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ON A MINIMALITY CONDITION

Let L be an arbitrary non-empty set, and let $\mathcal{R}$ be an arbitrary non-empty collection of its non-empty subsets. (Metaphysically, L is to be interpreted as the totality of ,,elementary situations", and $\mathcal{R}$ as the totality of their ,,realizations", i.e. maximally compatible sets of them.). For any $M \subset L$, set:

$$r(M) = \{R \in \mathcal{R} : M \cap R \neq \emptyset\}, \text{ and } \mathcal{B}_M = \{B \subset M : r(B) = r(M)\}$$

The question is: when does the collection $\mathcal{B}_M$ contain minimal members?

In [1] and [2] Hawranek and Zygmunt have answered our question by showing that a member B of $\mathcal{B}_M$ is minimal in it if and only if the following holds:
$\bigwedge x \in B \bigvee R \in \mathcal{R} : B \cap R = \{x\}$. Call that ,,the HZ-condition".

In [3] and [4] we have defined a relation of ,,entailment" between sets (of elementary situations) as follows: $A \vdash B$ iff $\bigwedge R \in \mathcal{R} : A \subset R \Rightarrow B \cap R \neq \emptyset$. Let ,,$B/x$" be short for ,,$B - \{x\}$", and ,,$x \vdash B$" short for ,,$\{x\} \vdash B$". Then we can see the HZ-condition to be equivalent to the following:

$$\bigwedge x \in B : \neg(x \vdash B/x) \tag{1}$$

Indeed, as for $HZ \Rightarrow (1)$ suppose the contrary: $x \vdash B/x$, for some $x \in B$. Then for any $R \in \mathcal{R}$, such that $x \in R$ there is an $y \in B/x$ such that $y \in R$ – which contradicts the HZ-condition.

Conversely, assuming (1) suppose $x \in B$. This means there is an $R \in \mathcal{R}$ such that $x \in R$ but $B/x \cap R = \emptyset$. Then clearly $B \cap R = \{x\}$ for that particular R. QED.

Observe that for any non-empty set $A \subset B$ we have: $X \vdash A \Rightarrow X \vdash B$. Hence condition (1) is equivalent to:

$$\bigwedge A \subset B/x : \neg(x \vdash A) \tag{2}$$

Thus we get eventually the proposition: a member B of the collection $\mathcal{B}_M$ is minimal in it if and only if no element of B entails a subset of B of which it is not part.

Hence, in particular, B is minimal only if none of its elements entails any other; and consequently, only if none is entailed by any other either. Thus in minimal members of $\mathcal{B}_M$ there cannot be any entailments between distinct elements: if $x, y \in B$, and $x \neq y$, then $\neg(x \vdash y)$.

References

[1] J. Hawranek, J. Zygmunt, *Comments on a Question of Wolniewicz's*, **Bulletin of the Section of Logic** 19:4 (1990).

[2] J. Hawranek, J. Zygmunt, *Wokół pewnego zagadnienia z dziedziny półkrat górnych z jednością*, **Logika**, vol. 15, Wrocław 1993.

[3] B. Wolniewicz, *Entailments and Independence in Join-Semilattices*, **Bulletin of the Section of Logic** 18:1 (1989).

[4] B. Wolniewicz, *Logic and Metaphysics. Studies in Wittgenstein's Ontology of Facts*, Warszawa 1999, Polish Semiotic Society (Warsaw University).