Andrzej Indrzejczak

SEQUENT CALCULI FOR MONOTONIC MODAL LOGICS

Abstract

It is well known that the epistemic or doxastic interpretation of modal constants leads to unintuitive results in the context of normal logics. This is the main reason that weaker logics are considered as better candidates. One can easily notice however, that investigations into proof methods and decision procedures for such logics are rather modest. There is a lot of books and papers devoted to exploration of nonaxiomatic formalizations of modal logic but, only Fitting [2] covers some regular ones. Lavendhomme and Lucas [5] offer cut-free sequent calculus only for the most basic classical and monotonic systems. In particular, the systems containing axioms 4 and 5 have been of no special interest up to now, despite their importance in epistemic and doxastic logics. In what follows we offer an extension of work in [5] covering all combinations of axioms D, T, 4, B and 5 over the weakest monotonic logic $M$. Our approach to the logics is, in contrast to that of [5], purely syntactical. First we present a class of sequent calculi that are equivalent to axiomatic formalizations of the logics we mentioned, and then show by standard Gentzen argument, that most of them are cut-free.

1 Introduction

By Monotonic Modal Logic we understand any structural (i.e. closed under substitution) logic $L$ that contains CPL (Classical Propositional logic) and is closed with respect to two rules: (MP) (Modus Ponens) and (M) $\Box \varphi \to \Box \psi \in L$ if $\varphi \to \psi \in L$ (see [1]). We assume the sole primitive modal constant is $\Box$. Let $M$ stand for the weakest monotonic logic. We will consider extensions of $M$ obtained by addition of any of the “big five”: 
Any extension of $M$ by some axioms $X$, $Y$, ..., $Z$ will be denoted as $MXY...Z$. Moreover, any logic containing $D$ will be called as a $D$-logic, containing $T$ – as a $T$-logic and so on. There are only 15 distinct monotonic logics axiomatized by possible combinations of these axioms over (and including) $M$, due to the following dependencies:

**Lemma 1.**

\[
\begin{align*}
\text{CPL} + T &\vdash D \\
\text{CPL} + T + 5 &\vdash B \\
\text{CPL} + D + 4 + B &\vdash T \\
\text{MT}5 &\vdash 4 \\
\text{MB}4 &\vdash 5 \\
\text{MB}5 &\vdash 4
\end{align*}
\]

In the first three cases we have used the form $\text{CPL} + X$ instead $MX$ because we want to stress that there is no use of $(M)$ in the proof.

The lattice of these logics is isomorphic to the well known lattice of normal logics obtained by the combination of these axioms over $K^1$.

## 2 Sequent Calculi

In the following we will examine sequent formalizations of the logics under consideration. As a basis we use a standard version of the sequent calculus for $\text{CPL}$ denoted by $\text{SC}$, where sequents are pairs of (finite) multisets of formulae, axioms are of the form $\varphi \Rightarrow \varphi$. Except of a pair of rules for every (nonmodal) constant we have structural rules of contraction, weakening (for both sides of a sequent) and cut. In order to get $\text{SC-L}$, where $L$ denotes any of the 15 logics under consideration, we must consider the following rules:

\footnote{It is not the case of regular logics, since – for example – any regular logic containing $B$ is normal.}
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\[(M) \varphi \Rightarrow \psi \Rightarrow \Box \varphi, \Box \psi \]

\[(T) \Box \varphi, \Gamma \Rightarrow \Delta \Rightarrow \Box \varphi, \Gamma \Rightarrow \Delta \]

\[(B) \Rightarrow \Box \varphi, \psi \Rightarrow \varphi, \Box \psi \]

\[(D) \varphi, \psi \Rightarrow \Box \varphi, \Box \psi \]

\[(4) \Box \varphi \Rightarrow \psi \Rightarrow \Box \psi \]

\[(5) \Rightarrow \Box \varphi, \psi \Rightarrow \Box \varphi, \Box \psi \]

(M) is the basic modal rule and it is enough to add it to SC in order to get SC-M (see [5]). We use the same name for this rule as for the rule applied in axiomatic formulation since this should lead to no troubles. Replacing (M) by the stronger regularity rule (R), where instead of a single formula \(\varphi\) in the antecedent we admit any nonempty multiset of formulae leads to the class of regular logics. The rest of rules correspond in modular fashion to respective axioms. We state this fact in the next

**Lemma 2.** SC-M+X is equivalent to SC-M+(X), where X is one of D, T, 4, B or 5.

**Proof.** We show the case of 5 and (5). From right to left we have the following proof:

\[(\text{AX}) \Box \varphi \Rightarrow \Box \varphi \]
\[\Rightarrow \neg \Rightarrow \varphi, \neg \Box \varphi \]
\[\neg \Rightarrow \varphi, \neg \Box \varphi \]
\[\Rightarrow \neg \Rightarrow \varphi \Rightarrow \neg \Box \varphi \]
\[\Rightarrow \neg \Rightarrow \Rightarrow \neg \Box \varphi \Rightarrow \neg \Box \varphi \]

From left to right: First note that in the standard way we can deduce \(\neg \Box \varphi \Rightarrow \Box \neg \Box \varphi\) from 5 by one use of \((\Rightarrow \Rightarrow)\) and (Cut). We denote this proof figure as \(D_1\). So we have:

\[\Rightarrow \Box \varphi, \psi \]
\[\neg \Box \varphi \Rightarrow \psi \]
\[(\text{Cut}) \neg \Box \varphi \Rightarrow \Box \psi \]
\[\Rightarrow \neg \Rightarrow \Box \psi, \neg \Box \varphi \]

\[(\text{M}) \Rightarrow \neg \Rightarrow \Box \varphi \Rightarrow \Box \psi \]

\[(\neg \Rightarrow) \Rightarrow \neg \Rightarrow \Box \varphi \Rightarrow \Box \psi \]

\[(\Rightarrow \neg) \Rightarrow \neg \Rightarrow \Box \psi, \neg \Box \varphi \]
From the last sequent by (Cut) on \( \neg \neg \varphi \Rightarrow \Box \varphi \) we obtain the desired conclusion \( \Rightarrow \Box \varphi, \Box \psi \) which shows that (5) is admissible in \( \text{SC-M}+5 \).

In this paper we are interested primarily in how many of these 15 logics satisfy the cut-elimination theorem. In order to maximize the number we need two more rules:

\[(D5) \varphi \Rightarrow \Box \psi \quad (D4) \Box \varphi, \Box \psi \Rightarrow \varphi, \Box \psi \Rightarrow \]

Both rules are dispensable in fact, since we can prove their admissibility but we need them to obtain cut-free formalizations of logics \( \text{MD}4 \), \( \text{MD}5 \) and \( \text{MD}45 \). We will denote \( \text{SC-MD}4+(D4) \) by \( \text{SC}^-\text{MD}4 \), \( \text{SC-MD}5+(D5) \) by \( \text{SC}^-\text{MD}5 \) and \( \text{SC-MD}45+(D4)+(D5) \) by \( \text{SC}^-\text{MD}45 \). One can prove:

**Lemma 3.** \( \text{SC-L} \) is equivalent to \( \text{SC}^-\text{L} \), for \( \text{L} \) being \( \text{MD}4 \), \( \text{MD}5 \) or \( \text{MD}45 \).

**Proof.** In one direction it is obvious, so it is sufficient to show that (D4) is admissible in \( \text{SC-MD}4 \) and (D5) is admissible in \( \text{SC-MD}5 \); construction of suitable proofs makes evident that both rules are admissible in \( \text{SC-MD}45 \). We show only admissibility of (D5); for (D4) the proof is identical, only the position of the parameter \( \Box \psi \) is switched to the antecedent of the sequent.

\[(\Rightarrow \neg) \varphi \Rightarrow \Box \psi \quad \neg \varphi, \varphi \Rightarrow \quad (\neg \Rightarrow) \Box \psi, \neg \varphi \Rightarrow \varphi, \Box \psi \Rightarrow \quad (\text{AX})
\]

\[(5) \Rightarrow \Box \psi, \neg \varphi \quad \Box \neg \varphi, \varphi \Rightarrow \quad (D) \Box \varphi \Rightarrow \Box \psi
\]

### 3 Cut-elimination

Now we prove that in \( \text{SC}^-\text{MD}45 \) (Cut) is admissible. We present a standard Gentzen’s argument by double induction on cut-rank and on the grade of the cut-formula (see e.g. [3], [4] for classical logic, [6], [7], and in particular [11] for modal logics). Recall that the cut-formula is simply the formula which is eliminated from both premises and the grade of the
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A detailed analysis of the whole proof is rather tiresome since we have a lot of cases to consider due to many modal rules. On the other hand suitable proof-figures showing how to reduce the grade or rank are quite similar, so we illustrate only some of them. Let’s start with the base step of induction on cut-rank. Here cut-rank=2 and we make an induction on the grade of the cut-formula and the only interesting case for us is when it is some $\Box \varphi$. In general we have 6 different rules introducing such formulae into sequents; fortunately some of them operate only on antecedents ((D), (D4)) whereas some only on succedents ((5)). Anyway, the number of possible combinations is 12, since 3 rules introduce box into succedent and 4 into antecedent. In case the left premise of a cut is by (M) we show the subcase where the right premise is by (D5). Here, the cut of the form:

\[
\varphi \Rightarrow \psi \\
\Box \varphi \Rightarrow \Box \psi \\
\Box \psi \Rightarrow \Box \chi
\]

is replaced by the cut of lower grade:

\[
\varphi \Rightarrow \psi \\
\Box \varphi \Rightarrow \Box \chi
\]

Subcases where the right premise comes by (M), (D) or (D4) are analysed in a similar way i.e. by permutation of cut with the application of the
rule in question. If the left premise is by (4) the most interesting subcase is with the right premise obtained by (D). Here the cut of the form:

\[
\frac{\Box \varphi \Rightarrow \psi, \chi \Rightarrow \psi, \chi \Rightarrow (D)}{(4) \frac{\Box \varphi \Rightarrow \Box \psi}{\Box \varphi, \Box \chi \Rightarrow (D)}}
\]

is replaced by:

\[
\frac{\Box \varphi \Rightarrow \psi, \chi \Rightarrow \psi, \chi \Rightarrow (D)}{(4) \frac{\Box \varphi \Rightarrow \Box \psi}{\Box \varphi, \Box \chi \Rightarrow (D)}}
\]

What is important in the above proof-figure is that we cannot obtain the desired sequent without the application of our additional rule (D4) which explains the failure of cut-elimination in SC-MD45. Other subcases are treated similarly, in particular, if the right premise is by (D4) or (D5) we obtain the desired sequent immediately after pushing cut upward.

When the left premise is by (5) we can see that also (D5) is necessary if the right premise is by (D),

\[
\frac{\Rightarrow \Box \varphi, \psi \Rightarrow \psi, \chi \Rightarrow \psi, \chi \Rightarrow (D)}{(5) \frac{\Rightarrow \Box \varphi, \Box \psi}{\Box \chi \Rightarrow \Box \varphi, \Box \chi \Rightarrow (D)}}
\]

is replaced by:

\[
\frac{\Rightarrow \Box \varphi, \psi \Rightarrow \psi, \chi \Rightarrow \psi, \chi \Rightarrow (D)}{(5) \frac{\Rightarrow \Box \varphi}{\Box \chi \Rightarrow \Box \varphi, \Box \chi \Rightarrow (D)}}
\]

The subcase with (M) on the right is reduced by permutation of (Cut) and (5), the other two lead to desired sequents immediately.

Next we go to induction on the rank. When one of the premises is generated by a classical rule the reduction is made on this premise, exactly as in the original proof of Gentzen for classical logic. So we have to analyse...
only these cases where both premises are obtained by modal rules. Let’s start with the situation when the right rank > 1. There are 8 cases: (4) or (D4) on the right versus (M), (4), (D5) or (5) on the left. Below we have a proof-figure where the left premise is by one of (M), (4) or (D5) (we denote any of these rule as (⋆)) and the right one is by (4):

\[(⋆) \quad \Box \varphi \Rightarrow \Box \psi \quad \Box \psi \Rightarrow \Box \chi \quad (4)\]

(Cut) \[\Box \varphi \Rightarrow \Box \chi\]

it is replaced by the following cut of lower rank:

\[(⋆) \quad \Box \varphi \Rightarrow \Box \psi \quad \Box \psi \Rightarrow \Box \chi \quad (4)\]

(Cut) \[\Box \varphi \Rightarrow \Box \chi\]

The cases with (⋆) on the left but (D4) on the right are treated in a similar way. Two subcases are left: with (5) applied on the left and (4) or (D4) on the right. We illustrate the latter as it shows the necessity of (D5) in our calculus for this reduction step.

\[(5) \quad \Rightarrow \Box \varphi, \psi \quad \Box \psi, \chi \Rightarrow \Box \chi \quad (D4)\]

(Cut) \[\Box \chi \Rightarrow \Box \varphi\]

is replaced by:

\[(5) \quad \Rightarrow \Box \varphi, \Box \psi \quad \Box \psi, \Box \chi \Rightarrow \Box \chi \quad (D4)\]

(Cut) \[\Box \chi \Rightarrow \Box \varphi\]

(D5) \[\Box \chi \Rightarrow \Box \varphi\]

When the left rank > 1 we have 10 cases to analyse: (5) or (D5) on the left versus (M), (4), (D), (D4) or (D5) on the right. Below we have the situation illustrating (5) versus (⋆) which is (M), (4) or (D5):

\[(5) \quad \Rightarrow \Box \varphi, \psi \quad \Box \psi, \Box \chi \Rightarrow \Box \chi \quad (⋆)\]

(Cut) \[\Rightarrow \Box \psi, \Box \chi\]
it is replaced by:

\[(\text{Cut}) \quad \frac{\Diamond \varphi, \psi \quad \Box \varphi \Rightarrow \Box \chi}{\Rightarrow \psi, \Box \chi \quad \Rightarrow \Box \psi, \Box \chi} \quad (\ast)\]

When the left premise is by (5) and the right by (D) or (D5) then the proof-figure:

\[(\text{Cut}) \quad \frac{\Rightarrow \Box \varphi, \psi \quad \Rightarrow \Box \varphi, \Box \psi \quad \Box \varphi, \Box \chi \Rightarrow \ast}{\Box \chi \Rightarrow \Box \psi} \quad (5)\]

is replaced by:

\[(\text{Cut}) \quad \frac{\Rightarrow \Box \varphi, \psi \quad \Box \varphi \Rightarrow \Box \chi \Rightarrow \ast}{\Box \chi \Rightarrow \Box \psi} \quad (4)\]

Note that even if the right premise is by (D) we must apply (4) in this reduction step and it is exactly the point showing the failure of cut-elimination in MD5. All subcases with (D5) on the left are treated similarly; also if the right premise is generated by (D) we need (D4) after transformation to get the desired sequent.

The full analysis of reduction on rank demands also consideration of cases where the cut-formula is not a parameter but a side-formula or principal-formula of the rule applied immediately before (Cut). In case of right rank \(> 1\) the only candidate is (D4). If the cut-formula is a side-formula and left premise is by (M), (4) or (D5) we have the following situation:

\[(\ast) \quad \frac{\Box \varphi \Rightarrow \Box \psi \quad \Box \Box \psi, \Box \psi \Rightarrow \ast}{\Box \varphi, \Box \Box \psi \Rightarrow \ast} \quad (4)\]

which is replaced by the cut of lower rank:
A similar reduction is performed when the left premise is by (5). When the cut-formula is principal and the same rules are involved the situation looks like this:

\[
\begin{array}{c}
\phi \Rightarrow \\
\psi, \ \phi \Rightarrow \\
\psi \end{array} \quad \text{(Cut)}
\]

\[
\begin{array}{c}
\phi, \ \psi \Rightarrow \\
\psi, \ \psi \Rightarrow \\
\phi \end{array} \quad \text{(Weakening)}
\]

it is replaced by the following figure:

\[
\begin{array}{c}
\phi \Rightarrow \\
\psi, \ \phi \Rightarrow \\
\psi \end{array} \quad \text{(Cut)}
\]

\[
\begin{array}{c}
\phi \Rightarrow \\
\phi, \ \phi \Rightarrow \\
\phi \end{array} \quad \text{(Contraction)}
\]

where both cuts are of lower rank and hence eliminable by the induction hypothesis. The cases with left rank $> 1$ are connected only with (5) and their treatment is symmetric to those discussed above.

\section{4 General results}

A detailed analysis of the proof allows to extract the cases sufficient for elimination of the cut in many sublogics of $\text{MD45}$. In the course of the proof we have pointed out that in $\text{SC-MD5}$ cut cannot be eliminated but all other logics are cut-free, so we have:

\textbf{Lemma 4:} $\text{SC-MD}$, $\text{SC-M4}$, $\text{SC-M5}$, $\text{SC}^*\text{-MD}$ and $\text{SC-M45}$ are cut-free.

It is worth noticing that in the context of normal logics standard Gentzen formalizations of $\text{K45}$ and $\text{KD45}$ were shown to be cut-free by
Shvarts [8], but both $K5$ and $KD5$ are not. Shvarts applies the following rule:

\[(K45) \quad \Gamma, \Box \Delta \Rightarrow \Box \Pi, \Sigma
\]

\[\Box \Gamma, \Box \Delta \Rightarrow \Box \Pi, \Box \Sigma
\]

with side-condition that $\Pi$ and $\Sigma$ is nonempty in $SC-K45$.

Takano [10] explores $SC-K5$ and $SC-KD5$ where Shvart’s rule is replaced by:

\[(K5) \quad \Gamma \Rightarrow \Box \Delta, \varphi
\]

\[\Box \Gamma \Rightarrow \Box \Delta, \Box \varphi
\]

Takano shows that although cut is not eliminable in his calculi, they satisfy an extended subformula property. His counterexample to cut-elimination in $SC-K5$ does not apply to $SC-M5$ because it is not a thesis of $M5$ (but it is a thesis of $MD5$!). The reason why cut-elimination fails in $SC-K5$ whereas it goes with no difficulty in $SC-M5$ is connected with the possibility of nonempty antecedents in (K5). Consider the case of cut with left rank $> 1$ of the form:

\[
\begin{align*}
(K5) & \quad \Gamma \Rightarrow \Box \Delta, \Box \varphi, \psi \\
(Cut) & \quad \Box \Gamma, \Box \Pi \Rightarrow \Box \Delta, \Box \Sigma, \Box \varphi, \Box \psi
\end{align*}
\]

In Shvart’s $SC-K45$ we can obtain the proper sequent (with no duplication of boxes on elements of $\Pi$) by the application of (K45) instead of (K5), so reduction is possible. In $SC-M5$ the problem does not arise because the antecedent of the premise for (5) must be empty. Even if we use on the right (M) instead of (K5) both $\Gamma$ and $\Pi$ must be empty but note that if we consider instead the use of regularity rule (R) mentioned in section 2 we have the same problem with reduction on the rank (nonempty
It shows that if we change SC-M5 on its regular counterpart SC-R5 we still fail with cut-elimination.

For B-logics (MB, MDB, MTB, MB4 and MTB4) we cannot prove cut-elimination, at least by standard methods. This fact corresponds exactly to the situation in normal logics, where only elimination of analytic cut is provable (see Takano [9]). We are left with T-logics. In case of MT and MT4 the proof of cut-elimination goes exactly as for normal logics T and S4. The case of MT5 (which is a B-logic as well since MT5=MT45=MTB4) is negative because the reduction of rank fails.

Consider the proof-figure:

\[
\frac{\Rightarrow \Box \varphi, \psi}{\Rightarrow \Box \varphi, \Box \psi} \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box \varphi, \Gamma \Rightarrow \Delta} \frac{\Gamma \Rightarrow \Delta}{(T)}
\]

\[(\text{Cut})\]

If we reduce the left rank we obtain:

\[
\frac{\Rightarrow \Box \varphi, \psi}{\Rightarrow \Box \varphi, \Box \psi} \frac{\Box \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Box \psi}
\]

\[(\text{Cut})\]

There is no way to obtain the desired sequent, namely \(\Gamma \Rightarrow \Delta, \Box \psi\).

As a consequence we have the following result:


## 5 Decision Procedure

We are not going to explore in detail the problem of constructive completeness proof for these calculi or how to extract decision procedures from them. Yet some remarks are in order. Note that if we apply to any sequent some procedure for downward saturation (only rules for classical constants performed until we have some compound nonmodal formulae) we may stop on some branch with the sequent of the form

\[
\frac{\vdash}{\Gamma, \Box \varphi_1, \ldots, \Box \varphi_k \Rightarrow \Box \psi_1, \ldots, \Box \psi_k, \Delta}
\]
where $\Gamma \cup \Delta \subseteq \text{VAR}$. Now, if any of them has $\Gamma \cap \Delta = \emptyset$ and $l + k > 0$ we can continue proof-search on this branch. Consider the following sets:

- $S_M = \{ \varphi_i \Rightarrow \psi_j : i \leq l, j \leq k \}$
- $S_4 = \{ \Box \varphi_i \Rightarrow \psi_j : i \leq l, j \leq k \}$
- $S_5 = \{ \Rightarrow \psi_i, \psi_j : i \leq k, j \leq k \}$
- $S_D = \{ \varphi_i, \varphi_j \Rightarrow : i \leq l, j \leq l \}$
- $S_{D5} = \{ \varphi_i \Rightarrow \Box \psi_j : i \leq l, j \leq k \}$
- $S_{D4} = \{ \varphi_i, \Box \varphi_j \Rightarrow : i \leq l, j \leq l \}$

For each logic having a cut-free formalization we define suitable set $S_L$ which is the union of some sets displayed above. Details are set forth in the following table:

<table>
<thead>
<tr>
<th>Calculus</th>
<th>$S_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC-M, SC-MT</td>
<td>$S_M$</td>
</tr>
<tr>
<td>SC-MD</td>
<td>$S_M \cup S_D$</td>
</tr>
<tr>
<td>SC-M4, SC-MT4</td>
<td>$S_M \cup S_4$</td>
</tr>
<tr>
<td>SC-M5</td>
<td>$S_M \cup S_5$</td>
</tr>
<tr>
<td>SC*-MD4</td>
<td>$S_M \cup S_D \cup S_4 \cup S_{D4}$</td>
</tr>
<tr>
<td>SC-M45</td>
<td>$S_M \cup S_4 \cup S_5$</td>
</tr>
<tr>
<td>SC*-MD45</td>
<td>$S_M \cup S_D \cup S_4 \cup S_5 \cup S_{D4} \cup S_{D5}$</td>
</tr>
</tbody>
</table>

Any element of $S_L$ is a sequent which generates a subtree. Roughly, the proof-search procedure works as follows. We saturate initial sequent applying successively classical rules and in case of MT and MT4 additionally contraction and (T) on each modal formula in the antecedent. Either each branch finishes with (AX) or some branch is finished with an atomic sequent (only variables) or with a sequent of the form $\parallel$. In the first case we have a proof, in the second we can falsify an initial sequent. The third alternative leads to the creation of subtrees, where we repeat the saturation of each nested tree. Note that the procedure sketched above must terminate even in the case of some 4-logics. It is well known fact that in normal (or regular) 4-logics we can generate infinite branches due to duplication of $\Box$-formulae. In case of monotonic logics M4 and MT4 this problem disappears since no such formula is put with a box and without it at the same sequent. Namely, we
never obtain at this stage a sequent of the form $\Box \varphi, \varphi \Rightarrow \psi$ but two different sequents: $\Box \varphi \Rightarrow \psi$ and $\varphi \Rightarrow \psi$. Due to this subformula property we always finish the procedure either with a proof or with an open construction, where each final tree has at least one atomic sequent. The problem of possibly infinite branches reappear only in case of serial transitive logics $\text{MD4}$ and $\text{MD45}$. The definition of $\mathcal{S}_{D4}$ allows the repetition of sequents of the form $\varphi, \Box \varphi \Rightarrow$ but such loops are easy to detect and to deal with.

6 Final comments

An exploration of the 15 monotonic logics generated by the 5 most popular axioms shows many similarities with the situation known for their normal counterparts. But there are at least 3 points worth noticing:

1. Combinations of D, 4 and 5 require additional rules in order to obtain cut-free calculi.
2. $\text{M5}$ has cut-free calculus in contrast to $\text{K5}$ (and $\text{R5}$).
3. Monotonic 4-logics which are nonserial do not require additional tech-
nics to obtain decision procedures.

We can also consider monotonic logics with addition of the rule (N) which is the sequential version of the Necessitation rule. Such a rule together with (R) gives a formalization of $\text{K}$ but with (M) we are still in the class of monotonic logics, even if we add other rules under consideration, except (B). In case of B-logics (N) is admissible which is easy to demonstrate. In such a way we obtain 10 (15 − 5 B-logics) $\text{MN}$-counterparts of monotonic logics considered above. One can quite easily check that the addition of (N) does not affect our proof of cut-elimination, so we can conclude with:

**Lemma 7.** Cut is admissible in: $\text{SC-MN, SC-MND, SC-MNT, SC-MN4, SC-MN5, SC-MNT4, SC*-MND4, SC-MN45}$ and $\text{SC*-MND45}$.

**Acknowledgments.** I would like to thank anonymous referees for important remarks.
References


Department of Logic
University of Łódź
Kopcińskiego 16/18
90–232 Łódź
Poland

e-mail: indrzej@uni.lodz.pl