw'. Indeed, if for some formula $C : C \notin w'$, then $C \notin \mathcal{W}$, but then also $\sim C \in w$ (if $\sim C$ were not in w, then C would belong to $\mathcal{W}$). By the inductive

hypothesis we have $w' \not\Vdash_v B$ and by the definition of the truth in a world we obtain that $w \Vdash_v {\sim} B$.

3. If A is of the form $B \wedge C$, for some formulae B and C, then: $w \Vdash_v B \wedge C$ iff $w \Vdash_v B$ and $w \Vdash_v C$. By the inductive hypothesis it is equivalent to the fact that $B \in w$ and $C \in w$. And by Lemma 7 the case 1 this holds iff $B \wedge C \in w$.
   The proofs for other cases go also in a standard way.

b) The proof is standard. We present it only for the sake of the completeness of our considerations.

($\Leftarrow$). If A is a theorem of the logic $\mathbf{L}$, then by Lemma 6 we have that $A \in w$, where w is a maximally $\mathbf{L}$-consistent set. But by the case a) it means that A is true in the canonical model.

($\Rightarrow$). Let us assume that $A \notin \mathbf{L}$. There is no $n \geq 0$ and $\{A_1, \ldots, A_n\} \subseteq \mathbf{L}$ such that: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$. Indeed, if for some $n \geq 0$ and a set $\{A_1, \ldots, A_n\} \subseteq \mathbf{L}$: $\vdash_{\mathbf{L}} (A_1 \wedge \ldots \wedge A_n) \rightarrow A$, then also $\vdash_{\mathbf{L}} A$. A contradiction.

Thus, by Lemma 8 there is a maximally $\mathbf{L}$-consistent set w, such that $A \notin w$ and by the case a) of the present lemma $w \not\Vdash_v A$ i.e. A is false in the canonical model. □

By the above lemma we obtain:

Corollary 4. *If a formula is true in every frame it is a theorem of the logic* $\mathbf{P_K}$.

4. General Completeness result

Let $\mathrm{For}^{\mathrm{M}}$ denotes the set of all propositional modal formulae in the langauge with logical constants: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box, \Diamond$.

Definition 13. Let $-^u : \mathrm{For}^{\mathrm{M}} \longrightarrow \mathrm{For}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}$ the following conditions:

1. $(a)^u = a$
2. $(\neg A)^u = ((A)^u \rightarrow {\sim}(p \rightarrow p))$
3. $(A \wedge B)^u = (A^u \wedge B^u)$
4. $(A \vee B)^u = (A^u \vee B^u)$
5. $(A \rightarrow B)^u = (A^u \rightarrow B^u)$

6. $(A \leftrightarrow B)^u = (A^u \leftrightarrow B^u)$
7. $(\Diamond A)^u = \sim((A)^u \to \sim(p \to p))$,
8. $(\Box A)^u = (\sim((A)^u) \to \sim(p \to p))$.

Lemma 10. *For any* $A \in \mathrm{For}^{\mathrm{M}}$*, any model* $\mathcal{M} = \langle W, R, v\rangle$*, and any* $w \in W$ *the following holds* $w \models_v A$ *iff* $w \Vdash_v (A)^u$.[5]

Proof. The proof proceeds by induction on the complexity of a formula. For the case of propositional letters the condition holds by the item 1 of Definition 13.

Assume that the equivalence under consideration holds for any formula of complexity smaller than the complexity od a given formula A. Let us consider the main functor of A. The cases of $\wedge$, $\vee$, $\to$, and $\leftrightarrow$ are obvious. Let us consider other cases.

Ad 2. A formula A has a form $\neg B$ for some formula B. We have $w \models_v \neg B$ iff $w \not\models_v B$. By the inductive hypothesis it is equivalent to the fact that $w \nVdash_v (B)^u$. The last statement can be equivalently rewritten in the following way: $w \nVdash_v (B)^u$ or there is $w' \in W$, such that wRw' and $w' \nVdash_v (p \to p)$. But this means that $w \Vdash_v ((B)^u \to \sim(p \to p))$

Ad 7. A formula A has a form $\Diamond B$ for some formula B. We have: $w \models_v \Diamond B$ iff there is $w' \in W$, such that wRw' and $w' \models_v B$ iff (by the inductive hypothesis) there is $w' \in W$, such that wRw' and $w' \Vdash_v (B)^u$. While the last statement can be trivially rewritten in the following way: there is $w' \in W$, such that wRw' and ($w' \Vdash_v (B)^u$ and (there is no $w'' \in W$, such that $w'Rw''$ and $w'' \nVdash_v (p \to p)$)). But the last statement holds iff there is $w' \in W$, such that wRw' and it is not the case that ($w' \nVdash_v (B)^u$ or there is $w'' \in W$, such that $w'Rw''$ and $w'' \nVdash_v (p \to p)$) iff $w \Vdash_v \sim((B)^u \to \sim(p \to p))$.

Ad 8. A formula A has a form $\Box B$ for some formula B. We have: $w \models_v \Box B$ iff for any $w' \in W$, such that wRw': $w' \models_v B$ iff (by the inductive hypothesis) for any $w' \in W$, such that wRw': $w' \Vdash_v (B)^u$. The statement can be trivially rewritten in the following way: (it is not the case that there is $w' \in W$, such that wRw' and $w' \nVdash_v (B)^u$) or there is $w'' \in W$, such that (wRw'' and $w'' \nVdash_v (p \to p)$). But the last statement holds iff $w \Vdash_v (\sim(B)^u \to \sim(p \to p))$. □

[5] '$\models$' is referred to standard truth in a model of a modal logic.

We will need also an operation from For into $\mathrm{For}^{\mathrm{M}}$:

DEFINITION 14. Let $-^{m}$: For $\longrightarrow \mathrm{For}^{\mathrm{M}}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}^{\mathrm{M}}$ the following conditions:

1. $(a)^{m} = a$,
2. $(\sim A)^{m} = \Diamond\neg((A)^{m})$,
3. $(A \wedge B)^{m} = (A^{m} \wedge B^{m})$,
4. $(A \vee B)^{m} = (A^{m} \vee B^{m})$,
5. $(A \to B)^{m} = (A^{m} \to B^{m})$,
6. $(A \leftrightarrow B)^{m} = (A^{m} \leftrightarrow B^{m})$.

Before we present the main theorem, we prove some other lemmas.

LEMMA 11. *For any $A \in$ For, any model $\mathcal{M} = \langle W, R, v\rangle$, and any $w \in W$ the following holds: $w \Vdash_{v} A$ iff $w \models_{v} (A)^{m}$.*

PROOF. Goes by induction. The case of propositional variable is obvious.

Assume that the equivalence under consideration holds for any formula of a complexity smaller than the complexity of a given formula A. Let us consider the main functor of A and any $w \in W$. The cases of $\wedge$, $\vee$, $\to$, and $\leftrightarrow$ are obvious. Let $A = \sim B$, for some $B \in$ For. We have $w \Vdash_{v} A$ iff $w \Vdash_{v} \sim B$ iff there is a world w' such that wRw' and $w' \nVdash_{v} B$. By the inductive hypothesis this holds iff there is a world w' such that wRw' and $w' \not\models_{v} (B)^{m}$ iff $w \models_{v} \Diamond\neg((B)^{m})$ iff $w \models_{v} (A)^{m}$. □

LEMMA 12. *For any $A \in$ For, any model $\mathcal{M} = \langle W, R, v\rangle$, and any $w \in W$ the following holds: $w \Vdash_{v} A$ iff $w \Vdash_{v} ((A)^{m})^{u}$.*

PROOF. Follows directly from Lemmas 11 and 10. □

From the last lemma and Corollary 4 we immediately obtain:

LEMMA 13. *For any $A \in$ For:*

$$\vdash_{\mathbf{P_K}} A \leftrightarrow ((A)^{m})^{u} \qquad (\mathrm{ext}_{\mathbf{P_K}})$$

Similarly we have

LEMMA 14. *For any $A \in \mathrm{For}^{\mathrm{M}}$, any model $\mathcal{M} = \langle W, R, v\rangle$, and any $w \in W$ the following holds: $w \models_{v} A$ iff $w \models_{v} ((A)^{u})^{m}$.*

PROOF. Follows directly from Lemmas 10 and 11. □

From the above lemma and the completeness result for the logic **K** we immediately obtain:

LEMMA 15. *For any* $A \in \mathrm{For}^{\mathrm{M}}$*:*

$$\vdash_{\mathbf{K}} A \leftrightarrow ((A)^u)^m \qquad \mathrm{ext}_{\mathbf{K}}$$

DEFINITION 15. For $X \subseteq \mathrm{For}^{\mathrm{M}}$, let $\mathbf{K}[X]$ be the smallest normal modal logic containing the logic **K** and the set X. For a given logic $\mathbf{S} = \mathbf{K}[X]$, let $\mathbf{P}_{\mathbf{K}[X]}$ be the smallest logic in the class $\mathcal{K}$ which contains $\mathbf{P}_{\mathbf{K}}$ and the set of 'new' axioms $X^u = \{A^u : A \in X\}$.

LEMMA 16. *Let* $\mathbf{S} = \mathbf{K}[X]$*. For* $A \in \mathrm{For}^{\mathrm{M}}$ *we have* $A \in \mathbf{S}$ *iff* $\vdash_{\mathbf{P_S}} (A)^u$.

PROOF. ($\Rightarrow$). Let $A \in \mathrm{For}^{\mathrm{M}}$ and $A \in \mathbf{S}$. Consider a proof of A: $C_1, \ldots, C_k$. We prove by induction on i that for any $1 \leq i \leq k$: $\vdash_{\mathbf{P_S}} (C_i)^u$. Let us take any i: $1 \leq i \leq k$. We consider the cases:

1. C_i is a thesis of the classical positive logic; then $(C_i)^u$ is just C_i and of course $\vdash_{\mathbf{P_S}} (C_i)^u$.
2. C_i is of the form $(\neg A \to \neg B) \to (B \to A)$ for some $A, B \in \mathrm{For}$. We prove that $\vdash_{\mathbf{P_S}} \big(((A)^u \to {\sim}(p \to p)) \to ((B)^u \to {\sim}(p \to p))\big) \to ((B)^u \to (A)^u)$ i.e. $\vdash_{\mathbf{P_S}} (C_i)^u$. By (‡) from Lemma 1 we have that $\vdash_{\mathbf{P_S}} ((A)^u \vee {\sim}(p \to p)) \to (A)^u$ and by the positive classical logic we have that $\vdash_{\mathbf{P_S}} \big(((A)^u \to {\sim}(p \to p)) \to {\sim}(p \to p)\big) \to (A)^u$, and by the positive logic we have the following theorems: $\vdash_{\mathbf{P_S}} (B)^u \to \Big(\big(((A)^u \to {\sim}(p \to p)) \to {\sim}(p \to p)\big) \to (A)^u\Big)$, $\vdash_{\mathbf{P_S}} \Big((B)^u \to \big(((A)^u \to {\sim}(p \to p)) \to {\sim}(p \to p)\big)\Big) \to \big((B)^u \to (A)^u\big)$ and finally, since for any $D, E, F, G \in \mathrm{For}$: $\vdash_{\mathbf{P_S}} \big((D \to (E \to F)) \to G\big) \leftrightarrow \big((E \to (D \to F)) \to G\big)$ we have the required theorem.
3. C_i is obtained by MP from $C_k = C_j \to C_i$, where $k, j < i$. By the inductive hypothesis $\vdash_{\mathbf{P_S}} (C_k)^u$ and $\vdash_{\mathbf{P_S}} (C_j \to C_i)^u$. By Definition 13 we have $(C_j \to C_i)^u = (C_j)^u \to (C_i)^u$ and $(C_i)^u$ also arises from $(C_k)^u$ and $(C_j)^u$ by MP.
4. C_i is of the form $\Diamond A \leftrightarrow \neg\Box\neg A$ for some $A \in \mathrm{For}^{\mathrm{M}}$. By (‡), the classical positive logic and substitution we have $\vdash_{\mathbf{P_S}} {\sim}((A)^u \to {\sim}(p \to p)) \leftrightarrow \big(({\sim}((A)^u \to {\sim}(p \to p)) \to {\sim}(p \to p)) \to {\sim}(p \to p)\big)$. Thus, $\vdash_{\mathbf{P_S}} (C_i)^u$.

5. C_i is of the form $\Box(A \to B) \to (\Box A \to \Box B)$ for some $A, B \in \mathrm{For}^{\mathrm{M}}$. By the positive classical logic we have: $\vdash_{\mathbf{P_S}} (\sim((A)^u \to (B)^u) \to \sim(p \to p)) \to \Big(\big(\sim((A)^u) \to \sim(p \to p)\big) \to \big(\big(\underline{\sim((A)^u \to (B)^u) \vee \sim((A)^u)}\big) \to \sim(p \to p)\big)\Big)$. But by the fact that $\vdash_{\mathbf{P_S}} (((A)^u \to (B)^u) \wedge (A)^u) \to (B)^u$, *via* (CONTR) and (dM1$_\to$) we see that $\vdash_{\mathbf{P_S}} \sim((B)^u) \to \big(\underline{\sim((A)^u \to (B)^u) \vee \sim((A)^u)}\big)$. And again by the positive classical logic we have $\vdash_{\mathbf{P_S}} (\sim((A)^u \to (B)^u) \to \sim(p \to p)) \to \Big(\big(\sim((A)^u) \to \sim(p \to p)\big) \to \big(\sim((B)^u) \to \sim(p \to p)\big)\Big)$ i.e. $\vdash_{\mathbf{P_S}} (\sim((A \to B)^u) \to \sim(p \to p)) \to \Big(\big(\sim((A)^u) \to \sim(p \to p)\big) \to \big(\sim((B)^u) \to \sim(p \to p)\big)\Big)$, i.e. $\vdash_{\mathbf{P_S}} (C_i)^u$.
6. C_i is obtained by necessity rule from some formula C_j where $j < i$. By the inductive hypothesis $\vdash_{\mathbf{P_S}} (C_j)^u$. By the positive classical logic we also have $\vdash_{\mathbf{P_S}} (p \to p) \to (C_j)^u$ while by (CONTR) we have $\vdash_{\mathbf{P_S}} \sim((C_j)^u) \to \sim(p \to p)$, thus, by Definition 13 we get $\vdash_{\mathbf{P_S}} (C_i)^u$.
7. C_i is obtained by a substitution of some formulae $D_1^i \ldots D_k^i \in \mathrm{For}^{\mathrm{M}}$ into a formula C_j where $j < i$. Since $\mathbf{P_S}$ is closed on any substitution it is enough to observe that $(C_i)^u$ is equivalent on the basis of $\mathbf{P_S}$ to a formula which arises from $(C_j)^u$ by the appropriate substitution of formulae $(D_1^i)^u, \ldots, (D_k^i)^u$. The last statement can be easily proved using the fact that $\sim(A \to A) \leftrightarrow \sim(B \to B) \in \mathbf{P_K}$.
8. C_i is a specific axiom of the logic $\mathbf{S}$ i.e. $C_i \in X$. By the definition of $\mathbf{P_S}$: $(C_i)^u \in \mathbf{P_S}$.

($\Leftarrow$). Let $A \in \mathrm{For}$ and $\vdash_{\mathbf{P_S}} A$. Consider a proof of A: $C_1, \ldots, C_k$. We prove by induction on i that for any $1 \leq i \leq k$: $(C_i)^m \in \mathbf{S}$. Let us take any i: $1 \leq i \leq k$. We consider the following cases:

1. C_i is a thesis of the classical positive logic; then $(C_i)^m$ is just C_i and of course $(C_i)^m \in \mathbf{K}$.
2. C_i is of the form $\sim(p \wedge q) \to (\sim p \vee \sim q)$. It is easy to see that on the basis of $\mathbf{K}$ the formula $(\sim(p \wedge q) \to (\sim p \vee \sim q))^m$ is equivalent to the formula $\Diamond(\neg p \vee \neg q) \to (\Diamond\neg p \vee \Diamond\neg q)$. The last one belongs to $\mathbf{K} \subseteq \mathbf{S}$.
3. C_i is of the form $\sim(p \to p) \to p$. On the basis of $\mathbf{K}$ the formula $(\sim(p \to p) \to p)^m$ is just the formula $\Diamond\neg(p \to p) \to p$ which belongs to $\mathbf{K} \subseteq \mathbf{S}$.

4. C_i is obtained by MP from $C_k = C_j \to C_i$, where $k, j < i$. By the inductive hypothesis $(C_k)^m \in \mathbf{S}$ and $(C_j \to C_i)^m \in \mathbf{S}$. By Definition 14 we have $(C_j \to C_i)^m = (C_j)^m \to (C_i)^m$ and $(C_i)^m$ also arises from $(C_k)^m$ and $(C_j)^m$ by MP.

5. $C_i = \sim B \to \sim A$ is obtained by (CONTR) from $C_k = A \to B$, where $k < i$ and $A, B \in$ For. By the inductive hypothesis $(A \to B)^m = A^m \to B^m \in \mathbf{S}$, while by necessity rule, the axiom K, MP, contraposition, and inter-definability of modal functors we have that $(C_i)^m = \Diamond\neg((B)^m) \to \Diamond\neg((A)^m) \in \mathbf{S}$.

6. C_i is obtained by a substitution of some formulae $D_1^i \dots D_k^i \in$ For into a formula C_j where $j < i$. Since $\mathbf{S}$ is closed under any substitution, it is enough to observe that $(C_i)^m$ arises from $(C_j)^m$ by the appropriate substitution of formulae $(D_1^i)^m, \dots, (D_k^i)^m$. This can be easily proved.

7. $C_i \in X^u$, i.e. C_i is of the form $(A)^u$, for some $A \in X$. Thus, $(C_i)^m = ((A)^u)^m$ and by Lemma 15 it is on the basis of $\mathbf{K}$ equivalent to A. So, also $(C_i)^m \in \mathbf{S}$.

Finally, let $A \in \text{For}^{\text{M}}$. If $\vdash_{\mathbf{P_S}} (A)^u$ than by the above reasoning we have $((A)^u)^m \in \mathbf{S}$ but by Lemma 15 also $A \in \mathbf{S} = \mathbf{K}[X]$. □

THEOREM 1. *Let* $\mathbf{S} = \mathbf{K}[X]$. *If logic* $\mathbf{S}$ *is complete with respect to some class of frames with accessibility relation fulfilling a given condition* C*, then for the logic* $\mathbf{P_S}$ *the following holds:*

For any $A \in$ *For:* A *is true in every frame with accessibility relation fulfilling the condition* C *iff* A *is a theorem of the logic* $\mathbf{P_S}$.

PROOF. Assume that A is true in every frame with accessibility relation fulfilling the condition C. By Lemma 11 it holds iff $(A)^m$ is true in every frame with accessibility relation fulfilling the condition C and by the assumption on $\mathbf{S}$ it is true iff $(A)^m \in \mathbf{S}$. By Lemma 16 it is equivalent to the fact that $\vdash_{\mathbf{P_S}} ((A)^m)^u$ while by Lemma 13 and the definition of $\mathbf{P_S}$ it holds iff $\vdash_{\mathbf{P_S}} A$. □

REMARK 1. $(\dagger)^m \in \mathbf{K}[\mathbf{X}]$ iff $p \to \Box p \in \mathbf{K}[\mathbf{X}]$.

PROOF. It is easy to see that on the basis of any normal modal logic the formula $(p \to (\sim p \to q))^m$ is equivalent to $(p \wedge \neg q) \to \Box p$. □

Remark 2. For any normal modal logic $\mathbf{K}[X]$, such that $p \to \Box p \notin \mathbf{K}[X]$ the logic $\mathbf{P}_{\mathbf{K}[X]}$ is paraconsistent.

Proof. Assume a given logic $\mathbf{P}_{\mathbf{K}[X]}$ is not paraconsistent, i.e. $(p \to ({\sim} p \to q)) \in \mathbf{P}_{\mathbf{K}[X]}$, but by a substitution it would mean that also $(p \to ({\sim} p \to {\sim}(p \to p))) \in \mathbf{P}_{\mathbf{K}[X]}$. Since by Definition 13: $(p \to \Box p)^u = (p \to ({\sim} p \to {\sim}(p \to p)))$, thus, by Lemma 16 this holds iff $p \to \Box p \in \mathbf{K}[X]$. $\Box$

As a direct corollary from the above remarks we have:

Remark 3. If $(\dagger)^m \notin \mathbf{K}[X]$, then $(\dagger) \notin \mathbf{P}_{\mathbf{K}[X]}$, i.e. $\mathbf{P}_{\mathbf{K}[X]} \in \mathcal{P}$. This explains why '**P**' has been used in the chosen notation for logic $\mathbf{P}_{\mathbf{K}[X]}$.

Let us recall that $\mathbf{D} = \mathbf{K}[\Box p \to \Diamond p]$ and $\mathbf{Triv} = \mathbf{K}[\Box p \to \Diamond p, p \to \Box p]$. By Remark 2 we have:

Theorem 2. *For any normal modal logic $\mathbf{S} \supseteq \mathbf{D}$, such that* $\mathrm{For}^{\mathrm{M}} \neq \mathbf{S} \neq \mathbf{Triv}$*, the logic $\mathbf{P}_{\mathbf{S}}$ is paraconsistent.*

5. Examples

5.1. The Logic $\mathbf{P}_{\mathbf{T}}$

Let us recall that the logic $\mathbf{P}_{\mathbf{T}}$ is obtained by adding to $\mathbf{P}_{\mathbf{K}}$ a single extra axiom:

$$(\Box p \to p)^u .$$

i.e. $({\sim} p \to {\sim}(p \to p)) \to p$.

Corollary 5. (Completeness for $\mathbf{P}_{\mathbf{T}}$) *A formula A is true in every frame with reflexive accessibility relation iff A is a theorem of the logic $\mathbf{P}_{\mathbf{T}}$.*

We can simplify the formulation of the logic $\mathbf{P}_{\mathbf{T}}$:

Theorem 3. *The logic $\mathbf{P}_{\mathbf{T}}$ is the smallest logic in $\mathcal{K}$ containing the formula $p \vee {\sim} p$.*

Proof. ($\supseteq$) We prove that the smallest logic containing $\mathbf{P}_{\mathbf{K}}$ and the formula $p \vee {\sim} p$ is contained in $\mathbf{P}_{\mathbf{T}}$, by referring to Corollary 5 and Lemma 3. Using them, it is enough to show that $p \vee {\sim} p$ is true in every frame with reflexive accessibility relation. Let us consider a model $\langle W, R, v\rangle$ where R

is reflexive on $W \times W$. Let $w \in W$ and $w \not\Vdash_v p$, since wRw, there is a world w' such that wRw' and $w' \not\Vdash_v p$ (of course we mean the world w). Therefore $w \Vdash_v \sim p$.

($\subseteq$) By Lemma 9 and Corollary 5 it is enough to show that the accessibility relation of the canonical model of Theorem's right-hand side logic is reflexive. Let $\langle W, R\rangle$ be the canonical frame, $w \in W$ and A be any formula. We assume that $A \notin w$. By Lemma 6: $A \vee \sim A \in w$ and by Lemma 7 case 3 either $A \in w$ or $\sim A \in w$, so $\sim A \in w$ i.e. wRw.

We can also prove the theorem syntactically.

It is easy to see right-to-left inclusion. Let us prove left-to-right inclusion. By $(\sim p \to p) \to \big((p \to \sim(p \to p)) \to (\sim p \to \sim(p \to p))\big)$ and $(\Box p \to p)^u$ with the help of transitivity we have $(\sim p \to p) \to \big((p \to \sim(p \to p)) \to p\big)$, while *via* Peirce's law $((p \to \sim(p \to p)) \to p) \to p$ and once again transitivity, we obtain $(\sim p \to p) \to p$ i.e. by the positive classical logic $p \vee \sim p$. □

5.2. The Logic $\mathbf{P_{KB}}$

The logic $\mathbf{P_{KB}}$ is obtained by adding to $\mathbf{P_K}$ an extra axiom:

$$(p \to \Box\Diamond p)^u\,.$$

Corollary 6. (Completeness for $\mathbf{P_{KB}}$) *A formula A is true in every frame with symmetric accessibility relation iff A is theorem of the logic* $\mathbf{P_{KB}}$.

Theorem 4. *The logic* $\mathbf{P_{KB}}$ *is the smallest logic in* $\mathcal{K}$ *containing the formula* $\sim\sim p \to p$.

Proof. ($\supseteq$) By Corollary 6 and Lemma 3 it is enough to show that the formula $\sim\sim p \to p$ is true in every frame with symmetric accessibility relation. Let us consider a model $\langle W, R, v\rangle$ where R is symmetric on $W \times W$. Let $w \in W$ and $w \Vdash_v \sim\sim p$. By the definition of the truth in a world there is a world $w' \in W$ such that wRw' and $w' \not\Vdash_v \sim p$. It means that for every world $w'' \in W$ such that $w'Rw''$ holds: $w'' \Vdash_v p$. But since R is symmetric we obtain $w'Rw$, so in particular for w we have $w \Vdash_v p$.

($\subseteq$) By Lemma 9 and Corollary 6 it is enough to show that the accessibility relation of the canonical model of the smallest logic in $\mathcal{K}$ containing

$\mathbf{P_K}$ and the formula $\sim\sim p \to p$ is symmetric. Let $\langle W, R\rangle$ be the canonical frame, $w, w' \in W$, such that wRw' and $A \in$ For. We prove that $w'Rw$. Assume that $A \notin w$. By Lemma 6 we have $\sim\sim A \to A \in w$ and by the Lemma 7 case 2 either $\sim\sim A \notin w$ or $A \in w$, so by our assumption $\sim\sim A \notin w$. By the definition of R, since wRw' we have $\sim A \in w'$. □

Below we skip similar proofs of Theorems 5 and 6.

5.3. The Logic $\mathbf{P_{KE}}$

The logic $\mathbf{P_{KE}}$ is obtained by adding to $\mathbf{P_K}$ an extra axiom:

$$(\Diamond p \to \Box\Diamond p)^u \,.$$

Corollary 7. (Completeness for $\mathbf{P_{KE}}$) *A formula A is true in every frame with Euclidean accessibility relation iff A is theorem of the logic $\mathbf{P_{KE}}$.*

Theorem 5. *The logic $\mathbf{P_{KE}}$ is the smallest logic in $\mathcal{K}$ containing the formula $\sim p \wedge \sim\sim p \to q$.*

5.4. The Logic $\mathbf{P_{D^*}}$

Here, we refer to the completeness result for the normal modal logic $\mathbf{D}^*$ presented in [10]. The logic $\mathbf{P_{D^*}}$ is obtained by adding to $\mathbf{P_K}$ an axiom:

$$(\Box\Diamond p \to \Diamond p)^u \,.$$

Corollary 8. *A formula A is true in every frame with accessibility relation fulfilling the following condition:*

$$\forall_w \exists_{w'} \Big(wRw' \wedge \forall_{w''}(w'Rw'' \to wRw'') \Big) \qquad *$$

iff A is theorem of the logic $\mathbf{P_{D^*}}$.

Theorem 6. *The logic $\mathbf{P_{D^*}}$ is the smallest logic in $\mathcal{K}$ containing the formula $\sim p \vee \sim\sim p$.*

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